STABLE MODULATING MULTI–PULSE SOLUTIONS FOR DISSIPATIVE SYSTEMS WITH RESONANT SPATIALLY PERIODIC FORCING

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Abstract

We show the existence and stability of modulating multi–pulse solutions for a class of bifurcation problems given by a dispersive Swift–Hohenberg type of equation with a spatially periodic forcing. Equations of this type arise as model problems for pattern formation over unbounded weakly oscillating domains and, more specifically, in laser optics. As associated modulation equation one obtains a nonsymmetric Ginzburg–Landau equation which has exponentially stable stationary n–pulse solutions. The modulating multi–pulse solutions of the original equation then consist of a traveling pulse–like envelope modulating a spatially oscillating wave train. They are constructed by means of spatial dynamics and center manifold theory. In order to show their stability we use Floquet–theory and combine the validity of the modulation equation with the exponential stability of the n–pulses in the modulation equation. The analysis is supplemented by a few numerical computations.
In addition we also show, in a different parameter–regime, the existence of exponentially stable stationary periodic solutions for the class of systems under consideration.

1 Introduction

As a model problem for the pattern formation in systems with a spatially periodic forcing we consider parabolic partial differential equations of type

$$\partial_t u = L(\partial_x)u + g(x)u + f(u, u_x). \qquad (1.1)$$

Here $u = u(t,x) \in \mathbb{R}$, $t>0$, $x \in \mathbb{R}$, L is a dissipative constant coefficient differential operator, g is a smooth spatially periodic function and f is a smooth nonlinear function with $|f(u,\partial_x u)| \leq C(|u|^2 + |\partial_x u|^2)$ for small $|u|$, $|\partial_x u|$. Below we choose L of Swift–Hohenberg type, and for this reason we call (1.1) a periodic Swift–Hohenberg equation (pSHe). Equations of type (1.1) arise in hydrodynamic bifurcation problems over oscillating domains, see e.g. [EK97, DS98]. Other interesting applications are electromagnetic waves in periodic media, see Remark 1.9.

For certain L, g and f we construct exponentially stable modulating n–pulse solutions $u_{\text{pu},n}$ to (1.1). These consist of a traveling n–pulse–like envelope modulating a spatially oscillating wave train, see figure 1. The method is inspired by [Sch99]. The first step consists in the derivation and justification

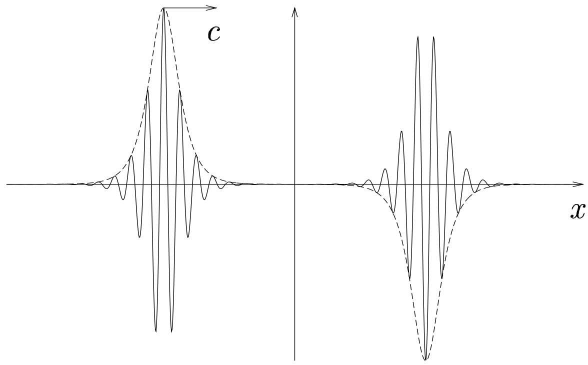

Figure 1: A modulating 2–pulse, traveling at speed c

of a non–symmetric Ginzburg–Landau equation (nsGLe) as the modulation equation associated to (1.1). For suitable choice of parameters the nsGLe has exponentially stable n–pulse–solutions [KS98]. This suggests the existence of $u_{\text{pu},n}$ in (1.1) and will be used later to show their stability. The construction of $u_{\text{pu},n}$ will be done via a spatial dynamics formulation and center manifold theory.

Our particular choice of L and g is as follows. We take

$$Lu = [-(1+\partial_x^2)^2 - \varepsilon^2\eta\alpha_0 + \beta_1\partial_x + \beta_3\partial_x^3]u,$$

with $\beta_1, \beta_3 \in \mathbb{R}$, $\alpha_0, \eta > 0$ and $0 < \varepsilon^2 \ll 1$. The operator $-(1+\partial_x^2)^2 - \varepsilon^2\eta\alpha_0$ is the linear part of, e.g., the (subcritical) Swift–Hohenberg equation, where the small bifurcation parameter ε^2 measures the distance from the onset of instability of the trivial solution $u \equiv 0$, see [SH77, Man92]. The terms $\beta_1\partial_x u + \beta_3\partial_x^3 u$ account for dispersion. For g we make the rather specific choice

$$g(x) = 2\varepsilon^2\eta\alpha_1\cos(2x) \tag{1.2}$$

with $\alpha_1 > 0$. Later we assume that $\alpha_1, \alpha_0 = \mathcal{O}(1)$ and take η as additional small parameter, introduced for convenience.

The equation $u_t = Lu$ possesses solutions $e^{\mathrm{i}kx+\lambda(k)t}$ where

$$\lambda(k) = -(1-k^2)^2 - \varepsilon^2\eta\alpha_0 + \mathrm{i}(\beta_1 k - \beta_3 k^3). \tag{1.3}$$

We call $k_c = 1$ the critical wavenumber and consider $0 < \varepsilon^2 \ll 1$ as the main small bifurcation parameter. Moreover, we assume that the phase velocity of the critical mode $e^{\mathrm{i}x}$ is of order $\mathcal{O}(\varepsilon^2)$, i.e.

$$\omega_0 = \beta_3 - \beta_1 = \nu_0\varepsilon^2 \text{ with } 0 < \nu_0 = \mathcal{O}(1). \tag{1.4}$$

Together with (1.2) this means that the periodic amplification $g(x)u$ is in spatial and temporal resonance with the critical wave $e^{\mathrm{i}x}$ of $L(\partial_x)$. For slightly non–resonant forcings see Remarks 2.1 and 2.2. Note that by (1.4) we could express, e.g., β_1 through ε^2, ν_0 and β_3. However, we keep all four parameters since this makes the analysis more transparent.

By the multiple scaling ansatz

$$u(t,x) = \psi_A(t,x) = \varepsilon A(T,X)e^{\mathrm{i}x} + \text{c.c.} + \text{h.o.t.}, \tag{1.5}$$

where $X = \varepsilon(x - \nu_1 t), T = \varepsilon^2 t, \nu_1 = 3\beta_3 - \beta_1$, c.c. means complex conjugate, and h.o.t. denotes higher order terms, we formally obtain the non–symmetric complex Ginzburg–Landau equation

$$A_T = c_1\partial_X^2 A - (\eta\alpha_0 + \mathrm{i}\nu_0)A + \eta\alpha_1\overline{A} + c_3|A|^2 A, \tag{1.6}$$

as the modulation equation for the complex envelope $A = A(T,X) \in \mathbb{C}$, where $c_j = c_{jr} + \mathrm{i}c_{ji} \in \mathbb{C}, j = 1,3$. Because of the term $\eta\alpha_1\overline{A}$ the usual $\mathcal{S}^1$ symmetry $A \mapsto e^{\mathrm{i}\theta}A$ of the Ginzburg–Landau equation is broken. Due to the special choice $g(x) = 2\varepsilon^2\eta\alpha_1\cos(2x)$ the derivation of (1.6) is straightforward, see section 2. In Remark 1.3 below we preview the coefficients c_1 and c_3 in (1.6) and comment on the peculiar choice of parameters in the model.

The next step is to show that solutions of (1.1) may be well approximated over an $\mathcal{O}(1/\varepsilon^2)$ timescale by the ansatz (1.5) if A solves (1.6), see Theorem 2.6. Here we use the uniformly local Sobolev spaces $H^m_{\text{lu}}(\mathbb{R})$, [MS95]. These Banach spaces contain all kinds of bounded functions and are defined as follows: fix the weight function $\rho(x) = 1/\cosh(x)$ and let

$$\|u\|^2_{L^2_{\text{lu}}} = \sup_{y\in\mathbb{R}} \int u^2(x)\rho(x+y)\,\mathrm{d}x, \quad \tilde{L}^2_{\text{lu}}(\mathbb{R}) = \{u \in L^2_{\text{loc}}(\mathbb{R}) : \|u\|_{L^2_{\text{lu}}} < \infty\},$$

$L^2_{\text{lu}}(\mathbb{R}) = \{u \in \tilde{L}^2_{\text{lu}}(\mathbb{R}) : \|T_y u - u\|_{L^2_{\text{lu}}} \to 0 \text{ as } y \to 0\}$, where $(T_y u)(x) = u(x-y)$. Then

$$H^m_{\text{lu}}(\mathbb{R}) := \{u \in L^2_{\text{lu}} : \partial_x^j u \in L^2_{\text{lu}} \text{ for } 0 \le j \le k\}. \tag{1.7}$$

Since the spaces $H^m_{\text{lu}}(\mathbb{R})$ are based on $L^2(\mathbb{R})$ the global existence of solutions for typical dissipative systems can be shown via Fourier transform methods and weighted energy estimates, see [MS95, Mie97a].

In the nonlinear Schrödinger limit with anomalous dispersion, that is for $c_{1r}, c_{3r}{=}0$ and $c_{1i}, c_{3i}{>}0$, the nsGLe (1.6) is called the parametrically forced Schrödinger equation (pfNLSe). This equation models the propagation of pulses in a nonlinear optical fiber with linear loss ($-\eta\alpha_0 < 0$), compensated by phase sensitive amplifiers ($\eta\alpha_1 > 0$) with non zero phase mismatch ($\nu_0 > 0$), see [KK96] and references therein.

In [KS98] it is shown that for small η and a suitable choice of α_0, α_1 the pfNLSe has exponentially stable pulse solutions in the form $\pm\sqrt{b_1}\mathrm{sech}(\sqrt{b_2}x)e^{\mathrm{i}\theta}$ with some fixed $b_1, b_2, \theta > 0$. Adding small dissipation $c_{1r} > 0$ there exist stationary n–pulse solutions of the pfNLSe. These n-pulses resemble n widely spaced concatenated single pulses of the form $\pm\sqrt{b_1}\mathrm{sech}(\sqrt{b_2}x)e^{\mathrm{i}\theta}$. Moreover, n–pulses of type up–down are exponentially orbitally stable while all other n–pulses are unstable. Here "up" means "+" and "-" means down, and "type up-down" means an n–pulse with alternating up and down. In section 3 we give a brief review of the analysis from [KS98]. Here we summarize the result as follows.

Theorem 1.1 *[KS98, Corollary 7.4, Theorem 7.5] Fix $\eta > 0$ small, $\nu_0 > 0$ and $n \geq 1$. There exists a set $\mathcal{P} = \mathcal{P}(\eta, \nu_0, n) \subset \mathbb{R}^6$ of parameters such that for $(c_{1r}, c_{1i}, c_{3r}, c_{3i}, \alpha_0, \alpha_1) \in \mathcal{P}$ there exists a one parameter family*

$$\mathcal{M}_{A,n} = \{A_{\mathrm{pu},n} = A_{\mathrm{pu},n}(\cdot - X_0) : X_0 \in \mathbb{R}\}$$

of n–pulse solutions to (1.6) of type up-down. This family is exponentially orbitally stable. This means that there exist $C_1, C_2, b_0 > 0$ such that if

$$\|A_0(\cdot) - A_{\mathrm{pu},n}(\cdot - X_0)\|_{H^1_{\mathrm{lu}}} \leq C_1$$

for some $X_0 \in \mathbb{R}$, then there exists an $X_1 \in \mathbb{R}$ such that

$$\|A(T, \cdot) - A_{\mathrm{pu},n}(\cdot - X_1)\|_{H^1_{\mathrm{lu}}} \leq C_2 e^{-b_0 T}.$$

Remark 1.2 In [KS98] the above theorem is proved for $c_{3r} = 0$. However, it is clear that the results extend to small $c_{3r} \neq 0$, see section 3. Moreover, in [KS98] the analysis is done in $L^2(\mathbb{R})$. The generalization to $H^1_{\mathrm{lu}}(\mathbb{R})$ is no problem since the linearization around an n–pulse $A_{\mathrm{pu},n}$ possesses exactly one simple eigenvalue 0 from the translation invariance of (1.6) and the rest of the spectrum is bounded away from the imaginary axis.

Remark 1.3 From (1.3) we obtain $c_1 = 4 + 3\mathrm{i}\beta_3$ in (1.6), see section 2, while c_3 depends on β_3 and the specific choice of f. We assume the nonlinearity f in (1.1) to be of the form $f(u, u_x) = f_1 u^2 + f_2 u u_x$ with $f_1, f_2 \in \mathbb{R}$. This gives

$$c_3 = (2f_1 + \mathrm{i}f_2)\left(2f_1 + \frac{f_1 + \mathrm{i}f_2}{9 + 6\mathrm{i}\beta_3}\right), \tag{1.8}$$

cf. (2.3), and hence we have sufficient degrees of freedom to obtain any c_3 we like. Higher order terms could be easily included into f. In [KS98] the analysis is done with the normalization c_{1i}=1 and c_{3i}=4. Clearly this can be achieved in (1.6) by rescaling

$$A(T,X) = \sqrt{\frac{4}{c_{3i}}}\widetilde{A}(T, X/\sqrt{c_{1i}}),$$

which could also be included directly into the ansatz (1.5). Small c_{1r} then means $1 \gg \eta \gg c_{1r}/c_{1i} = 4/(3\beta_3) > 0$, see [KS98, section 7.2], and hence $\beta_3 \gg 1/\eta$, which together with (1.4) also implies that the group velocity of e^{ikx} has to be large, i.e.

$$\nu_1 = 3\beta_3 - \beta_1 \gg 1/\eta. \tag{1.9}$$

Similarly, $\beta_3 \gg 1/\eta$ implies $c_3 \approx 4f_1^2 + 2\mathrm{i}f_1 f_2$, and hence small c_{3r} means $1 \gg \eta \gg 2f_1/f_2 > 0$, giving $f_2 \gg f_1/\eta$. Finally, another basic assumption for our analysis is

$$\nu_0 - \eta\alpha_1 > 0, \tag{1.10}$$

i.e, the phase mismatch has to be greater than the parametric gain. Thus, alltogether we need a rather specific choice of parameters, taking into account the three different scales $1 \gg \eta \gg \varepsilon > 0$, the resonance conditions (1.2) and (1.4), and the set $\mathcal{P}$ from Theorem 1.1, in particular the smallness of c_{1r}, c_{3r} discussed above. However, in numerical simulations it turns out that modulating multi-pulses can be found numerically in a wide range of parameters, see section 6 for some specific examples.

We may now state our existence theorem for the modulating n–pulse solutions for the original equation (1.1). They are constructed in section 4 by means of spatial dynamics and center manifold theory.

