helfcyl reference sheet, Alexander Meiners, Hannes Uecker, March 6, 2025

Institut für Mathematik, Universität Oldenburg, alexander.meiners@uol.de, hannes.uecker@uol.de

In [MU25] we consider the constrained L^2–gradient flow

$$\begin{pmatrix} V_t(t) \\ 0 \end{pmatrix} = \begin{pmatrix} G(X(t), \Lambda(t)) \\ q(X(t)) \end{pmatrix} \tag{1}$$

associated to the Helfrich energy

$$E(X) = \int_X (H - c_0)^2 \, \mathrm{d}S, \tag{2}$$

where $X = X(t)$ is a 2D manifold embedded in $\mathbb{R}^3$ with mean curvature H, and where the parameter $c_0 \in \mathbb{R}$ models an energetically favorable spontaneous (or preferred) curvature. In (1), $V_t(t)$ is the normal velocity of X, $q_1(X) = \mathcal{A}(X) - \mathcal{A}_0$ and $q_2(X) = \mathcal{V}(X) - \mathcal{V}_0$ are area and volume constraints, and $\Lambda = \Lambda(t) = (\lambda_1(t), \lambda_2(t))$ are Lagrange multipliers for these constraints. We consider manifolds X of cylindrical topology, with a "trivial" branch (parameterized by $\lambda_2 \in \mathbb{R}$) of steady states is given by the straight cylinders $\mathcal{C}_L$ with radius 1 and length L and $\lambda_1 = \frac{1}{2}\left(\frac{1}{2r^2} - 2r\lambda_2 - 2c_0^2\right)$. Via $X = X_0 + uN$ with u the normal displacement of X and N the (here inner) normal, (1) can be expressed as a quasilinear parabolic constrained problem

$$\eta(u)\partial_t u = G(u, \Lambda), \tag{3a}$$

$$0 = q(u), \tag{3b}$$

where $\eta(u) = g^{-1/2}(1+u)$ with $g = \det(g_{ij}) = (r+u)^2(1+u_x^2) + u_\varphi^2$. In [MU25] we in particular study the bifurcations from $\mathcal{C}_L$, with periodic BCs for u along the cylindrical axis x, leading to four basic classes of branches: pearling, wrinkling, buckling and coiling. In the numerics we focus on three particular values of c_0, namely

$$\text{(a) } c_0 = 0, \quad \text{(b) } c_0 = -1, \quad \text{(c) } c_0 = 0.48,$$

and present basic bifurcation diagrams BDs of steady states, including some secondary bifurcations, and also some numerical flows.

Here we comment on the associated `pde2path` implementation in the folder `helfcyl/`, available at [pde25], essentially listing the purpose of the script files `cmds*.m` and what Figures from [MU25] they produce, and giving some comments on "special" files for (3). See also `aaa.m` included in `helfcyl/` for similar (and slightly more detailed and possibly more up–to–date) information. For background and usage of `pde2path` we refer to [Uec21] and the tutorials at [pde25], and to [MU24] for further applications of `pde2path` in the geometric setting.

[MU24] A. Meiners and H. Uecker. Numerical continuation and bifurcation for differential geometric PDEs. *Numerical Mathematics – Theory Methods Applications*, OA-2024:0005, 2024.

[MU25] A. Meiners and H. Uecker. Helfrich cylinders – bifurcation analysis and numerics, 2025. Preprint, `https://arxiv.org/abs/2502.21137`.

[pde25] pde2path. `https://pde2path.uol.de/`, 2025.

[Uec21] H. Uecker. *Numerical continuation and bifurcation in Nonlinear PDEs*. SIAM, Philadelphia, PA, 2021.

