CH on cones, reference sheet for `CHcone`, October 14, 2024

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In [GHUV24] we discuss the Cahn–Hilliard problem

$$\begin{aligned} &\text{(a)} \quad -\varepsilon^2\Delta u + W'(u) - \lambda = 0 \text{ in } M, \quad \partial_\nu u = 0 \text{ on } \partial M,\\ &\text{(b)} \quad q(u) := \langle u\rangle - m = 0, \end{aligned} \tag{1}$$

on cones M with metric g, where $\varepsilon > 0$ is an interfacial energy parameter, Δ is the (negative) Laplace–Beltrami operator, W is a *double well potential*, for definiteness $W(s) := \frac{1}{4}(s^2-1)^2$ for $s \in \mathbb{R}$, $\langle u\rangle := \frac{1}{\mathrm{vol}_g(M)}\int_M u \,\mathrm{d}S$, m is a prescribed mass, and λ is the Lagrange multiplier λ for the mass constraint $q(u) = 0$; (1) is the first order necessary condition for minimizing

$$E_\varepsilon(u) := \frac{1}{2\sigma}\int_M \frac{\varepsilon}{2}|\nabla u|_g^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}S, \tag{2}$$

under $q(u) = 0$, with normalization $\sigma := \int_{-1}^1 \sqrt{W(s)/2}\,\mathrm{d}s = \sqrt{2}/3$. For smooth M, it is well known that minimizing (2) is closely related to constant mean curvature surfaces in M, i.e. surfaces which have minimal area among those satisfying a volume constraint, and under mild conditions sequences of minimizers u_ε of E_ε for $\varepsilon \to 0$ converge to a function u_0 which only takes values in the *pure phases* $u = \pm 1$, and $\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) = |I_0|$ where $|I_0|$ is the interface length, see, e.g., [Uec21, §6.9] and the references therein.

We show that this also holds for manifolds (M, g) with conical singularities and illustrate this by numerical continuation and bifurcation results obtained with `pde2path`. We choose truncated cones of height h with elliptic base at $\tilde z = 0$ of semi axes 1 and $a \ge 1$, parameterized over the unit disk, i.e.

$$\begin{pmatrix}\tilde x\\ \tilde y\\ \tilde z\end{pmatrix} = \phi(x,y) := \begin{pmatrix} ax\\ y\\ h(1-(x^2+y^2)^{1/2})\end{pmatrix}, \quad (x,y) \in \Omega = \{x^2+y^2 \le 1\}. \tag{3}$$

With a slight abuse of notation, the metric determinant is $g = a^2 + h^2(x^2+ay^2)/r^2$, $r^2 = x^2+y^2$, and $dS = \sqrt{g}\,\mathrm{d}(x,y)$, and the corresponding Laplace Beltrami operator is given by

$$\begin{aligned}\Delta u(x,y) = \frac{1}{\sqrt{g}}\Big[&\partial_x\big(\frac{1}{\sqrt{g}}(1+h^2y^2r^{-2})\partial_x u\big) - \partial_y\big(\frac{1}{\sqrt{g}}h^2xyr^{-2}\partial_x u\big)\\ &- \partial_x\big(\frac{1}{\sqrt{g}}h^2xyr^{-2}\partial_y u\big) + \partial_y\big(\frac{1}{\sqrt{g}}(a^2+h^2x^2r^{-2})\partial_y u\big)\Big].\end{aligned} \tag{4}$$

Table 1 gives an overview of the files used in the folder `CHcone`, available at [Uec24]. For general background and usage of `pde2path` we refer to [Uec21] and the tutorials at [Uec24].

[GHUV24] D. Grieser, S. Held, H. Uecker, and B. Vertman. Phase transitions on manifolds with conical singularities, *preprint*, 2024.

[Uec21] H. Uecker. *Numerical continuation and bifurcation in Nonlinear PDEs.* SIAM, Philadelphia, PA, 2021.

[Uec24] H. Uecker. `www.staff.uni-oldenburg.de/hannes.uecker/pde2path`, 2024.

Table 1: Scripts and functions in CHcone. All figure numbers refer to [GHUV24]. 1st block: scripts; 2nd block: problem describing functions, overloads of pde2path library functions, and convenience functions.

file	purpose, remarks
cmds1	First main script starting with (a,h,ε)=$(1.05,1,0.15)$ and the homogeneous solution $u\equiv m$, λ=$W'(m)$ at m=-0.8. Continuation in m yields bifurcations to various primary patterned branches in the spinodal region, roughly $m\in(-1+\delta,1-\delta)$ for some small $\delta>0$. Subsequently we do some continuation in $\varepsilon\to 0$, and fold-point continuation in m and h. Plotting in cmds1plot.m, yielding Figs.8–10.
cmds2	Similar to cmds1.m but starting at $(a,h,\varepsilon)=(1.05,0.25,0.15)$ and focussing on continuation in a,h, and ε. Plotting in cmds2plot.m, yielding Figs.1 and 11.
cmdsaux1	Interface lengths on circular cone as functions of h or a, yielding Fig.4, independent of FEM for 1.
cmdsaux2	Illustration of cusp, Fig.7, independent of 1.
chcinit	Initialization of problem struct p with standard parameter values, call of diskpdeo2 to generate a disk PDE object, initialization $u\equiv m$, resetting of some pde2path parameters and function handles in p to problem adapted values.
oosetfemops	Here a dummy function as both, the FEM mass and stiffness matrices are (re)assembled in sG as parameters change. In particular, p.mat.M=[1;1] is set to deliberately produce an error if p.mat.M is falsely used instead of getM.
getM	overload of library function as the mass matrix M is not pre-assembled here.
sG,sGjac	rhs of (1a), and Jacobian, including assembling of mass matrix M and stiffness matrix K in each call, i.e., [K,M]=LBcone3(p,h,ell).
bpjac,spjac	Jacobians for branch–point–continuation and fold–point–continuation.
qf2, qf2der	mass constraint (1b) and Jacobian.
chbra2	branch output [pars,E_ε, $\max(u-m),\min(u-m)$].
cplot	plot of u over actual cone (3), see levplots for level–line plot of u.
tricontour	helper for levplots, here slightly modified from library version wrt colors.
e2rs	elements–to–refine–selector for mesh–refinement based on Δu.
e2rs_rad	dummy elements–to–refine–selector for initial mesh–refinement near cone–tip.