Theorem 1.4 *Fix $\eta > 0$ small, $\nu_0 > 0$ and $n \geq 1$. Assume that $\alpha_0, \alpha_1, \beta_1, \beta_3$ and f are chosen in such a way that in (1.6) we have $(c_{1r}, c_{1i}, c_{3r}, c_{3i}, \alpha_0, \alpha_1) \in \mathcal{P}(\eta, \nu_0, n)$ with $\mathcal{P}(\eta, \nu_0, n)$ from Theorem 1.1. Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ there exist a $c = \nu_1 + \mathcal{O}(\varepsilon) \in \mathbb{R}$ and a one dimensional family of modulating up–down n–pulse–solutions of the pSHe (1.1) in the form*

$$\begin{gathered}
\mathcal{M}_{u,n} = \{u(t,x) = u_{\mathrm{pu},n}(x, x - ct - x_0) : x_0 \in \mathbb{R}\},\\
u_{\mathrm{pu},n}(p,\xi) = \varepsilon A_{\mathrm{pu},n}(\varepsilon\xi)e^{\mathrm{i}p} + \mathrm{c.c.} + \mathcal{O}(\varepsilon^2),\\
u_{\mathrm{pu},n}(p,\xi) = u_{\mathrm{pu},n}(p+2\pi,\xi), \qquad \lim_{\xi\to\pm\infty} u_{\mathrm{pu},n}(p,\xi) = 0.
\end{gathered}$$

with $A_{\mathrm{pu},n}$ from Theorem 1.1.

In a moving frame the modulating n–pulses are time periodic. Thus we use Floquet theory to prove the stability of the family $\mathcal{M}_{u,n}$. In section 5 we combine the validity of (1.6) as a modulation equation for (1.1) with the exponential orbital stability of the n–pulses for the nsGLe to obtain the following result.

Theorem 1.5 *For each $\eta, \nu_0, n, \alpha_0, \alpha_1, \beta_1, \beta_3$ and f from Theorem 1.4 there exist $C_1, C_2, b, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let*

$$\|u_0(\cdot) - u_{\mathrm{pu},n}(\cdot, \cdot - x_0)\|_{H^1_{\mathrm{lu}}} \leq C_1\varepsilon$$

for some $x_0 \in \mathbb{R}$, and let u be the solution (1.1) with $u|_{t=0} = u_0$. Then u exists for all $t > 0$, and there exists an $x_1 \in \mathbb{R}$ such that

$$\|u(t, \cdot) - u_{\mathrm{pu},n}(\cdot, \cdot - ct - x_1)\|_{H^1_{\mathrm{lu}}} \leq C_2 \varepsilon e^{-b\varepsilon^2 t} \quad \text{as} \quad t \to \infty. \tag{1.11}$$

Remark 1.6 In Theorem 1.4, we may construct modulating n–pulses with any combinations of ups and downs. Here we focus on modulating up–down n–pulses since they inherit the stability of the up–down n–pulse solutions of the nsGLe, and from section 5 it will be clear, that all other modulating n–pulses inherit the instability of the other n–pulses. The difference between, e.g., a modulating up–down 2–pulse and a modulating up–up 2–pulse is of course less clear than for the 2–pulses in the nsGLe itself. Here the point is, that also for the multi–pulses in the nsGLe, it is rather the way that the individual humps are linked together in between the humps that decides on the stability of the multi–pulse solutions, not the signs of the humps themselves, see the discussion in [SJA97, section 2.3].

However, for the evolution from, e.g., an up–up pulse in the nsGLe and a modulating up–up pulse in the pSHe we conjecture two rather different scenarios: for the up–up pulse in the nsGLe, numerical simulations reveal that generically the flat region in between the two humps raises up and the solution converges to a one–pulse. For the modulating up–up pulse in the pSHe we expect that the solution only slightly changes the shape of the tails of the humps and rearranges in phase at the humps, thus converging to a modulating up–down 2–pulse.

Remark 1.7 In the nsGLe there exist n–pulses for various n in overlapping parameter regimes, cf. [KS98]. Thus, by Theorems 1.4 and 1.5 we have asymptotically stable modulating n–pulses for various n, and hence the coexistence of multiple attractors in the pSHe. A numerical example for this is given in section 6.

Remark 1.8 In a different parameter regime, the nsGLe has nontrivial exponentially stable homogenous fixed points, see Appendix C. These provide exponentially stable stationary periodic solutions for (1.1), so called rolls. This behaviour is quite different from, e.g., the diffusive stability of rolls in the classical Swift–Hohenberg equation, see [Sch96, Uec99]. This shows again the significant influence of the spatially periodic forcing $g(x)u$ in the resonant case.

Remark 1.9 Pulse–like modulations of electromagnetic waves are used to transport digital information in nonlinear optics. In an optical fiber we have dissipation and dispersion of energy. Recently, see [MS99] and the references therein, it was proposed to use a supporting periodic structure of the wave-length of light in the fiber to compensate for loss an jitter. In this sense (1.1) may be considered as a phenomenological model of such a device. See also, e.g., [LMN95] for the derivation of equations of Swift–Hohenberg type from the Maxwell–Bloch equations.

The plan of the paper is as follows. In section 2 we derive the nsGLe (1.6) as the modulation for the pSHe (1.1), comment on the existence of solutions to the (time–dependent) nsGLe, Lemma 2.4, and show the approximation property Theorem 2.6. In section 3 we review the results from [KS98] concerning the existence and stability of stationary n–pulses for the pfNLSe. This also shows how these results extend to the nsGLe. The proof of Theorem 1.4, i.e. the construction of the modulating n–pulses $u_{\mathrm{pu},n}$ for the pSHe (1.1) is done in section 4. The stability of the n–pulses $u_{\mathrm{pu},n}$ is shown in section 5. In section 6 we give a few numerical results confirming and illustrating our analysis.

In Appendix A we calculate the spectrum of the spatially periodic operator $L + g(x)$. Strictly speaking, this and the calculation of the linearly most unstable waves should be the first step in the derivation of the modulation equation for (1.1). However, due to our special choice of g this derivation is straightforward as given in section 2. The spectrum of $L + g(x)$ is needed in the proof of Theorem 1.5.

In Appendix B we show an attractivity result for the set of modulated patterns for the linearization of (1.1) around an n–pulse $u_{\mathrm{pu},n}$. This is also needed in the proof of Theorem 1.5. From the proof of Lemma B.2 it will be clear that a similar result also holds for (1.1). This, together with the approximation property shows the so called validity of the nsGLe as the modulation equation for (1.1).

2 Derivation and justification of the nsGLe as the modulation equation for the pSHe

2.1 Derivation

We use multiple scaling analysis to formally derive (1.6) as a modulation equation for (1.1). In order to keep the calculations simple we consider the particular spatially–periodic function $g(x) = 2\varepsilon^2\alpha_1\cos(2x)$ in (1.1). Moreover we assume that f is in the form $f(u,u_x) = f_1u^2 + f_2uu_x$. It is well known [Sch94a], that in the nondegenerated case only the quadratic and cubic terms in the nonlinearity of the original equation determine the coefficients of the cubic term in the associated modulation equation. Moreover, for the question of validity cubic terms in f are no problem, see [KMS92]. Therefore we restrict ourselves to quadratic f.

As noted in the Introduction, the equation $u_t = Lu$ possesses solutions $e^{\mathrm{i}kx+\lambda(k)t}$ where $\lambda(k) = -(1-k^2)^2 - \varepsilon^2\alpha_0 + \mathrm{i}(\beta_1 k - \beta_3 k^3)$. Expanding λ around $k_c = 1$ we obtain

$$\lambda(1+\varepsilon K) = -(\alpha_0 + \mathrm{i}\nu_0)\varepsilon^2 - i\nu_1\varepsilon K - (\lambda_2 + \mathrm{i}\nu_2)\varepsilon^2K^2 + \mathcal{O}(\varepsilon^3) \tag{2.1}$$

where ν_1=$3\beta_1 - \beta_1, \lambda_2$=$4, \nu_2$=$3\beta_3$. The derivation of the modulation equation now proceeds in the usual way, see for instance [Sch94c, MS96]. We let

$$u(t,x) = \frac{\varepsilon^2}{2}A_0(T,X)e_0 + \varepsilon A(T,X)e_1 + \varepsilon^2A_2(T,X)e_2 + \text{c.c.} + \text{h.o.t.}, \tag{2.2}$$

where $X = \varepsilon(x - \nu_1 t)$, $e_j = e^{\mathrm{i}jx}$. Inserting this into (1.1), using $f(u,u_x) = f_1u^2 + f_2uu_x$, and equating coefficients in front of $\varepsilon^j e_j$ we obtain the closed system of equations

$$\begin{aligned}
\varepsilon^2 e_0: &\quad 0 = -A_0 + 2f_1|A|^2\\
\varepsilon^2 e_2: &\quad 0 = -9A_2 + (f_1 + \mathrm{i}f_2)A^2 + \mathrm{i}(2\beta_1 - 8\beta_3)A_2\\
\varepsilon^3 e_1: &\quad A_T = (4 + 3\mathrm{i}\beta_3)\partial_X^2A - (\eta\alpha_0 + \mathrm{i}(\beta_3 - \beta_1)/\varepsilon^2)A + \eta\alpha_1\overline{A}\\
&\qquad + (2f_1 + \mathrm{i}f_2)(A_0A + A_2\overline{A}).
\end{aligned}$$

Eliminating A_0 and A_2 and using ω_0=β_3−β_1=$\varepsilon^2\nu_0$ we obtain the nsGLe (1.6), i.e. $A_T = c_1\partial_X^2A - (\eta\alpha_0 + \mathrm{i}\nu_0)A + \eta\alpha_1\overline{A} + c_3|A|^2A$, with

$$c_1 = 4 + 3\mathrm{i}\beta_3 \quad \text{and} \quad c_3 = (2f_1 + \mathrm{i}f_2)\left(2f_1 + \frac{f_1 + \mathrm{i}f_2}{9 + 6\mathrm{i}\beta_3}\right). \tag{2.3}$$

Remark 2.1 This straightforward formal derivation of the nsGLe is due to our special choice of $g(x) = 2\varepsilon^2\eta\alpha_0\cos(2x) = \varepsilon^2\eta\alpha_0(e_2 + e_{-2})$. For more general $2\pi/p$–periodic functions $g(x) = \gamma \sum_{j\in\mathbb{Z}} h_j e^{\mathrm{i}jpx}$ with $|h_j| \leq 1$ for all j, the derivation of the modulation equation is more complicated. Interesting phenomena occur in case that the resonance condition $p \approx nk_c$ for some $n \in \mathbb{N}$ holds, see [DS98], where in particular relations between γ and n for which one obtains the nsGLe as the modulation equation are analyzed in detail.

Remark 2.2 If the phase velocity $\omega_0 = \beta_3 - \beta_1$ of $e^{\mathrm{i}x}$ is of order $\mathcal{O}(\varepsilon^{2+\delta})$ for some $\delta > 0$ then it obviously drops out of the modulation equation, i.e. we obtain ν_0=0 in (1.6). In this case there may exist n–pulse–solutions to (1.6), but if they do, we do not know whether or not they are stable, see section 3. In case that $\omega_0 = \nu_0\varepsilon^{2-\delta}$ we basically obtain a symmetric Ginzburg–Landau equation. This can be seen from inserting the ansatz $u(t,x) = \varepsilon B(T,X)\tilde{e}_1$ with $T = \varepsilon^2 t,\ X = \varepsilon(x - \nu_1 t)$ and $\tilde{e}_j = e^{\mathrm{i}j(x-\omega_0 t)}$ into (1.1). Comparing coefficients at order $\varepsilon^3\tilde{e}_1$ we obtain

$$B_T = c_1\partial_X^2 B - \eta\alpha_0 B + \eta\alpha_1\overline{B}e^{2\mathrm{i}\omega_0 t} + c_3|B|^2 B. \tag{2.4}$$

Substituting $B(T,X) = A(T,X)e^{\mathrm{i}\omega_0 t} = A(T,X)e^{\mathrm{i}\nu_0 T/\varepsilon^\delta}$ into (2.4) we obtain $A_T = c_1\partial_X^2 A - (\eta\alpha_0 + \mathrm{i}\varepsilon^{-\delta}\nu_0)A + \alpha_1\overline{A} + c_3|A|^2 A$ and this limits to the nsGLe (1.6) as $\delta \to 0$. However, for $\delta > 0$ it is more instructive to consider directly (2.4). Since $\omega_0 t = \nu_0 T/\varepsilon^\delta$, the term $\eta\alpha_1\overline{A}e^{2\mathrm{i}\omega_0 t}$ is highly oscillatory and may be averaged out, thus giving a symmetric Ginzburg–Landau equation for B.

2.2 The approximation property

To show that a solution A of the nsGLe gives a good approximation of a solution u of (1.1) via (1.5), we have to bound the error $R(t,x) = u(t,x) - \psi_A(t,x)$ over an $\mathcal{O}(1/\varepsilon^2)$ timescale in the original equation. This timescale is necessary in order to see interesting modulations. This question was first treated in [CE90] for the Swift–Hohenberg equation, see also [vH91]. In [KMS92] a simple proof for the cubic case was given.

Here we adopt the method presented in [Sch94c] to handle the quadratic nonlinearity and the spatially periodic part. The analysis is done in the spaces $H^m_{\mathrm{lu}}(\mathbb{R})$, see (1.7). We do not distinguish real spaces $H^m_{\mathrm{lu}}(\mathbb{R}) = H^m_{\mathrm{lu}}(\mathbb{R},\mathbb{R})$ and complex spaces $H^m_{\mathrm{lu}}(\mathbb{R},\mathbb{C}) = H^m_{\mathrm{lu}}(\mathbb{R}) + \mathrm{i}H^m_{\mathrm{lu}}(\mathbb{R})$. However, in the following we write $Z^m = H^m_{\mathrm{lu}}(\mathbb{R})$ for the phase space for (1.1), and $Y^m = H^m_{\mathrm{lu}}(\mathbb{R}) + \mathrm{i}H^m_{\mathrm{lu}}(\mathbb{R})$ for the function space for the Ginzburg–Landau equation. Thus, we consider $A \mapsto \psi_A$ in (1.5) as a mapping from Y^m into Z^m. For this mapping we have the estimates

$$\|\psi_A\|_{L^\infty} \leq 2\varepsilon\|A\|_{L^\infty}, \quad \|\psi_A\|_{Z^0} \leq 2\sqrt{\varepsilon}\|A\|_{Y^0}, \quad \text{and } \|\psi_A\|_{Z^m} \leq C\varepsilon\|A\|_{Y^m}$$

for $m \geq 1$. In the second estimate a factor $\sqrt{\varepsilon}$ is lost by scaling. The third estimate holds since $\|A\|_{L^\infty(\mathbb{R})} \leq C\|A\|_{Y^1}$ by Sobolev's imbedding theorem, and since $\partial_x A(\varepsilon x) = \varepsilon \partial_X A(X)$, i.e., each derivative generates a factor ε.

Remark 2.3 Strictly speaking, the subsequent results in this subsection are not used in our further analysis. In fact, to prove the stability of the modulating n–pulses we only need a linear variant of the approximation property Theorem 2.6 below, i.e., the approximation property for the linearization around a modulating pulse, see section 5. However, we think that it is helpful to realize the general nonlinear case, and also to see (very briefly) the (local) existence theory for solutions to the nsGLe.

We obtain the following result concerning the existence of solutions to (1.6).

