**Origins of jumping oscillons**, reference sheet for `JO1D`, March 6, 2025

Hannes Uecker, Institut für Mathematik, Universität Oldenburg, hannes.uecker@uol.de

In [KUY21] we discuss the bifurcation of jumping oscillons (JOs) as modulated traveling waves (MTWs) from traveling waves (TWs) in the 3–component FHN type model,

$$\begin{aligned} \partial_t u &= k_1 + k_2 u - u^3 - k_3 v - k_4 w + D_u \nabla^2 u + s\partial_x u, \\ \tau \partial_t v &= u - v + D_v \nabla^2 v + \tau s \partial_x v, \\ \vartheta \partial_t w &= u - w + D_w \nabla^2 w + \vartheta s \partial_x w, \end{aligned} \tag{1}$$

following [YZE06], where $k_{1,2,3,4} \in \mathbb{R}$ are parameters, $D_{u,v,w} > 0$ are diffusion constants, $\tau, \vartheta > 0$ are timescales, $u = u(t,x)$ is an activator variable while $v = v(t,x)$ and $w = w(t,x)$ are inhibitors. The term $s\partial_x U$, $U = (u,v,w)$ in (1) comes from going into a frame moving with speed $s$, such that we can compute TWs and MTWs as relative equiibria. The system has a unique uniform steady state $U_* \equiv (u_*, v_*, w_*)^{\mathrm{T}}$, and we mostly study bifurcations from $U_*$ in $\Omega = (-40, 40)$ with periodic boundary conditions (BCs), though for some computations of standing waves (SWs) and localized SWs (LSWs) (not discussed in [KUY21]) we consider (1) in $\Omega = (-40, 40)$ with Neumann BCs. We refer to [KUY21] and the references therein for further discussion of the model, and to [Uec21] and the tutorials at [pde25] for general background and usage of `pde2path`, and here only comment on the implementation of (1) in `pde2path` underlying the results in [KUY21]. Table 1 lists the pertinent files.

**Remarks.** The `JO1D` folder contains some somewhat advanced and experimental `pde2path` usage; most importantly, we use a quadratic finite element method (P2FEM, see `assem1d2`), which is in particular helpful to obtain a smooth behavior of the comoving frame speed $s$ during continuation of TWs as relative equilibria with a non–small stepsize. Moreover, starting points for continuation of traveling pulses (TPs), aka localized traveling waves (LTWs), are obtained from first "chopping–off" TWs (i.e., replacing the spatially periodic TWs by $U_*$ in parts of the domain) and then running numerical time integration, aka direct numerical simulation (DNS), to converge to TPs. A similar "chopping–off" is also used for SWs to obtain initial guesses for Newton–loops converging to localized standing waves (LSWs), which then yield snaking branches of LSWs (not shown in [KUY21]). However, in particular the convergence to LSWs after chopping–off SWs depends rather sensitively on the exact way of chopping–off. Finally, we also use chopping off and DNS to obtain convergence to TP–JO mixed mode bound states [KUY21, Fig.1], which again depends sensitively on the chop–off details. Thus, though the continuation and bifurcation methods in `JO1D` are standard, the folder contains some somewhat experimental (or "trial–and–error" and "trial–and–success") procedures to obtain initial points for continuation. ⌋

Table 1: Scripts and functions in `JO1D`. Associated to some `cmds*`–scripts are `cmds*plot` scripts for plotting; all figure numbers refer to [KUY21]. 1st block: scripts; 2nd block: problem describing functions and overloads of `pde2path` library functions; 3rd block: convenience functions.