Table 1: Scripts and functions in `helfcyl`; all figure numbers refer to [MU25]. First two blocks: Main scripts; 3rd block: Problem describing functions, incl. overloads of `pde2path` library functions and convenience functions; 4th block: Further auxiliary scripts.

file	purpose, remarks
`cmds0`	$c_0 = 0$, $L = 10$, output to "0/", Fig.4 and 5 (plotting in `cmds0plot.m`).
`cmds0fl`	Flows from (perturbations of) selected states computed in `cmds0`, Fig.6.
`cmdsm1`	$c_0 = -1$, $L = 10$, output to "m1/", Fig.7 (plotting in `cmdsm1plot.m`). Flows from (perturbations of) selected $c_0 = -1$ states in `cmdsm1fl`, Fig.8.
`cmds05`	$c_0 = 0.48$, $L = 10$, output to "05/", Fig.9 (plotting in `cmds05plot.m`). Flows from (perturbations of) selected states in `cmds05fl`, results not in paper.
`cmds05b`	$c_0 = 0.48$, $L = 15$ (natural length for primary pearling branch), output to "05b/", Fig.10 (plotting in `cmds05bplot.m`). Flows from (perturbations of) selected states in `cmds05bfl`, Fig.11.
`cmds05c`	$c_0 = 0.48$, $L = 22.5$, output to "05c/", Fig.12 (plotting in `cmds05cplot.m`). Flows from (perturbations of) selected states in `cmds05cfl`.
`cylinit`	initialization of struct `p` with standard parameter values incl. X–cont setting; call of `stanpdeo2D` to generate a 2D PDE mesh subsequently mapped to cylinder; idenfication of left/right boundary faces for periodicity, call of oosetfemops to generate "fill" matrices.
`oosetfemops`	here only identification of boundaries, no FEM matrices generated.
`sG,sGjac`	rhs G in (3), and Jacobian; note that the 4th order problem $\eta(u)\partial_t u = G(u,\Lambda)$ is written as two 2nd order problems with $u_2 = H$ (mean curvature of $X + uN$). Additionally, depending on the symmetry of the current branch the respective terms for phase conditions (PCs) are computed.
`qAV, qAVder`	area $\mathcal{A}$ and volume $\mathcal{V}$ constraints, and their u–derivatives; versions without further constraints, i.e., for continuation of straight cylinder, and for numerical flows.
`qAVxtr`	$\mathcal{A}$ and $\mathcal{V}$ constraints together with x–translational PC (used for pearling and coiling branches). u–derivative in `qAVxtrder`. Switched on via `stanxtr`. Similarly, `qAVxro, qAVxroder, stanxrot` for rotational PC (for wrinkling branches), and `qAVxtrro, qAVxtrroder, stanxtrrot` for translational *and* rotational PC (used for buckling branches).
`hvesbra`	branch output, inter alia the bending energy (2) computed in `benE`.
`getA, getV`	area and volume of X; the latter is somewhat special since here we do not have a closed X and hence need to apply Gauss with care, see [MU25, App.A.2].
`getN`	normal N to X, controlled by `p.ncurv`. For robustness it is convenient to average N between left and right boundary, and to keep N at boundary in the y–z plane (done if `p.ncurv`=1).
`getM`	FEM mass matrix; here directly in system form and with `filltrafo` to identify periodic boundaries.
`getbdnn`	helper function to identify BooundaryNearestNeighbor indizes `p.bdnn`, which are used in derivative of volume constraint, e.g., `qAVder`.
`refufu`	REFinementUserFUnction, mod of `stanufu` and linked into `p.fuha.ufu`; refines and coarsens the mesh controlled by `p.nc.qbound` (mesh–quality bound) and `p.nc.Ab` (area bound).
`upX`	overload of library update-X function taking care of periodic boundaries.
`imdnsX`	IMplicitDirectNum.Sim.X–setting, see [MU25, App.A.2]; mod of library `imdnsX` with control of E decay.
`Xflplot`	minor convenience mod of library version to plot flows of X computed in `imdnsX`.
`stanxtr`	convenience function to switch on translational x PC; similar `stanxrot, stanxtrrot`.
`cmdsstab`	script to plot stability ranges, Fig.2(a–c).
`cmdsdisp`	script to plot (selected) dispersion relations, using `md`, Fig.2(d).