Lemma 2.4 *For all $A_0 \in Y^1$ there exist a $T_0 > 0$ and a unique local solution*

$$A \in C([0,T_0), Y^1) \cap C((0,T_0), Y^2) \cap C^1((0,T_0), Y^0)$$

to (1.6) with $A(0,X) = A_0(X)$.

Proof. We only sketch the main steps. Letting $A(T,X) = U(T,X) + \mathrm{i}V(T,X)$ we obtain

$$\partial_T W = L^{\mathrm{GL}}(\partial_X) W + F(W), \tag{2.5}$$

where $W = (U,V)^T$ and

$$L^{\mathrm{GL}}(\partial_X) = \begin{pmatrix} \eta(\alpha_1 - \alpha_0) + c_{1r}\partial_X^2 & -c_{1i}\partial_X^2 + \nu_0 \\ c_{1i}\partial_X^2 - \nu_0 & -\eta(\alpha_1 + \alpha_0) + c_{1r}\partial_X^2 \end{pmatrix},$$

$$F(W) = (U^2 + V^2) \begin{pmatrix} c_{3r} & -c_{3i} \\ c_{3i} & c_{3r} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

The linear operator $L^{\mathrm{GL}}(\partial_X)$ is sectorial with the spectrum given as two curves

$$\lambda_{1,2}(k) = -\eta\alpha_0 - c_{1r}k^2 \pm \mathrm{i}\sqrt{(c_{1i}k^2 + \nu_0)^2 - \eta^2\alpha_1^2}, \quad k \in \mathbb{R}. \tag{2.6}$$

Note that we assume $\nu_0 \geq \eta\alpha_1$, cf. (1.10), such that the root is real for all k. Therefore $L^{\mathrm{GL}}(\partial_X)$ generates a holomorphic semigroup

$$e^{L^{\mathrm{GL}}(\partial_X)T} : Y^0 \to Y^m \text{ with } \|e^{L^{\mathrm{GL}}(\partial_X)T}\|_{Y^0 \to Y^m} \leq Ce^{-\eta\alpha_0 T}(1 + T^{-m/2}).$$

Since $Y^1 \hookrightarrow L^\infty$ the nonlinearity F is locally Lipschitz from Y^1 into Y^0 and we obtain the local existence using standard theory for parabolic equations, see, e.g., [Hen81]. □

Remark 2.5 Since the origin is exponentially stable in (2.5), it is easy to see, that there exists an $R > 0$ such that for all $A_0 \in Y^1$ with $\|A_0\|_{Y^1} \leq R$ the solution exists globally in time and satisfies $\|A(T)\|_{Y^1} \leq R$ for all times.

Theorem 2.6 *Let $A \in C([0,T_0],Y^1)$ be a solution of the nsGLe. Then for all $d > 0$ there exist $\varepsilon_0, D > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the following holds. For $\|u_0(\cdot) - \psi_A(0,\cdot)\|_{Z^1} \leq d\varepsilon^{5/4}$ there exist a solution u to the pSHe (1.1) for $0 \leq t \leq T_0/\varepsilon^2$ such that $u|_{t=0} = u_0$ and*

$$\sup_{0\leq t\leq T_0/\varepsilon^2} \|u(t,\cdot) - \psi_A(t,\cdot)\|_{Z^1} \leq D\varepsilon^{5/4}. \tag{2.7}$$

Proof. We write $f(u,u_x)$ as symmetric bilinear form $N(u,u) = f(u,u_x)$. Letting $u(t,x) = \varepsilon\tilde{\psi}_A(t,x) + \varepsilon^{5/4}R(t,x)$ we obtain

$$\partial_t R = LR + g(x)R + 2\varepsilon N(\tilde{\psi}_A, R) + \varepsilon^{5/4}N(R,R) + \varepsilon^{-5/4}\text{Res}(\varepsilon\tilde{\psi}_A), \tag{2.8}$$

where $\text{Res}(\psi) = -\psi_t + L\psi + g(x)\psi + f(\psi,\psi_x)$ is called the residuum. From (2.8) we need to bound the error $R(t)$ by some kind of Gronwall estimate. The linear operator $\mathcal{L} = L + g(x)$ generates a (weakly) exponentially damped holomorphic semigroup $e^{\mathcal{L}t}$, i.e.

$$\|e^{\mathcal{L}t}u\|_{Z^m} \leq Ce^{-\varepsilon^2\eta\alpha_0 t}(1 + t^{-(m-n)/2})\|u\|_{Z^n}, \tag{2.9}$$

cf. (A.2). However, because of the term $\varepsilon N(\tilde{\psi}, R)$, and since moreover

$$\varepsilon^{-5/4}\text{Res}(\varepsilon\tilde{\psi}) = \varepsilon^{7/4}\eta a_1 Ae_3 + \text{c.c.} + \mathcal{O}(\varepsilon^{9/4})$$

this does not allow to bound R by $\mathcal{O}(1)$ on an $\mathcal{O}(1/\varepsilon^2)$ timescale via naive integration of (2.8).

This problem can be solved by using so called mode filters which are defined via smooth cut–off functions in Fourier space. The idea is to write u as

$$u = \varepsilon\tilde{\psi}_A^c + \varepsilon^2\tilde{\psi}_A^s + \varepsilon^{5/4}R_c + \varepsilon^{9/4}R_s,$$

where $\tilde{\psi}_A^c$ and R_c correspond to critical Fourier modes, and $\tilde{\psi}_A^s$ and R_s contain stable modes. In the proof of Theorem 2.6 the only substantial difference compared to, e.g., [Sch94c] is the term $\varepsilon^3\eta\alpha_1 Ae_3 + \text{c.c.}$ in

$$\text{Res}(\varepsilon\tilde{\psi}_A) = \varepsilon^3\eta a_1 Ae_3 + \text{c.c.} + \mathcal{O}(\varepsilon^4).$$

But Ae_3 only contributes to the stable part R_s of the error. Therefore the proof of Theorem 2.6 works as in the case $\alpha_1 = 0$ and we refer to [Sch94c] for the details. □

Remark 2.7 Due to the smoothing properties of the pSHe we may replace Z^1 in (2.7) by Z^4 for $t > 0$, see again [Sch94c].

3 The pulses for the pfNLSe and the nsGLe

This section reviews the results from [KS98] concerning the existence and stability of stationary n–pulse solutions for the pfNLSe ((1.6) with c_{1r}=c_{3r}=0 and $c_{1i}, c_{3i} > 0$). This also shows how they extend to the nsGLe. Without loss of generality we assume $c_{1i} = 1, c_{3i} = 4$, see Remark 1.3. Letting $A = U + \mathrm{i}V$ the stationary problem written as a first order equation for $W = (U, U', V, V') \in \mathbb{R}^4$ reads

$$\partial_X W = MW + N(W), \text{ where} \tag{3.1}$$

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \nu_0 & 0 & \eta(\alpha_1 + \alpha_0) & 0 \\ 0 & 0 & 0 & 1 \\ \eta(\alpha_1 - \alpha_0) & 0 & \nu_0 & 0 \end{pmatrix}, N(W) = -4(W_1^2 + W_3^2) \begin{pmatrix} 0 \\ W_1 \\ 0 \\ W_3 \end{pmatrix}.$$

The (primary) 1–pulse solution A^0_{pu} is explicitly given by

$$A^0_{\mathrm{pu}}(X) = \sqrt{b_1}\mathrm{sech}(\sqrt{b_2}X)e^{\mathrm{i}\theta}, \quad b_1 = b_2/2,$$
$$b_2 = (\nu_0 + \eta\alpha_1 \sin(2\theta)), \quad \cos(2\theta) = \alpha_0/\alpha_1.$$

For $\sin(2\theta)$>0 and η sufficiently small the pulse is stable. The spectrum of the linearization $L^{\mathrm{GL}}_{\mathrm{pu}}$ of the pfNLSe around A^0_{pu} as obtained in [KS98] is sketched in figure 2. The spectrum is symmetric to the line $\mathrm{Re}\lambda = -\eta\alpha_0$ and the

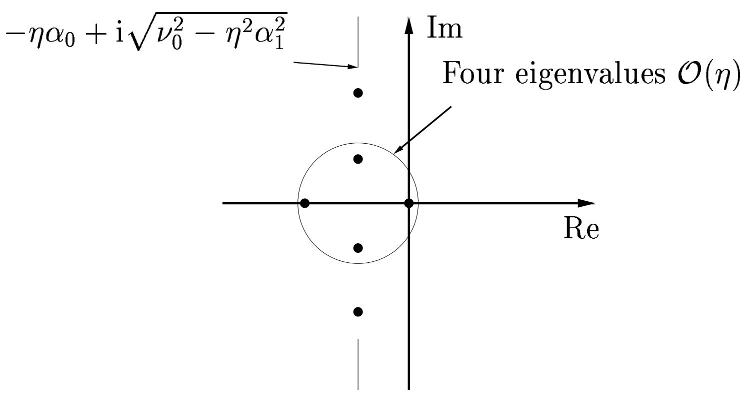

Figure 2: The spectrum of the linearization of the pfNLSe around A^0_{pu}

essential spectrum is given by the lines $-\eta\alpha_0 \pm \mathrm{i}\sqrt{(k^2 + \nu_0)^2 - \eta^2\alpha_1^2}$, cf. (2.6).

There is one simple eigenvalue $\lambda_1 = 0$ with associated eigenfunction $\partial_X A^0_{\mathrm{pu}}$, one eigenvalue $\lambda_2 = -2\eta\alpha_0$ and a pair of complex conjugate eigenvalues $\lambda_{3,4}$

with $\mathrm{Re}\lambda = -\eta\alpha_0$ and imaginary part of order $\mathcal{O}(\eta)$. These four eigenvalues bifurcate out of the fourfold eigenvalue 0 for the linearization around the pulse $A_{\mathrm{pu}}^{\mathrm{NLS}} = \sqrt{\nu_0/2}\,\mathrm{sech}(\sqrt{\nu_0}x)$ in the nonlinear Schrödinger equation ($\eta = 0$).

Moreover, for η sufficiently small, only two additional eigenvalues pop out of the essential spectrum. They are $\mathcal{O}(\eta^2)$ close to the points $-\eta\alpha_0 \pm \mathrm{i}\sqrt{\nu_0^2 - \eta^2\alpha_1^2}$.

Combining this spectral analysis for the primary pulse A_{pu}^0 with additional resolvent estimates one finds that A_{pu}^0 is exponentially orbitally stable.

These results extend to the nsGLe ($c_{1r}, c_{3r} \neq 0$) as follows. The stationary problem (3.1) is reversible under $RW = (W_1, -W_2, W_3, -W_4)$. Since we have a simple eigenvalue 0 it follows that A_{pu}^0 is an elementary homoclinic orbit, see, e.g., [FV92]. Therefore it persists under small reversible perturbations of (3.1). Hence, for $c_{1r}, c_{3r} = o(\eta)$ there exists a pulse A_{pu} which approaches A_{pu}^0 for $c_{1r}, c_{3r} \to 0$. Obviously the pulse A_{pu} is stable for $c_{1r} > 0$.

The stationary nsGLe (and hence (3.1)) is equivariant under the transformation $A \mapsto -A$ ($W \mapsto -W$). The eigenvalues of the matrix M are given by

$$\lambda_{\pm,\pm} = \pm\sqrt{\nu_0 \pm \eta\sqrt{\alpha_1^2 - \alpha_0^2}} = \pm\sqrt{\nu_0 \pm \eta\alpha_1 \sin(2\theta)},$$

where $\cos(2\theta) = \alpha_0/\alpha_1$, i.e., the primary pulse A_{pu}^0 is contained in the strong stable manifold of 0. These are the basic conditions for the occurrence of a so called orbit–flip bifurcation [SJA97] creating n–pulses that resemble n widely–spaced copies of $\pm A_{\mathrm{pu}}^0$. In [KS98] it is shown that for $c_{3r} = 0$ and small $c_{1r} > 0$ these n–pulses $A_{\mathrm{pu},n}$ indeed exist. The n–pulses $A_{\mathrm{pu},n}$ of the form up–down ($+-$) are stable and all other n–pulses are unstable. For stable $A_{\mathrm{pu},n}$ we again have one isolated eigenvalue 0 and now $6n-1$ eigenvalues with negative realparts. Therefore, $A_{\mathrm{pu},n}$ again persists for sufficiently small $c_{3r} \neq 0$.

4 Existence of the modulating multi-pulses

In order to construct modulating multi–pulse–solutions to (1.1) we use a spatial dynamics formulation and center manifold theory for elliptic systems as introduced in [Kir82]. This method has also been used in a variety of similar problems, see, e.g., [EW91, IM91, HCS98, Sch99]. The idea is as follows. Let

$$u(t,x) = v(x, x - ct) = v(p, \xi) \tag{4.1}$$

with $p \in \mathcal{T}_{2\pi}$ and $\xi \in \mathbb{R}$. Here $\mathcal{T}_\alpha$ is the one dimensional torus of length α, and $c = \nu_1 + \varepsilon\tilde{c}_1$ where $\tilde{c}_1$ is a priori unknown. Inserting this into (1.1) we obtain the elliptic problem

$$\begin{aligned}-c\partial_\xi v =& -(1-(\partial_p+\partial_\xi)^2)^2 v - \varepsilon^2\eta\alpha_0 v + \beta_1(\partial_p+\partial_\xi)v + \beta_3(\partial_p+\partial_\xi)^3 v\\ &+ 2\varepsilon^2\eta\alpha_1\cos(2p)v + f(v,(\partial_p+\partial_\xi)v)\end{aligned} \tag{4.2}$$

on the infinite cylinder $(p,\xi)\in\mathcal{T}_{2\pi}\times\mathbb{R}$. Writing (4.2) as first order equation in ξ, the linearization around the trivial solution $v\equiv 0$ has four eigenvalues $\mathcal{O}(\varepsilon)$ close to the origin with the rest of the spectrum bounded away from the imaginary axis. This gives a four dimensional center manifold $\mathcal{M}_c$. The reduced equation on $\mathcal{M}_c$ is given in lowest order by the stationary nsGLe in a frame moving with velocity $\varepsilon\tilde{c}_1$. Thus, for $\varepsilon=0$ we have n–pulses for the reduced equation. We then show that these pulses persist for the higher order perturbations for suitable $\tilde{c}_1=\mathcal{O}(1)$, and this will prove Theorem 1.4.