| file | purpose, remarks |
|---|---|
| `cmdsTW` | Main script for TWs, localized TWs (LTWs) and mTWs, i.e., JOs, over $\Omega = (-40, 40)$ with periodic BCs. First we continue the trivial branch $U_*$ in $k_1$ (output to `b/0pbc`) and then use `twswibra` to switch to (spatially periodic) TWs, with the speed $s$ as a secondary parameter, using the standard phase condition (PC) $q(u) = \int (\partial_x U_{\mathrm{old}}) U \, \mathrm{d}x$ [KUY21, (3)], where $U_{\mathrm{old}}$ is the solution from the last step. |

| | |
|---|---|
| | We then obtain "localized" TWs (LTWs, aka traveling pulses, TPs) by "chopping off" TWs, i.e., replacing by $U_*$ all but one of the waves in a periodic TW and runing DNS. After convergence, we can then continue the LTWs again as relative equilibria, with some snaking behavior as additional pulses are added to the initial 1–pulse, and on the resulting branch there are Hopf points (HPs) to MTWs. After `hoswibra` from the HPs on the LTW branch, we switch to the average PC [KUY21, (4)]; the resulting MTW branch turns into a branch of stable 1–peak JOs after a first fold, and subsequently shows some further snaking behavior where the 1–peak JOs turn into $m$–peak JOs ($m \geq 2$). |
| `cmdsTWplot` | plotting of results from `cmdsTW`, Figs.1–2. |
| `cmdsSW` | compute the trivial branch with Neumann BCs (output to `b/0`, and then use `hoswibra` to switch to SWs (no PC needed due to the Neumann BCs). Then use some cut–off SW as an initial guess for a Newton loop which converges to a LSW. Continuing this in $k_1$ yields a snaking LSW branch which adds/takes away another wave in every second folds. Plotting of this directly at the end of `cmdsSW` (not shown in [KUY21]). |
| `cmdsdisp` | plot dispersion relations for linearization of (1) around $U_*$ [KUY21, Fig.S1]. Independent of domain, hence using a small domain. |
| `cmdsdns1` | DNS from various initial guesses, yielding various $n$TP–$m$JO bound states as in Fig.1. |
| `dns2hopf` | sample script how to use DNS to generate time periodic states (JOs) for continuation. |
| `yzeinit` | initialization of problem struct `p` with standard parameter values, call of `stanpdeo1D` to generate a 1D PDE object (interval, with mesh), initialization of $u$ with $u^*$, call of `assem1d2` to generate a P2–FEM mesh from the original (P1) mesh and from the P2–mesh generate the stiffness matrix $K$, the mass matrix $M$ and the convection matrix $K_x$, and finally resetting of some `pde2path` parameters to problem adapted values. |
| `oosetfemops` | assemble the mass matrix $M$, and the (1-component) Neumann-Laplacian $K$ and the (1–component) convection matrix $K_x$. For the case of periodic BCs, these are then also converted via `filltrafo`, and then stored in `p.mat`. `oosetfemops` is called after loading a point since `p.mat` is not saved to disk. |
| `sG,sGjac` | rhs of (1), and Jacobian; these here have a simple standard structure. |
| `nodalf` | "nonlinearity", i.e., terms without spatial derivatives, called in `hotintxs`. |
| `nodaljac` | Jacobian of "nonlinearity", called in `sGjac`. |
| `hobra,` `hobratw` | mods of respective library functions. |
| `assem1d2` | quadratic FEM |
| bdmovJO | Tsss |
| bdmovSW | |


[KUY21] E. Knobloch, H. Uecker, and A. Yochelis. Origin of jumping oscillons in an excitable reaction-diffusion system. *Phys. Rev. E*, 104:L062201, Dec 2021.

[pde25] pde2path. `https://pde2path.uol.de/`, 2025.

[Uec21] H. Uecker. *Numerical continuation and bifurcation in Nonlinear PDEs*. SIAM, Philadelphia, PA, 2021.

[YZE06] L. Yang, A. M. Zhabotinsky, and I. R. Epstein. Jumping solitary waves in an autonomous reaction-diffusion system with subcritical wave instability. *Phys. Chem. Chem. Phys.*, 8:4647–4651, 2006.