4.1 The spatial dynamics formulation

Letting $\mathcal{V}=(\mathcal{V}_0,\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3)^T=(v,\partial_\xi v,\partial_\xi^2 v,\partial_\xi^3 v)^T$ we write (4.2) as first order system

$$\partial_\xi\mathcal{V}=\mathcal{M}\mathcal{V}+\eta\alpha_1\varepsilon^2\mathcal{P}\mathcal{V}+\mathcal{N}(\mathcal{V}),\ \text{where} \tag{4.3}$$

$$\mathcal{M}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a&b&c&d\end{pmatrix},\quad \begin{aligned}\mathcal{P}\mathcal{V}&=(0,0,0,(e^{-2\mathrm{i}p}+e^{2\mathrm{i}p})\mathcal{V}_0)^T\\ \mathcal{N}(\mathcal{V})&=(0,0,0,f(\mathcal{V}_0,\partial_p\mathcal{V}_0+\mathcal{V}_1))^T\end{aligned},$$

$$\begin{aligned}a&=-\eta\alpha_0\varepsilon^2-(1+\partial_p^2)^2+\partial_p(\beta_1+\beta_3\partial_p^2),\\ b&=-4\partial_p(1+\partial_p^2)+\beta_1+3\beta_3\partial_p^2+c,\\ c&=-6\partial_p^2-2+3\beta_3\partial_p,\quad d=-4\partial_p+\beta_3.\end{aligned}$$

Expanding $\mathcal{V}(p,\xi)=\sum_{m\in\mathbb{Z}}V_m(\xi)e^{imp}$ with $V_m(\xi)\in\mathbb{C}^4$, $V_{-m}(\xi)=\overline{V_m}(\xi)$ we obtain

$$\partial_\xi V_m=M_mV_m+\eta\alpha_1\varepsilon^2P_m(V)+N_m(V), \tag{4.4}$$

where $V=(V_m)_{m\in\mathbb{Z}}$,

$$M_m=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\a_m&b_m&c_m&d_m\end{pmatrix},\quad P_m(V)=\begin{pmatrix}0\\0\\0\\V_{m-2,0}+V_{m+2,0}\end{pmatrix},$$

$N_m(V) = \big(0,0,0,\sum_{p+q=m} V_{p,0}(f_1 V_{q,0} + f_2(V_{q,1} + \mathrm{i}qV_{q,0}))\big)^T$, and

$$\begin{aligned} a_m &= -\eta\alpha_0\varepsilon^2 - (1-m^2)^2 + \beta_1 im - \mathrm{i}\beta_3 m^3, \\ b_m &= -4\mathrm{i}m(1-m^2) + \beta_1 - 3\beta_3 m^2 + c, \\ c_m &= 6m^2 - 2 + 3\mathrm{i}\beta_3 m, \quad d_m = -4\mathrm{i}m + \beta_3. \end{aligned}$$

The main difference to previous work is the linear coupling $P_m(V)$ coming from the spatial periodic forcing. We finally write (4.4) as

$$\partial_\xi V = (M + \eta\alpha_1\varepsilon^2 D)V + N(V) \tag{4.5}$$

where $M = \mathrm{diag}(\dots, M_{-1}, M_0, M_1, \dots)$, $N(V) = (\dots, N_{-1}, N_0, N_1, \dots)^T$,

$$D = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & \tilde{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{D} & & & \\ & & \tilde{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{D} & & \\ & & & \tilde{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{D} & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tilde{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and in D empty places and $\mathbf{0}$ both denote the 4×4 zero matrix.

4.2 The construction of the center manifold

In order to construct a center manifold $\mathcal{M}_c$ for (4.5) we introduce the Hilbert spaces

$$\begin{aligned} \mathcal{E}^\alpha = \{V = (V_m)_{m\in\mathbb{Z}} : V_0 \in \mathbb{R}^4, V_m \in \mathbb{C}^4 \text{ for } m \in \mathbb{Z}\setminus\{0\}, V_{-m} = \overline{V_m}, \\ \|V\|_\alpha = \sum_{m\in\mathbb{Z}} \sum_{j=0,\dots,3} |V_{m,j}|^2 (1+m^2)^{(4-j)\alpha} < \infty\}. \end{aligned}$$

Then $V(\xi) \in \mathcal{E}^1$ means $\mathcal{V}(\xi) \in H^4(\mathcal{T}_{2\pi}) \times H^3(\mathcal{T}_{2\pi}) \times H^2(\mathcal{T}_{2\pi}) \times H^1(\mathcal{T}_{2\pi})$ and $V(\xi) \in \mathcal{E}^0$ corresponds to $\mathcal{V} \in [L^2(\mathcal{T}_{2\pi})]^4$. Starting with the linear problem we show that for small ε and $c = \nu_1 + \mathcal{O}(\varepsilon)$ the operator $M + \eta\alpha_1\varepsilon^2 D : \mathcal{E}^1 \to \mathcal{E}^0$ has four eigenvalues $\mathcal{O}(\varepsilon)$ close to zero while the rest of the spectrum is bounded away from the imaginary axis. Here we basically recall the analysis from [Sch99] for M, since obviously D is a compact perturbation of M with

$$\|D\|_{\mathcal{E}^1 \to \mathcal{E}^0} \leq 2. \tag{4.6}$$

The operator M decouples into the 4×4 blocks M_m. For all $m \in \mathbb{Z}$ the four eigenvalues $\lambda = \lambda_{m,j}$, $j = 1, \dots, 4$, of M_m fulfill

$$-c\lambda = -(1 + (\mathrm{i}m + \lambda)^2)^2 - \varepsilon^2\eta\alpha_0 + \beta_1(\mathrm{i}m + \lambda) + \beta_3(\mathrm{i}m + \lambda)^3. \tag{4.7}$$

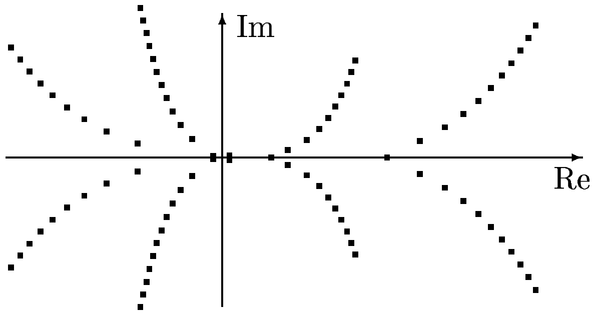

Figure 3: The spectrum of the operator M

Figure 3 shows a plot of $\lambda_{m,j}$ for $m = -10, \dots, 10$, $j = 1, \dots, 4$. To calculate the central part of the spectrum of M_m we substitute $\lambda = \mathrm{i}k$ and l=m+k in (4.7) to obtain

$$-\mathrm{i}ck = -(1-l^2)^2 - \varepsilon^2\alpha_0 + i\beta_1 l - i\beta_3 l^3.$$

For $\varepsilon = 0$ the real part gives $l = \pm 1$. From the imaginary part we obtain $c(m-l) = 0$ and hence the conditions $l = m = 1$ or $l = m = -1$. Here we used the assumption that $\beta_1 - \beta_3 = \nu_0\varepsilon^2$ while $c = \nu_1 + \mathcal{O}(\varepsilon)$ with $\nu_1 = 3\beta_3 - \beta_1 \neq 0$. In summary, it is easy to see that for $\varepsilon = 0$ and $m = \pm 1$ we have an algebraically double eigenvalues 0 for M_m corresponding to a Jordan block of size 2. The associated eigenvector is $\phi_{m,1} = (1,0,0,0)^T$ and the generalized eigenvector is $\phi_{m,2} = (0,1,0,0)^T$, such that $M_m\phi_{m,1} = 0$ and $M_m\phi_{m,2} = \phi_{m,1}$. Using perturbation analysis as in [EW91] we obtain that for $\varepsilon \neq 0$ we have four distinct eigenvalues $\lambda_{\pm 1,1}, \lambda_{\pm 1,2}$ of order $\mathcal{O}(\varepsilon)$.

The hyperbolic part of the spectrum of the matrices M_m can be estimated using the scaling $\lambda = \mu|m|^{1/4} - im$. This gives $\mu^4 = \mathrm{i}c + \mathcal{O}(|m|^{-1/4})$ as $m \to \infty$, and thus $\lambda = m^{1/4}(\mathrm{i}c)^{1/4} - \mathrm{i}m + \mathcal{O}(1)$. Combining these results with (4.6) we obtain the following lemma.

Lemma 4.1 *There exist $C_1, C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the operator $M + \eta\alpha_1\varepsilon^2 D$ has four central eigenvalues $\lambda_{\pm 1,1}, \lambda_{\pm 1,2}$ of size $\mathcal{O}(\varepsilon)$. All other eigenvalues $\lambda_{m,j}$, $m \in \mathbb{Z}$, $j = 1, \dots, 4$ satisfy $|\mathrm{Re}\lambda_{m,j}| \geq C_1 + C_2|m|^{1/4}$.*

Now the center manifold can be contructed in a standard way, see, e.g., [IV92]. By $\mathcal{E}_c = \mathcal{E}_c(\varepsilon) = \mathrm{span}\{\tilde{\phi}_{1,1}(\varepsilon), \tilde{\phi}_{1,2}(\varepsilon), \tilde{\phi}_{-1,1}(\varepsilon), \tilde{\phi}_{-1,2}(\varepsilon)\} \subset \mathcal{E}^0$ we denote the center subspace associated to the eigenvalues with realpart of order $\mathcal{O}(\varepsilon)$. We may identify $\mathcal{E}_c$ with $\mathbb{C}^2$. The associated projection on $\mathcal{E}_c$ is denoted by $P_c(\varepsilon) : \mathcal{E}^0 \to \mathcal{E}_c$, and we write $\mathcal{E}_h$ for the hyperbolic subspace $(\mathrm{Id} - P_c(\varepsilon))\mathcal{E}^0$. We choose a smooth cut–off function $\chi \in C^\infty(\mathbb{R}, [0,1])$ with $\chi(x) = 1$ for $|x| \leq 1$

and $\chi(x)=0$ for $|x|\geq 2$. Then by a version of Sobolev's imbedding theorem the modified nonlinearity $N_r:\mathcal{E}^{1/4}\to\mathcal{E}^0$, $N_r(V)=N(u\,\chi(\|u\|_{\mathcal{E}}/r))$ is smooth and globally Lipschitz. From this we obtain the following theorem.

Theorem 4.2 *For each $k\in\mathbb{N}$ there exist $r,\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ there exists a unique global center manifold $\mathcal{M}_c$ for the modified equation*

$$\partial_\xi V=(m+\varepsilon^2\eta\alpha_1 D)V+N_r(V),$$

which is tangential to $\mathcal{E}_c$. This means that there exist a C^k–function $h_k:\mathcal{E}_c\to\mathcal{E}_h\cap\mathcal{E}^1$, $\|h_k(V_c)\|_{\mathcal{E}^1}=\mathcal{O}((\|V_c\|+\varepsilon^2)^2)$, such that the manifold

$$\mathcal{M}_c=\{V=V_c+h_k(V_c):V_c\in\mathcal{E}_c\}$$

contains all small bounded solutions of (4.5). Moreover, every solution of the reduced equation

$$\partial_\xi V_c=(M+\varepsilon^2\eta\alpha_1 D)V_c+P_cN_r(V_c+h_k(V_c)) \tag{4.8}$$

gives a solution of (4.5) via

$$V(\xi)=V_c(\xi)+h_k(V_c(\xi)). \tag{4.9}$$

4.3 The reduced equation

The reduced equation (4.8) is related to the stationary nsGLe in a frame moving with velocity $\varepsilon\tilde{c}_1$ as follows. For $\varepsilon=0$ the operator $M+\varepsilon^2\eta\alpha_1 D$ has the four central eigenvectors

$$\tilde{\phi}_{-1,j}(0)=(\ldots|\mathbf{0}|\phi_{-1,j}|\underset{\substack{\uparrow\\ m=0}}{\mathbf{0}}|\mathbf{0}|\ldots)^T,\quad \tilde{\phi}_{1,j}=(\ldots|\underset{\substack{\uparrow\\ m=0}}{\mathbf{0}}|\phi_{1,j}|\mathbf{0}|\mathbf{0}|\ldots)^T,$$

$j=1,2$, where $\mathbf{0}=0\in\mathbb{C}^4$, and $\phi_{\pm1,j}$ is the eigenvector respectively generalized eigenvector associated to the Jordan block of $M_{\pm1}$. For $\varepsilon>0$ we choose $\psi_{\pm1,j}(\varepsilon)$ as basis for $\mathcal{E}_c$ where $\psi_{\pm1,j}(\varepsilon)=P_c(\varepsilon)\tilde{\phi}_{\pm1,j}(0)$. Thus we write

$$V_c(\xi)=\tilde{A}(\xi)\psi_{1,1}+\tilde{B}(\xi)\psi_{1,2}+\overline{\tilde{A}}(\xi)\psi_{-1,1}+\overline{\tilde{B}}(\xi)\psi_{-1,2}.$$

The vectors $\psi_{\pm1,1}$ are of the form

$$\begin{aligned}\psi_{-1,1}&=(\ldots|\boldsymbol{\mathcal{O}}(\varepsilon^2)|\underbrace{(\mathcal{O}(1),\mathcal{O}(\varepsilon),\mathcal{O}(\varepsilon^2),\mathcal{O}(\varepsilon^3))}_{m=-1}|\underset{\substack{\uparrow\\ m=0}}{\boldsymbol{\mathcal{O}}(\varepsilon^2)}|\boldsymbol{\mathcal{O}}(\varepsilon^2)|\ldots)^T,\\ \psi_{1,1}&=(\ldots|\mathcal{O}(\varepsilon^2)|\underset{\substack{\uparrow\\ m=0}}{\boldsymbol{\mathcal{O}}(\varepsilon^2)}|\underbrace{(\mathcal{O}(1),\mathcal{O}(\varepsilon),\mathcal{O}(\varepsilon^2),\mathcal{O}(\varepsilon^3))}_{m=1}|\boldsymbol{\mathcal{O}}(\varepsilon^2)|\ldots)^T,\end{aligned} \tag{4.10}$$

and similarly for $\psi_{\pm 1,2}$. Scaling $\tilde{A}(\xi) = \varepsilon A(X)$, $\tilde{B}(\xi) = \varepsilon^2 B(X)$ where $X = \varepsilon\xi$ we obtain

$$\begin{aligned} \partial_X A &= B + \mathcal{O}(\varepsilon), \\ c_1 \partial_X B &= \varepsilon \tilde{c}_1 B + (\eta\alpha_0 + \mathrm{i}\nu_0)A - \eta\alpha_1 \overline{A} - c_3|A|^2 A + \mathcal{O}(\varepsilon), \end{aligned} \tag{4.11}$$

where c_1, c_3 are the Ginzburg–Landau coefficients obtained in section 2. This can be seen from inserting the infinite Ginzburg–Landau ansatz

$$v(p,\xi) = \sum_m V_{m,0}(\xi) e^{imp} = \sum_m \varepsilon^{\alpha_m} A_m(\varepsilon\xi) e^{imp}$$

with $\alpha_m = 1 + |1 - |m||$ into (4.2). Equating coefficients in front of $\varepsilon^j e^{imp}$ as usual we see that A_1 has to satisfy the stationary nsGLe with a drift term $\varepsilon \tilde{c}_1 \partial_X A_1$ and additional terms of formal higher order $\mathcal{O}(\varepsilon)$. These terms are of order $\mathcal{O}(\varepsilon)$ since $V_{m,j}(\xi) = \varepsilon^{\alpha_m + j} \partial_X A_m(X)$ and since $V = (V_m)_m$ lies on the center manifold. Due to the form of the eigenvectors $\psi_{\pm 1,j}$ given in (4.10), the coordinates A, B have to satisfy the reduced equation (4.11).

4.4 Existence of n–pulses for the reduced equation

For notational simplicity we take $n = 1$. For ε=0 and $(c_{1r}, c_{1i}, c_{3r}, c_{3i}, \alpha_0, \alpha_1) \in \mathcal{P}$ the reduced equation (4.11) has exponentially stable stationary pulse solutions $A_{\mathrm{pu}}(\cdot - X_0)$, $X_0 \in \mathbb{R}$. It remains to show that these pulses persist for $\varepsilon > 0$ and suitable $\tilde{c}_1 = \mathcal{O}(1)$. This will be done via a Liapunov–Schmidt type argument.

We apply the implicit function theorem to the first equation in (4.11) to obtain $B = \partial_x A + \mathcal{O}(\varepsilon)$. Inserting this into the second equation in (4.11) we obtain

$$L^{\mathrm{GL}} A - \varepsilon \tilde{c}_1 \partial_X A + N_1(A) + \tilde{L}_1(\varepsilon) A + \tilde{N}_1(\varepsilon, A) = 0, \tag{4.12}$$

where

$$L^{\mathrm{GL}} A = c_1 \partial_X^2 A - (\eta\alpha_0 + \mathrm{i}\nu_0) A + \eta\alpha_1 \overline{A}, \quad N_1(A) = c_3 |A|^2 A,$$

and where $\tilde{L}_1(\varepsilon)$ and $\tilde{N}_1(\varepsilon, \cdot)$ fulfill

$$\|\tilde{L}_1(\varepsilon)\|_{H^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(\varepsilon), \quad \|\tilde{N}_1(\varepsilon, A)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon \|A\|^2_{H^1(\mathbb{R})}).$$

These estimates are the essential consequence of the spatial dynamics ansatz and the center manifold reduction we use here. Letting $A = A_{\mathrm{pu}} + U$ we obtain

$$[L^{\mathrm{GL}}_{\mathrm{pu}} - \varepsilon \tilde{c}_1 \partial_X + \tilde{L}_2(\varepsilon)] U + N_2(U) + \tilde{N}_2(\varepsilon, U) + r(\varepsilon) - \varepsilon \tilde{c}_1 \partial_X A_{\mathrm{pu}} = 0, \tag{4.13}$$

where $L_{\rm pu}^{\rm GL}$ is the linearization around $A_{\rm pu}$ in the unperturbed nsGLe and

$$\begin{aligned} N_2(U) &= N_1(A_{\rm pu}+U) - N_1(A_{\rm pu}) - N_1'(A_{\rm pu})U, \\ \tilde{L}_2(\varepsilon)U &= \tilde{L}_1(\varepsilon)U + \tilde{N}_1'(\varepsilon, A_{\rm pu})U, \\ r(\varepsilon) &= \tilde{L}_1(\varepsilon)A_{\rm pu} + \tilde{N}_1(\varepsilon, A_{\rm pu}), \\ \tilde{N}_2(\varepsilon, U) &= \tilde{N}_1(\varepsilon, A_{\rm pu}+U) - \tilde{N}_1(\varepsilon, A_{\rm pu}) - \tilde{N}_1'(\varepsilon, A_{\rm pu})U. \end{aligned} \tag{4.14}$$

From section 3 we know that $L_{\rm pu}^{\rm GL}$ has one simple eigenvalue 0 with associated eigenfunction $A_{\rm pu}' = \partial_X A_{\rm pu}$, while the rest of the spectrum of $L_{\rm pu}^{\rm GL}$ has $\mathcal{O}(\eta)$ distance to 0. Moreover, $L_{\rm pu}^{\rm GL} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is Fredholm with index 0 since it is a compact perturbation of $L^{\rm GL}$. Let P be the orthogonal projection from $L^2(\mathbb{R})$ onto the critical eigenspace span$\{A_{\rm pu}'\}$ of $L_{\rm pu}^{\rm GL}$. Then (4.13) is equivalent to

$$P\big[(L_{\rm pu}^{\rm GL} - \varepsilon\tilde{c}_1\partial_X + \tilde{L}_2(\varepsilon))U + N_2(U) + N_2(\varepsilon, U) + r(\varepsilon) - \varepsilon\tilde{c}_1 A_{\rm pu}'\big] = 0, \tag{4.15}$$

$$(\mathrm{Id}-P)\big[(L_{\rm pu}^{\rm GL} - \varepsilon\tilde{c}_1\partial_X + \tilde{L}_2(\varepsilon))U + N_2(U) + N_2(\varepsilon, U) + r(\varepsilon)\big] = 0, \tag{4.16}$$

where we already used that $(\mathrm{Id}-P)A_{\rm pu}' = 0$. For ε suffiently small and $\tilde{c}_1 = \mathcal{O}(1)$ the operator

$$(\mathrm{Id}-P)(L_{\rm pu}^{\rm GL} - \varepsilon\tilde{c}_1\partial_X + \tilde{L}_2(\varepsilon)) : (\mathrm{Id}-P)H^2(\mathbb{R}) \to (\mathrm{Id}-P)L^2(\mathbb{R})$$

is invertible. Therefore (4.16) may be solved for $U = U(\varepsilon, \tilde{c}_1) \in (\mathrm{Id}-P)H^2(\mathbb{R})$ due to the implicit function theorem, with $\|U\|_{H^2} = \mathcal{O}(\varepsilon)$. Inserting this into the scalar equation (4.15) we obtain

$$P[r(\varepsilon) - \varepsilon\tilde{c}_1 A_{\rm pu}'] = -P[(-\varepsilon\tilde{c}_1\partial_X + \tilde{L}_2(\varepsilon))U + N_2(U) + N_2(\varepsilon, U)], \tag{4.17}$$

where we used $PL_{\rm pu}^{\rm GL}U = 0$. This equation has a unique solution $\tilde{c}_1 = \mathcal{O}(1)$ due to the fact that $PA_{\rm pu}' = A_{\rm pu}'$ and that the left hand side is $\mathcal{O}(\varepsilon)$ and moreover linear in $\varepsilon\tilde{c}_1$, while the right hand side is of higher order in ε and $\varepsilon\tilde{c}$.

Hence we obtain a family of pulse like solutions

$$A_{{\rm pu},\varepsilon}(\cdot - X_0) = A_{\rm pu}(\cdot - X_0) + U(\varepsilon, \tilde{c}_1) = A_{\rm pu}(\cdot - X_0) + \mathcal{O}(\varepsilon)$$

of (4.12). This gives the desired family of modulating (n–)pulse–solutions

$$\begin{aligned} u_{\rm pu}(x, x-ct-x_0) =& v(x, x-ct-x_0) \\ =& \varepsilon A_{{\rm pu},\varepsilon}(\varepsilon(x-ct-x_0))e^{{\rm i}x} + \text{c.c.} \\ &+ [h_k(\varepsilon A_{{\rm pu},\varepsilon}, \varepsilon^2 A_{{\rm pu},\varepsilon}', \varepsilon\overline{A}_{{\rm pu},\varepsilon}, \varepsilon^2\overline{A}_{{\rm pu},\varepsilon}')]|_{m=0,j=0}, \\ =& \varepsilon A_{\rm pu}(\varepsilon(x-ct-x_0))e^{{\rm i}x} + \text{c.c.} + \mathcal{O}(\varepsilon^2) \end{aligned}$$

to (1.1) via (4.9) and (4.1). Thus the proof of Theorem 1.4 is complete. ◻

5 Stability of the modulating n–pulses

For notational simplicity we continue to take $n = 1$. The solutions u_{pu} of (1.1) are time periodic in a frame moving with velocity c. To prove their stability we proceed as in [Sch99]. Inserting $v(t, y) = u(t, y + ct)$ into (1.1) we obtain

$$v_t(t, y) = Lv(t, y) + c\partial_y v(t, y) + 2\varepsilon^2\eta\alpha_1 \cos(y + ct)v(t, y) + N(v)(t, y), \quad (5.1)$$

where $N(v)(t, y) = f(v(t, y), \partial_y v(t, y))$. We let $v(t, y) = v_{\mathrm{pu}}(t, y)+w(t, y)$, with

$$v_{\mathrm{pu}}(t, y) = u_{\mathrm{pu}}(y + ct, y) = \varepsilon A_{\mathrm{pu}}(\varepsilon y)e^{\mathrm{i}(y+ct)} + \text{c.c.} + \mathcal{O}(\varepsilon^2).$$

Then the perturbation w satisfies

$$\begin{aligned} \partial_t w &= L_{\mathrm{pu}} w + N_1(w), \quad \text{where} \\ L_{\mathrm{pu}} w &= (L + \varepsilon^2\eta\alpha_1 \cos(2y + ct) + c\partial_y + DN(v_{\mathrm{pu}}))w, \\ N_1(w) &= N(v_{\mathrm{pu}} + w) - N(v_{\mathrm{pu}}) - DN(v_{\mathrm{pu}})w, \end{aligned} \quad (5.2)$$

and $DN(v_{\mathrm{pu}})w = 2f_1 v_{\mathrm{pu}} w + f_2(v_{\mathrm{pu}}\partial_y w + (\partial_y v_{\mathrm{pu}})w)$ for our choice of f. Since L_{pu} and N_1 are $2\pi/c$ periodic in t we use Floquet theory to show the stability of $w = 0$ in (5.2) and hence the stability of v_{pu}. We define the linear flow $\psi_{t,s} : H^1_{\mathrm{lu}}(\mathbb{R}) \to H^1_{\mathrm{lu}}(\mathbb{R})$ by the solution $w(t) = \psi_{t,s} w_0$ of the linear problem

$$w_t = L_{\mathrm{pu}} w, \quad w|_{t=s} = w_0. \quad (5.3)$$

The stability of w will follow from the following lemma concerning the eigenvalues of the associated Floquet (time $2\pi/c$) operator.

Lemma 5.1 *The operator $\Lambda = \psi_{2\pi/c,0}$ has one simple eigenvalue $\mu_1 = 1$. There exist $\varepsilon_0, b > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the rest of the spectrum of Λ is contained in $\{z \in \mathbb{C} : |z| < e^{-b\varepsilon^2}\}$.*

Proof. The function $w^*(t, y) = \partial_t v_{\mathrm{pu}}(t, y) - c\partial_y v_{\mathrm{pu}}$ solves $w_t = L_{\mathrm{pu}} w$ which gives the Floquet multiplier $\mu_1 = 1$.

The continuous Floquet spectrum $\sigma_{\mathrm{c}}(\Lambda)$ is determined by the continuous spectrum of $\mathcal{L}^* = L+2\varepsilon^2\eta\alpha_1 \cos(2(y+ct))+c\partial_y$ since the compact perturbation $DN(v_{\mathrm{pu}})$ does not change the essential spectrum. Since $c\partial_y$ does not contribute to the real part, by (A.2) we obtain

$$\sigma_{\mathrm{c}}(\Lambda) \subset \{z \in \mathbb{C} : |z| \leq e^{-\varepsilon^2\eta\alpha_0 2\pi/c+\mathcal{O}(\varepsilon^4)}\} \subset \{z \in \mathbb{C} : |z| \leq e^{-\varepsilon^2\eta\alpha_0\pi/c}\}$$

for ε sufficiently small.

To control the rest of the Floquet spectrum we use the Ginzburg–Landau formalism and the exponential stability of $A_{\rm pu}$. As a consequence of Theorem B.1 we find that every eigenfunction of Λ can be written as

$$w(y) = A_1(\varepsilon y)e^{\mathrm{i}y} + \text{c.c.} + \varepsilon R(y) \tag{5.4}$$

with $\|A_1\|_{H^1_{\rm lu}}, \|R\|_{H^1_{\rm lu}} \leq 1$. This is the so called attractivity of the set of modulated patterns, here for the linear equation (5.3). Moreover, the approximation property Theorem 2.6 holds in an analogous way for (5.3), since this is basically a linear result, cf. the proof of Theorem 2.6. This means that the dynamics of the eigenfunctions w can be approximated by the solutions of the linearization $A_T = L^{\rm GL}_{\rm pu} A$ of the nsGLe around $A_{\rm pu}$. In detail, let $A \in C([0, T_1], H^1_{\rm lu})$, $T_1 = \mathcal{O}(1)$, be the solution to $A_T = L^{\rm GL}_{\rm pu} A$ with $A|_{T=0} = A_1$ from (5.4). Then there exists $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{0\leq t\leq T_1/\varepsilon^2} \|w(t,\cdot) - \tilde{\psi}_A(t,\cdot)\|_{H^1_{\rm lu}} \leq C\varepsilon, \quad \tilde{\psi}_A(t,y) = A(\varepsilon^2 t, \varepsilon y)e^{\mathrm{i}(y+ct)} + \text{c.c.}.$$

Now let $\mu \neq 1$ be in $\mathrm{spec}_{\rm d}(\Lambda)$ and let $t_0 = 2\pi/c$ and $T_1 = mT_0 = m\varepsilon^2 t_0 = \mathcal{O}(1)$ for some $m \in \mathbb{N}$ with $m = \mathcal{O}(1/\varepsilon^2)$. Then

$$\begin{aligned} 0 = \|w(mt_0) - \mu^m w_0\|_{H^1_{\rm lu}} &= \|\psi_A(T_1) - \mu^m \psi_A(0)\|_{H^1_{\rm lu}} + \mathcal{O}(\varepsilon) \\ &= (e^{\lambda T_1} - \mu^m)\|A_1\|_{H^1_{\rm lu}} + \mathcal{O}(\varepsilon) \end{aligned}$$

where $\lambda \in \mathrm{spec}_{\rm d}(L^{\rm GL}_{\rm pu})$. Since $\mathrm{Re}(\lambda) < -b_0$ except for the one simple eigenvalue $\lambda = 0$, we find $|\mu|^m \leq e^{-b_0 T_1} + C\varepsilon \leq e^{-\frac{b_0}{2}T_1}$ for ε sufficiently small. Thus defining $b = \min\{\eta\alpha_0\pi/c, b_0 t_0/2\}$ the proof of Lemma 5.1 is complete. □

Proof of Theorem 1.5. We define the nonlinear flow $\phi_{t,s} : H^1_{\rm lu}(\mathbb{R}) \to H^1_{\rm lu}(\mathbb{R})$ by the solution $w(t) = \phi_{t,s} w_0$ of (5.2) with $w|_{t=s} = w_0$. The long time dynamics of (5.2) for small $w|_{t=0}$ can be described by iteration of the nonlinear Floquet operator $\Gamma = \phi_{2\pi/c,0} : \mathcal{U} \to H^1_{\rm lu}(\mathbb{R})$, where $\mathcal{U}$ is a neighborhood of 0 in $H^1_{\rm lu}(\mathbb{R})$. The mapping Γ exists for $\mathcal{U}$ sufficiently small since L is sectorial and hence (5.2) defines a local semigroup in $H^1_{\rm lu}(\mathbb{R})$. Thus we consider the discrete dynamical system

$$w^{(n+1)}(y) = \Gamma w^{(n)}(y). \tag{5.5}$$

The linearization of Γ is given by $w^{(n+1)} = \Lambda w^{(n)}$. From Lemma 5.1 it follows that there exists a one–dimensional center manifold $\mathcal{M}_c$ of size $\mathcal{O}(\varepsilon)$ for (5.5). This manifold is tangential to $\mathrm{span}\{w^*\}$ and contains all small solutions of (5.5). Hence it coincides with the family

$$\{u_{\rm pu}(\cdot, \cdot - x_0) - u_{\rm pu}(\cdot,\cdot) : x_0 \in \mathbb{R}\}$$

of fixed points of Γ and the flow on $\mathcal{M}_c$ is trivial. Now assume that $\|w^{(0)}(\cdot) - u_{\rm pu}(\cdot, \cdot - x_0)\|_{H^1_{\rm lu}} \leq C_1\varepsilon$ for some $C_1 > 0$ sufficiently small. Then for $t = m2\pi/c$ and some $x_1 \in \mathbb{R}$ we obtain

$$\|w^{(m+1)}(\cdot) - (u_{\rm pu}(\cdot + ct, \cdot - x_1) - u_{\rm pu}(\cdot + ct, \cdot))\|_{H^1_{\rm lu}} \leq C_2\varepsilon e^{-b\varepsilon^2 t}$$

for $m \to \infty$. Hence $\|v(t, \cdot + ct) - u_{\rm pu}(\cdot + ct, \cdot - x_1)\|_{H^1_{\rm lu}} \leq C_2\varepsilon e^{-b\varepsilon^2 t}$, which gives

$$\|u(t, \cdot) - u_{\rm pu}(\cdot, \cdot - ct - x_1)\|_{H^1_{\rm lu}} \leq C_2\varepsilon e^{-b\varepsilon^2 t}. \tag{5.6}$$

Since (5.2) defines a local semigroup in $H^1_{\rm lu}(\mathbb{R})$ we obtain (5.6) for all $t \in [m2\pi/c, (m+1)2\pi/c)$ and all $m \in \mathbb{N}$ and the proof of Theorem 1.5 is complete. □

6 A few numerical results

In order to calculate modulating multi–pulse solutions of (1.1) numerically we proceed similar to their analytic construction. First we integrate numerically (1.6) to obtain stationary n–pulse solutions of the nsGLe. Using (1.5) these are taken as initial conditions for the pSHe, which we integrate in the moving frame $y = x + ct$, see (5.1). Here we first choose $c = \nu_1 = 3\beta_3 - \beta_1$, thus ignoring the $\mathcal{O}(\varepsilon)$ correction of c. This leads to modulating multi–pulse with a small drift. Choosing different ε we calculate the correction coefficient $\tilde{c}_1$.

6.1 Results for the nsGLe

For the numerical integration of the nsGLe we use finite differences in space, projection boundary conditions [LPSS99], and the implicit NAG–routine d02ebf for time integration. The strategy to obtain n–pulse solutions is as follows. We fix $(\nu_0, \eta, \alpha_0, \alpha_1, c_{1i}, c_{3i})$ which we choose as $(1, 1, 0.7, 0.8, 9, 4)$. As system–size we take $X \in (-L, L)$ with $L = 50$ for 1–pulse solutions and $L = 75$ for 2–pulse solutions. These values have been chosen after a few test–runs with different values of L with no essential changes of the shapes of the pulses obtained. We start with $c_{1r} = c_{3r} = 0$ by placing n–copies of the analytical solution

$$\begin{aligned} &A_{\rm pu,1}(c_{1r} = 0, c_{3r} = 0)(X) = \sqrt{b_1}\,{\rm sech}(\sqrt{b_2}X)e^{i\theta}, \\ &\cos(2\theta) = \alpha_0/\alpha_1, \quad b_2 = (\nu_0 + \eta\alpha_1\sin(2\theta))/c_{1i}, \quad b_1 = 2c_{1i}b_2/c_{3i}. \end{aligned} \tag{6.1}$$

at $X_1, \ldots, X_n$. We then successively increase (c_{1r}, c_{3r}) and integrate the nsGLe until a stationary n–pulse is reached.

Figure 4 shows the result for the 1–pulse obtained for $(c_{1r}, c_{3r}) = (4, 0.5)$. This solution will be used as the first initial condition for the pSHe.

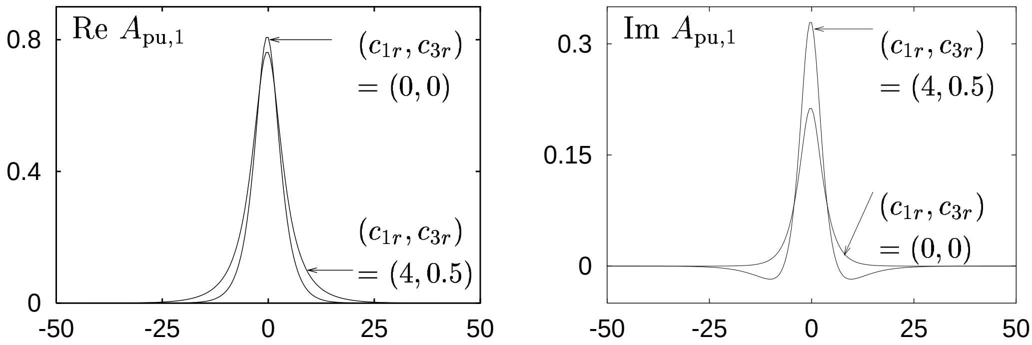

Figure 4: The 1–pulses $A_{\text{pu},1}(\cdot)$ for $(c_{1r}, c_{3r}) = (0, 0)$ and $(c_{1r}, c_{3r}) = (4, 0.5)$.

Next we consider 2–pulses for the nsGLe. Here we let $L = 75$ and place as initial condition one pulse at $X_1 = -7.5$ and one pulse at $X_2 = +7.5$. A priori these values are arbitrary. As expected (cf. [SJA97] for similar results), in case up–up the solution converges to the 1–pulse centered at $X_0 = 0$. In case down–up the real part of the solution at times $T = 0, 500, \dots, 6000$ is presented in figure 5. Here c_{1r}, c_{3r} were slowly increased to $(c_{1r}, c_{3r}) = (4, 0.5)$ at $T = 100$ and then kept fixed. Initially the two humps are too close to each other. They separate at speed $\mathcal{O}(e^{-\gamma T})$ and get stuck at $X_{1,2} \approx \pm 22.5$. Similarly, starting with a down–up–pulse with too widely spaced humps, the two humps approach each other and in the limit we obtain the same final position as in figure 5.

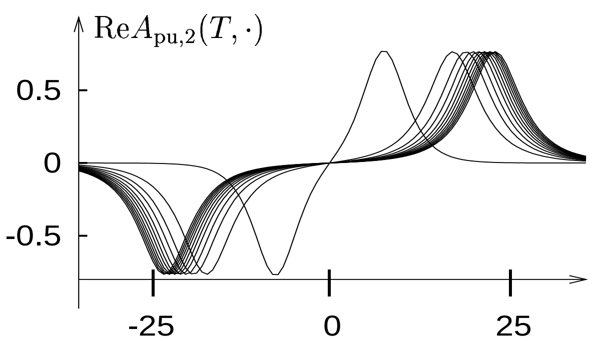

Figure 5: Separation of the humps when starting with a too narrow 2–pulse: $(c_{1r}, c_{3r}) = (0.5, 4)$, $T = 0, 500, \dots, 6000$, $X \in (-75, 75)$ all in all.

6.2 Results for the pSHe

We integrate the pSHe (5.1) using a semi–implicit Fourier–Galerkin pseudo-spectral code [CHQZ88] with $2000/\varepsilon$ Fourier modes and periodic boundary conditions. First we take $u_{\text{pu},1}(0,y) = \varepsilon A_{\text{pu},1}(c_{1r} = 4, c_{3r} = 0.5)(\varepsilon y)e^{\text{i}y} + \text{c.c.}$ as initial condition, $y \in (-50/\varepsilon, 50/\varepsilon)$ and let $c = 3\beta_3 - \beta_1 = \nu_1$ and

$$(\eta, \alpha_0, \alpha_1, f_1, f_2, \beta_3, \beta_1) = (1, 0.7, 0.8, 0.4, 4, 3, \beta_3 - \varepsilon^2). \tag{6.2}$$

These values correspond to

$$(c_{1r}, c_{1i}, c_{3r}, c_{3i}, \eta, \alpha_0, \alpha_1, \nu_0) = (4, 9, 0.504889, 4.00356, 1, 0.7, 0.8, 1) \tag{6.3}$$

in the nsGLe. Therefore $u_{\text{pu},1}(0)$ is an $\mathcal{O}(\varepsilon^2)$ approximation of a modulating 1–pulse in the pSHe. Figure 6 shows the $\mathcal{O}(1/\varepsilon^2)$ time scale evolution for $\varepsilon = 0.2$. For $t > 0$ only the envelope $\text{Env}(u_{\text{pu},1}(t))$ is plotted which is obtained from all local maxima of $u_{\text{pu},1}(t)$. Due to the missing $\mathcal{O}(\varepsilon)$ correction of c we obtain a drift and an asymmetric shape of the modulating pulse. Defining $y_1(t) = \max_y \text{Env}(u_{\text{pu},1}(t,y))$ we obtain, e.g., $y_1(500) \approx -235$, from which we may refine c to the value $c = \nu_1 + y_1(500)/500 \approx 5.57$. Restarting the integration with this c we obtain an envelope which is almost stationary (thus giving a time periodic modulating 1–pulse) and symmetric.

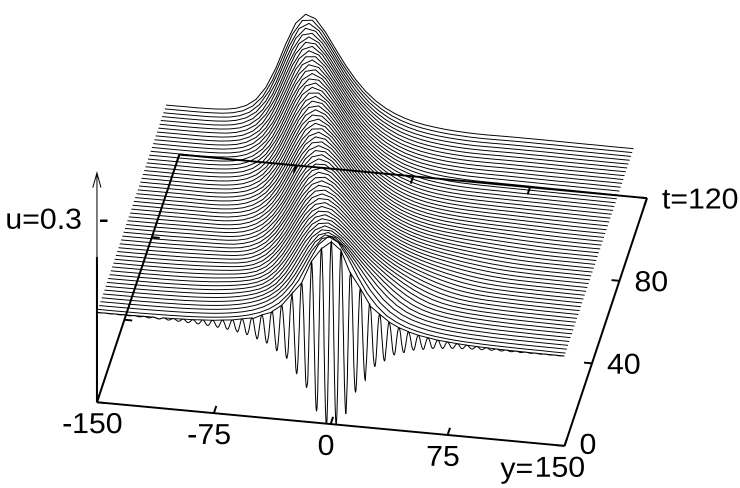

Figure 6: Numerical integration of the pSHe with $\varepsilon = 0.2$, $c = \nu_1$, parameters from (6.2), and initial condition $u_{\text{pu},1}(0,y) = \varepsilon A_{\text{pu},1}(4, 0.5)(\varepsilon y)e^{\text{i}y} + \text{c.c.}$: drift and asymmetric shape of the envelope due to missing $\mathcal{O}(\varepsilon)$–correction of c.

In order to calculate the coefficient $\tilde{c}_1$ we keep the initial condition $u_{\text{pu},1}(0)$ and the parameters from (6.2) fixed and integrate (5.1) with different ε and

$c{=}\nu_1$. For $\varepsilon = 1/3, 1/4, 1/5, 1/6, 1/7$ we obtain $y_1(1000)/(1000\,\varepsilon) = -2.7, -2.44, -2.35, -2.31, -2.3$. From this we may conclude that for the given parameters we have $\tilde{c}_1 \approx -2.3$.

Finally, figure 7 shows the numerical integration of the pSHe with parameters from (6.2), initial condition $u_{\text{pu},2}(0,y) = \varepsilon A_{\text{pu},2}(4, 0.5)(\varepsilon y)e^{\text{i}y} + \text{c.c.}$, $\varepsilon = 0.2$, and $c = 5.56$ which is slightly below the c calculated above for the 1–pulse.

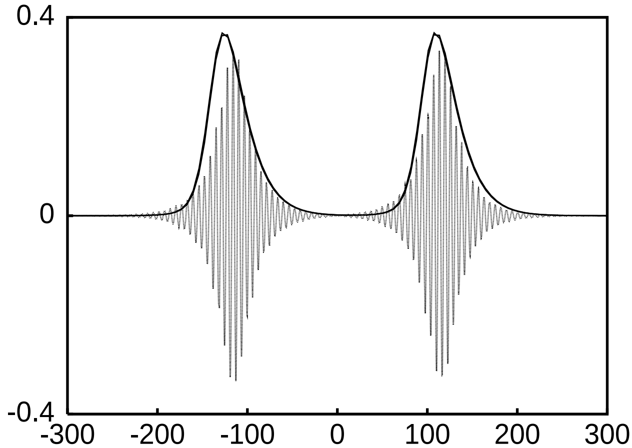

Figure 7: Numerical integration of the modulating 2–pulse, parameters from (6.2), $\varepsilon = 0.2$, $c = 5.56$: initial condition, and envelope for $t = 1000, 2000$.

A The spectrum of the operator $\mathcal{L} = L(\partial_x) + g(x)$.

We calculate the spectrum of the operator $\mathcal{L} : H^4_{\text{lu}}(\mathbb{R}) \to L^2_{\text{lu}}(\mathbb{R})$ with

$$\mathcal{L}u = [-(1+\partial_x^2)^2 - \varepsilon^2\eta\alpha_0 + \beta_1\partial_x + \beta_3\partial_x^3]u + 2\varepsilon^2\eta\alpha_1\cos(2x)u. \tag{A.1}$$

In particular we show that

$$\text{spec}\mathcal{L} \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -\varepsilon^2\eta\alpha_0 + \mathcal{O}(\varepsilon^4)\}, \tag{A.2}$$

which was used in the proof of Lemma 5.1. This is done in a way similar to [DS98], however, we rather adopt the setup from [Mie97b]. There, the spectrum of the linearization around a spatially periodic stationary solution of the Swift–Hohenberg equation is calculated by Liapunov–Schmidt reduction in Bloch wave space.

Bloch transform has its origin in Quantum mechanics. It is a generalization of Fourier transform which is adapted to spatially periodic operators. The setup is as follows. The periodicity of $\mathcal{L}$ is characterized by the lattice group $\mathcal{G} = \{l \in \mathbb{R} : T_l\mathcal{L} = \mathcal{L}T_l\} = \pi\mathbb{Z}$ where $(T_l u)(x) = u(x-l)$. We continue to write $\mathcal{T}_\alpha$ for the one dimensional torus of length α, and define the periodicity domain $\mathcal{T} = \mathbb{R}/\mathcal{G} = \mathcal{T}_\pi$, the dual lattice group $\mathcal{G}^* = \{h \in \mathbb{R} : h \cdot l \in 2\pi\mathbb{Z} \;\; \forall l \in \mathcal{G}\} = 2\mathbb{Z}$ and the wave number domain $\mathcal{T}^* = \mathbb{R}/\mathcal{G}^* = \mathcal{T}_2$. The space $L^2(\mathcal{T}^*, L^2(\mathcal{T}))$ is called Bloch space, and a function u given in the form $u(x) = e^{\mathrm{i}kx}U(x)$ with $k \in \mathcal{T}^*$ and $U \in L^2(\mathcal{T})$ is called a Bloch wave. The Bloch transform $D : L^2(\mathcal{T}^*, L^2(\mathcal{T})) \to L^2(\mathbb{R})$, defined as

$$u(x) = D(U)(x) = \int_{k\in\mathcal{T}^*} e^{\mathrm{i}kx}U(k,x)\,\mathrm{d}k,$$

is an isomorphism, its inverse given by means of Fourier–transform as

$$U(k,x) = D^{-1}(u)(k,x) = \sum_{j\in\mathcal{G}^*=2\mathbb{Z}} e^{\mathrm{i}jx}\hat{u}(k+j).$$

The Bloch operators $B(\varepsilon,k) : H^4(\mathcal{T}_\pi) \to L^2(\mathcal{T}_\pi)$ are defined by

$$\begin{aligned} B(\varepsilon,k)U(k,x) :=& e^{-\mathrm{i}kx}\mathcal{L}(\varepsilon,\kappa)[e^{\mathrm{i}kx}U(k,x)] \\ =& L(\partial_x + \mathrm{i}k)U(k,x) + \varepsilon^2\eta\alpha_1(e_2 + e_{-2})U(k,x), \end{aligned}$$

and we have the identity [Mie97b, Theorem 2.1]

$$L^2_{\mathrm{lu}}\text{–spec}(\mathcal{L}) = L^2\text{–spec}(\mathcal{L}) = \mathrm{closure}\Big(\bigcup_{k\in\mathcal{T}^*} \mathrm{spec}(B(\varepsilon,k))\Big).$$

For every fixed $k \in \mathcal{T}_2$ the operator $B(\varepsilon,k)$ has a discrete set of eigenvalues $\{\mu_j(\varepsilon,k) : j \in \mathbb{Z}\}$. We now use the fact that $B(\varepsilon,k)$ is a small perturbation of $B(0,k)$, i.e.

$$\|B(\varepsilon,k) - B(0,k)\|_{H^4(\mathcal{T}_{2\pi})\to L^2(\mathcal{T}_{2\pi})} \le 2\varepsilon^2\eta\alpha_1. \tag{A.3}$$

For $B(0,k)$ we have $B(0,k)e_j = \mu_j(0,k)e_j$, $j \in 2\mathbb{Z}$ where $e_j(x) = e^{\mathrm{i}jx}$ and

$$\mu_j(0,k) = \lambda(0,k+j) = -(1-(j+k)^2)^2 + \mathrm{i}\beta_1(k+j) - \mathrm{i}\beta_3(k+j)^3. \tag{A.4}$$

This basically means that $\{\mu_j(0,k) : j \in 2\mathbb{Z}\}$ is obtained from rolling up $\{\lambda(0,k) : k \in \mathbb{R}\}$ on a cylinder, see figure 8. We define the set of non-critical wavenumbers k as $G_\delta = \{k \in [0,2] : |k - k_c| \ge \delta\}$. Choosing $\delta = \varepsilon\sqrt{\eta(\alpha_0+\alpha_1)}/2$ using (A.3) and (A.4) we obtain

$$\mathrm{spec}(B(\varepsilon,k)) \subset \{\mu \in \mathbb{C} : \mathrm{Re}\,\mu \le -\varepsilon^2\eta\alpha_0 + \mathcal{O}(\varepsilon^4)\} \text{ for } k \in G_\delta.$$

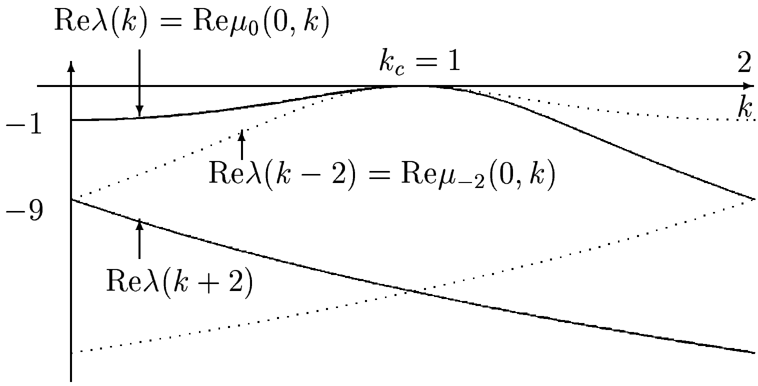

Figure 8: $\mathrm{Re}\lambda(0, k+j)$, logarithmic y–axis

In order to prove (A.2) it remains to calculate $\mu_{0,-2}(\varepsilon, k)$ for $k \in G_c = [0,2] \setminus G_\delta$. This is done by Liapunov–Schmidt reduction, and is also the first step in the derivation of the modulation equation for (1.1) with arbitrary (π–periodic) g, see [DS98]. We let

$$P : L^2(\mathcal{T}_\pi) \to \mathrm{span}\{e_0, e_{-2}\}, \quad PU(x) = \frac{1}{\pi}\Big(\int_0^\pi U \,\mathrm{d}x + \int_0^\pi U\overline{e_{-2}}\,\mathrm{d}x e_{-2}(x)\Big)$$

be the orthogonal projection of $L^2(\mathcal{T}_\pi)$ onto the critical eigenfunctions of $B(0,k)$ at $k = k_c$ and write

$$U(k,x) = \gamma_0(k)e_0 + \gamma_{-2}(k)e_{-2} + V(k,x)$$

with $V(k,\cdot) \in (\mathrm{Id} - P)L^2(\mathcal{T}_\pi)$. Then the eigenvalue problem $[B(\varepsilon,k) - \mu\mathrm{Id}]U = 0$ is equivalent to

$$P[B(\varepsilon,k) - \mu\mathrm{Id}]U = 0 \quad \text{and} \quad (\mathrm{Id} - P)[B(\varepsilon,k) - \mu\mathrm{Id}]U = 0. \tag{A.5}$$

Since $B(\varepsilon,k) : (\mathrm{Id} - P)H^4(\mathcal{T}_\pi) \to (\mathrm{Id} - P)L^2(\mathcal{T}_\pi)$ is invertible the second equation can be solved uniquely for $V(k,\cdot) = \mathcal{O}(\varepsilon^2|\gamma|)$ due to the implicit function theorem, where $\gamma = \gamma(k) = (\gamma_0(k), \gamma_{-2}(k))$. Inserting this into the first equation in (A.5) we obtain the reduced eigenvalue problem

$$[L(\partial_x + \mathrm{i}k) - \mu](\gamma_0 e_0 + \gamma_{-2}e_{-2}) + \varepsilon^2\eta\alpha_1(\gamma_0 e_{-2} + \gamma_{-2}e_0) + \mathcal{O}(\varepsilon^4|\gamma|) = 0.$$

From this we obtain the algebraic eigenvalue problem $M(\varepsilon,k,\mu)\gamma(k)^T = 0$ where $M(\varepsilon,k,\mu) = M_0(\varepsilon,k,\mu) + \mathcal{O}(\varepsilon^4) \in \mathbb{C}^{2\times 2}$ with

$$M_0(\varepsilon,k,\mu) = \begin{pmatrix} \mu_0(0,k) - \varepsilon^2\eta\alpha_0 - \mu & \varepsilon^2\eta\alpha_1 \\ \varepsilon^2\eta\alpha_1 & \mu_{-2}(0,k) - \varepsilon^2\eta\alpha_0 - \mu \end{pmatrix} + \mathcal{O}(\varepsilon^4).$$

With $l=k-1$ the condition $\det M_0=0$ gives

$$\begin{aligned}\mu_\pm(\varepsilon,1+l)=&-\varepsilon^2\eta\alpha_0\pm \mathrm{i}\varepsilon^2\sqrt{\nu_0^2-\eta^2\alpha_1^2}-\nu_1\mathrm{i}l\\&+\left(-4\pm 3\mathrm{i}\beta_3\nu_0/\sqrt{\nu_0^2-\eta^2\alpha_1^2}\right)l^2+\mathcal{O}(l^3)+\mathcal{O}(\varepsilon^4)\end{aligned}\qquad\text{(A.6)}$$

and hence $\mathrm{Re}\mu_\pm\le -\varepsilon^2\eta\alpha_0+\mathcal{O}(\varepsilon^4)$ which proves (A.2).

B The attractivity property

We prove that every eigenfunction of Λ in Lemma 5.1 is of the form (5.4). This follows from the attractivity of the set of modulated patterns

$$\mathrm{MP}=\{w(y)=\psi_A(y):\|A\|_{H^1_{\mathrm{lu}}}\le 1\},\quad \psi_A(y)=\varepsilon A(\varepsilon y)e^{\mathrm{i}x}+\mathrm{c.c.},$$

for the linear equation (5.3), i.e.

$$w_t=L_{\mathrm{pu}}w=(L+\varepsilon^2\eta\alpha_1\cos(2y+ct)+c\partial_y+DN(v_{\mathrm{pu}}))w.\qquad\text{(B.1)}$$

For consistency with the usual nonlinear case we consider initial conditions of order ε.

Theorem B.1 *There exist $\varepsilon_0,C,T_1>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and all $w_0\in H^1_{\mathrm{lu}}(\mathbb{R})$ with $\|w_0\|_{H^1_{\mathrm{lu}}}\le\varepsilon$ there exist an $A\in H^1_{\mathrm{lu}}(\mathbb{R})$ with $\|A\|_{H^1_{\mathrm{lu}}}=\mathcal{O}(1)$ such that for the solution w of (B.1) we have*

$$\|w(T_1/\varepsilon^2,\cdot)-\psi_A(\cdot)\|_{H^1_{\mathrm{lu}}}\le C\varepsilon^2.$$

The basic idea is that the solution w of (5.3) develops peaks of width $\mathcal{O}(\varepsilon)$ in Fourier space, i.e., $\hat{w}(j)=\mathcal{O}(\varepsilon^{\gamma_j})$ for $j\in\mathbb{Z}$, $\gamma_j=|1-j|$, and $\hat{w}(k)\approx 0$ in between the peaks, see figure 9a). Results of this type have been shown in different function spaces for the classical Swift–Hohenberg/Kuramoto–Shivashinsky equation and similar translation invariant system, see [Eck93, BvHS95] for an L^1–version and [Sch94b] for the result in local uniform spaces. In [Sch95] the stronger result of the analyticity of the function A (and of higher order modes) is shown.

To explain the mechanism, here we show a somewhat simplified version of Theorem B.1, namely the peak formation for initial conditions w_0 such that the Fourier transform of w_0 is in L^1. Since the only difference to earlier work are the terms coming from $DN(v_{\mathrm{pu}})$ and from the periodic term $2\varepsilon^2\eta\alpha_1\cos(2(y+ct))$, we mainly want to show how to treat these terms. One key feature of the usual

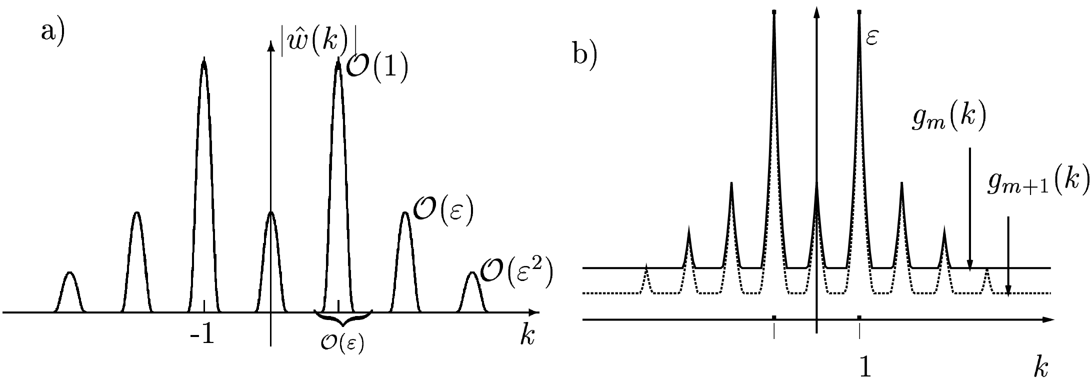

Figure 9: a) Mode distribution in Fourier space, b) Two consecutive weights g_m, g_{m+1}

translational invariant case is that the quadratic interaction of critical modes gives stable modes. In our case we find that the terms $(\mathcal{F}DN(v_{\rm pu})) * \hat{w}$ and $\mathcal{F}(2\cos(2(y+ct))w)$ preserve the mode structure in a similar way.

Finally, we have a linear problem which makes the proof more simple. For the nonlinear problem attractivity is a local result, i.e., it holds for suffiently small initial conditions, while for the linear problem the size of w_0 is arbitrary. However, for the sake of consistency with previous work we assume below that $\|\hat{w}_0\|_{L^1} = \mathcal{O}(\varepsilon)$.

Following [BvHS95] we introduce for $m \geq 1$ the weights

$$g_m(k) = \varepsilon^{(m+3)/2} \max_{j=\pm1,\pm2,l=\pm3,\dots,\pm m} \{2, (\varepsilon + |k-j|)^{|j|-m/2+3/2}, \\ (\varepsilon+|k|)^{1/2-m/2}, (\varepsilon+|k-l|)^{|l|/2-m/2-1/2}\},$$

and let $h_m(k) = 1/g_m(k)$, see figure 9b). Then $\|\hat{w}h_m\|_1 = \mathcal{O}(1)$ means that $\hat{w}$ has a mode distribution similar to the one shown in figure 9a). We now show the following result:

Lemma B.2 *For all $m \in \mathbb{N}$ there exists $C_m, T_m, \varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and $\|\hat{w}_0\|_{L^1} \leq \varepsilon$ the solution w of (B.1) fulfills $\|\hat{w}(T_m/\varepsilon^2, \cdot)h_m\|_1 \leq C_m$.*

Proof. The modulating pulse $v_{\rm pu}(t,y)$ is given as

$$v_{\rm pu}(t,y) = \sum_{j\in\mathbb{Z}} \varepsilon^{1-|1-|j||} A_{\rm pu}^{(j)}(\varepsilon y) e^{{\rm i}j(y+ct)} + \text{c.c.}$$

with $\|A_{\rm pu}^{(j)}\|_{H^1_{\rm lu}} = \mathcal{O}(1)$. From the proof it will be clear that we can ignore the higher modes $A_{\rm pu}^{(j)}$, $j \neq \pm 1$ in the proof of Lemma B.2. Using this simplification and writing $A_{\rm pu}$ for $A^{(1)}_{\rm pu}$, in Fourier space equation (B.1) for $\hat w = \hat w(t,k)$ reads

$$\partial_t \hat w(t,k) = (\lambda(k)+c\mathrm{i}k)\hat w(t,k) + \varepsilon I_1(\hat w)(t,k) + \varepsilon^2 I_2(\hat w)(t,k), \text{ where} \tag{B.2}$$

$$I_1(\hat w)(t,k) = (f_1+\mathrm{i}kf_2)\left((\frac{1}{\varepsilon}\hat A_{\rm pu}(\cdot/\varepsilon) * \hat w)(t,k-1)e^{\mathrm{i}ct} + \text{c.c.f}\right),$$

$$I_2(\hat w)(t,k) = \alpha_1\eta\left(\hat w(t,k-2)e^{2\mathrm{i}ct} + \hat w(t,k+2)e^{-2\mathrm{i}ct}\right),$$

and c.c.f stands for $(\hat{\overline{A}}_{\rm pu}(\cdot/\varepsilon) * \hat w)(t,k+1)e^{-\mathrm{i}ct}$. Clearly $\|I_2(\hat w)\|_1 = 2\alpha_1\eta\|\hat w\|_1$, and

$$\|\frac{1}{\varepsilon}\hat A(\cdot/\varepsilon) * \hat w\|_1 \le \|\frac{1}{\varepsilon}\hat A(\cdot/\varepsilon)\|_1\|\hat w\|_1 \le \|\hat A\|_1\|\hat w\|_1, \tag{B.3}$$

by Young's inequality. Thus we have $\|I_1(\hat w)\|_1 = \mathcal{O}(\|\hat w\|_1)$ and a priori we obtain $\|\hat w(t)\|_1 = \mathcal{O}(\varepsilon)$ only for $t = \mathcal{O}(1/\varepsilon)$.

The proof now proceeds in two steps. First we show that $\|\hat w(T_1/\varepsilon^2)h_1\|_1 \le C_1$ for some $T_1, C_1 > 0$. Second we show that $\|\hat w(T_m/\varepsilon^2)h_m\|_1 \le C_m$ implies $\|\hat w(T_{m+1}/\varepsilon^2)h_{m+1}\|_1 \le C_{m+1}$ for some $T_{m+1} > T_m$.

We call $I_c = [-5/4, -3/4] \cup [3/4, 5/4]$ the set of critical wavenumbers since in I_c we have $\mathrm{Re}\lambda(k) = \mathcal{O}(\varepsilon^2)$. Its complement is the set of stable wavenumbers I_s where $\mathrm{Re}\lambda_1 \le -\sigma$ for some $\sigma > 0$ independent of ε. By $\chi_c = \chi_{I_c}$ we denote the characteristic function of I_c and by χ_s the one of I_s. We then have the following elementary estimates,

$$\begin{aligned} &\sup_{k\in\mathbb{R}} |e^{(\lambda(k)+cik)t}| \le e^{-\alpha_0\eta\varepsilon^2 t}, \quad \sup_{k\in\mathbb{R}} |e^{(\lambda(k)+cik)t}k| \le C(1+t^{-1/4}), \quad \text{and} \\ &\sup_{k\in I_s} |e^{(\lambda(k)+cik)t}| \le e^{-\sigma t} \text{ with } \sigma > 0 \text{ independent of } \varepsilon. \end{aligned} \tag{B.4}$$

Step 1 itself consist of a preliminary step followed by two parts: in the **Preliminary step** we first note that obviously $\sup_{0\le t\le\varepsilon^{-1}} \|\hat w(t)\|_1 \le C\varepsilon$. Next we have $\|\hat w(\varepsilon^{-1/4})h_0\|_1 \le C$ where $h_0 = 1/(\varepsilon\chi_c + \varepsilon^2\chi_s)$. This holds since

$$\|\chi_c\hat w(\varepsilon^{-1/4})h_0\| = \|\chi_c\hat w(\varepsilon^{-1/4})/\varepsilon\| \le C, \text{ and}$$

$$\|\chi_s\hat w(\varepsilon^{-1/4})h_0\| \le \varepsilon^{-1}e^{-\sigma/\varepsilon^{-1/4}}\|\hat w(\varepsilon^{-1/4})/\varepsilon\|_1 + C\int_0^{\varepsilon^{-1/4}} e^{-\sigma(t-s)}\,\mathrm{d}s \le C$$

for $\varepsilon \le \varepsilon_0$ with $\varepsilon_0^{-1}e^{-\sigma\varepsilon_0^{-1/4}} \le 1$.

Step 1, Part 1: In (B.2) we start again with $\hat w_0 = \hat w(\varepsilon^{-1/4})$ and show that there exist $T_1, C_1 > 0$ such that $\sup_{t\le T_1/\varepsilon^2} \|\hat w(s)h_0\|_1 \le C_1$. Therefore we let $S_j(t) = \sup_{\tau\le t} \|\chi_j\hat w(\tau)h_0\|_1$, $j = c, s$. Then using

$$\begin{aligned} &\chi_c \max\{h_0(k)/h_0(k+1), h_0(k)/h_0(k-1))\} = \varepsilon, \\ &\chi_c \max\{h_0(k)/h_0(k+2), h_0(k)/h_0(k-2))\} \le 1, \end{aligned} \tag{B.5}$$

and (B.4) we obtain

$$\begin{aligned} S_c(t) \le \|\chi_c\hat{w}_0 h_0\|_1 + \int_0^t \Big\| e^{(\lambda(k)+cik)(t-\tau)}\chi_c(k)\Big\{\Big(\frac{h_0(k)}{h_0(k-1)}(f_1+\mathrm{i}f_2k) \\ (\hat{A}_{\mathrm{pu}}(\cdot/\varepsilon)*\hat{w})(\tau,k-1)h_0(k-1)+\text{c.c.f}\Big) + \varepsilon^2 I_2(\tau,k)\Big\|_1\Big\}\,\mathrm{d}\tau \\ \le \|\hat{w}_0 h_0\|_1 + \int_0^t \Big(\sup_{k\in I_c}\big|e^{(\lambda(k)+cik)(t-\tau)}(f_1+\mathrm{i}f_2k)\big| + 1\Big)\varepsilon^2\|\hat{w}(\tau)h_0\|_1\,\mathrm{d}\tau \\ \le C + C\varepsilon^2\int_0^t S_c(\tau)+S_s(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Similarly we obtain

$$S_s(t) \le e^{-\sigma t}\|\chi_s\hat{w}_0h_0\|_1 + \int_0^t \Big(e^{-\sigma(t-\tau)}(|f_1|+|f_2|t^{-1/2})+\varepsilon^2\Big)(S_c+S_s)\,\mathrm{d}\tau,$$

where we used

$$\sup_{k\in I_s} h_0(k)/h_0(k\pm 1) = 1/\varepsilon, \quad \sup_{k\in I_s} h_0(k)/h_0(k\pm 2) = 1/\varepsilon. \tag{B.6}$$

Thus, for $S(t) = S_c(t) + S_s(t)$ we have

$$S(t) \le \|\hat{w}_0h_0\| + \int_0^t (\varepsilon^2 + e^{-\sigma(t-\tau)}(|f_1|+|f_2|t^{-1/2})S(\tau)\,\mathrm{d}\tau \tag{B.7}$$

and hence $\sup_{s\le t}\|\hat{w}(s)h_0\|_1 \le S(t) \le C$ for $t = T_1/\varepsilon^2$.

Step 1, Part 2: We show that $\|\hat{w}(T_1/\varepsilon^2)h_1\|_1 \le C_1$ for some $C_1 > 0$. We have

$$\begin{aligned} &\chi_c h_1(k)/h_0(k) = 1 + ||k|-1|/\varepsilon, \quad \chi_s h_1(k)/h_0(k) = 1, \\ &\chi_c \max\{h_1(k)/h_0(k-1), h_1(k)/h_0(k+1)\} = \varepsilon + ||k|-1|, \\ &\chi_c \max\{h_1(k)/h_0(k-2), h_1(k)/h_0(k+2)\} = 1 + ||k|-1|/\varepsilon, \\ &\chi_s \max\{h_1(k)/h_0(k-1), h_1(k)/h_0(k+1)\} = 1/\varepsilon, \\ &\chi_s \max\{h_1(k)/h_0(k-2), h_1(k)/h_0(k+2)\} = 1/\varepsilon. \end{aligned} \tag{B.8}$$

Using these identities, for $t_* = T_1/\varepsilon^2$ we obtain

$$\begin{aligned}
\|\chi_c \hat{w}(t_*) h_1\|_1 &\leq C \sup_{k\in I_c} |e^{(\lambda(k)+cik)t_*} h_1(k)/h_0(k)| \; \|\hat{w}(t_*)h_0\|_1 \\
&\quad + \int_0^{t_*} \sup_{k\in I_c} |e^{(\lambda(k)+cik)(t_*-\tau)}(f_1+ikf_2)\frac{h_1(k)}{h_0(k\pm 1)}| \; \varepsilon\|\hat{w}(\tau)h_0\|_1 d\tau \\
&\quad + \int_0^{t_*} \sup_{k\in I_c} |e^{(\lambda(k)+cik)(t_*-\tau)}\frac{h_1(k)}{h_0(k\pm 2)}| \; \varepsilon^2\|\hat{w}(\tau)h_0\|_1 d\tau \\
&\leq C(1+T_1^{-1/2})\|\hat{w}(t_*)h_0\|_1 + \varepsilon S(t_*) \int_0^{t_*} (1+\tau^{-1/2})\, \mathrm{d}\tau \\
&\leq C, \\
\|\chi_s \hat{w}(t_*) h_1\|_1 &\leq C \sup_{k\in I_s} |e^{-\sigma t_*}\frac{h_1(k)}{h_0(k)}| \; \|\hat{w}_0 h_0\| + S(t_*) \int_0^{t_*} e^{-\sigma^2(t_*-\tau)}(t_*-\tau)^{-1/4}\, \mathrm{d}\tau \\
&\leq C.
\end{aligned}$$

Step 2: The step from (T_m, h_m) to (T_{m+1}, h_{m+1}) works as Part 1 and Part 2 in Step 1. We only have to replace (B.5), (B.6) and (B.8) by slightly weaker estimates of the form

$$\chi_c \max\{\frac{h_m(k)}{h_m(k+1)}, \frac{h_m(k)}{h_m(k-1)}\} = \varepsilon + ||k|-1|,$$

which gives

$$\int_0^t \sup_{k\in I_c} \left| e^{(\lambda(k)+cik)t} \max\{\frac{h_m(k)}{h_m(k-1)}, \frac{h_m(k)}{h_m(k+1)}\} \right| dt \leq C(\varepsilon t + t^{1/2}), \tag{B.9}$$

and similarly

$$\sup_{k\in I_c} |\frac{h_m(k)}{h_m(k\pm 2)}| \leq 1, \quad \sup_{k\in I_s} |\frac{h_m(k)}{h_m(k\pm 1)}| = 1/\varepsilon, \quad \sup_{k\in I_s} |\frac{h_m(k)}{h_m(k\pm 2)}| \leq \varepsilon^{-3/2}, \tag{B.10}$$

$$\begin{aligned}
&\sup_{k\in I_c} |\frac{h_{m+1}(k)}{h_m(k)}| = \sup_{k\in I_c} (1+||k|-1|/\varepsilon)^{1/2}, \quad \sup_{k\in I_s} |\frac{h_{m+1}(k)}{h_m(k)}| \leq \varepsilon^{-1/2}, \\
&\sup_{k\in I_c} |\frac{h_{m+1}(k)}{h_m(k\pm 1)}| = \sup_{k\in I_c} \varepsilon(1+||k|-1|/\varepsilon)^{3/2}, \\
&\sup_{k\in I_c} |\frac{h_{m+1}(k)}{h_m(k\pm 2)}| = \sup_{k\in I_c} (1+||k|-1|/\varepsilon)^{1/2}, \\
&\sup_{k\in I_s} |\frac{h_{m+1}(k)}{h_m(k\pm 1)}| = 1/\varepsilon, \quad \sup_{k\in I_s} |\frac{h_{m+1}(k)}{h_m(k\pm 2)}| = \varepsilon^{-3},
\end{aligned} \tag{B.11}$$

Using (B.9) and (B.10) we obtain $\sup_{T_m/\varepsilon^2 \leq t \leq T_{m+1}/\varepsilon^2} \|\hat{w}(t)h_m\|_1 \leq \tilde{C}_{m+1}$ and from (B.11) we get $\|\hat{w}(T_{m+1}/\varepsilon^2)h_{m+1}\|_1 \leq C_{m+1}$. □.

C Further solutions of the nsGLe and rolls for (1.1)

The stabilizing effect of a resonant spatially periodic forcing for certain spatially periodic solutions of systems of the form (1.1) is well known in the physics literature, see for instance [Wal97] and the references therein. Here we briefly comment on the existence and stability of spatially periodic solutions to (1.1) stated in Remark 1.8. These correspond to spatially homogenous equilibria A of the nsGLe (1.6) and may be constructed using the method of section 4, i.e., they are obtained as constant solutions of the reduced equation (4.11).

Here we first ignore the $\mathcal{O}(\varepsilon)$ terms. Letting $A(T) = U(T) + \mathrm{i}V(T)$ and $W(T) = (U(T), V(T))^T$ we obtain the Landau equation

$$\partial_T W = L^{\mathrm{Lan}} W + F(W), \quad \text{where} \tag{C.1}$$

$$L^{\mathrm{Lan}} = \begin{pmatrix} \eta(\alpha_1 - \alpha_0) & \nu_0 \\ -\nu_0 & -\eta(\alpha_1 + \alpha_0) \end{pmatrix},$$
$$F(W) = (U^2 + V^2) \begin{pmatrix} c_{3r} & -c_{3i} \\ c_{3i} & c_{3r} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

For $c_{3r}c_{3i} = 0$ the equilibria of (C.1) may be easily calculated. For $c_{3r} = 0$ we obtain four nontrivial fixed points $W^{(j)} = (U_j, V_j)$, $j = 1, \dots, 4$, where

$$(U_1, V_1) = \left(\sqrt{\frac{\alpha_1 + \alpha_0}{2c_{3i}\alpha_1}(\nu_0 + \eta\sqrt{\alpha_1^2 - \alpha_0^2})}, \sqrt{\frac{\alpha_1 - \alpha_0}{2c_{3i}\alpha_1}(\nu_0 + \eta\sqrt{\alpha_1^2 - \alpha_0^2})} \right),$$
$$(U_3, V_3) = \left(\sqrt{\frac{\alpha_1 + \alpha_0}{2c_{3i}\alpha_1}(\nu_0 - \eta\sqrt{\alpha_1^2 - \alpha_0^2})}, -\sqrt{\frac{\alpha_1 - \alpha_0}{2c_{3i}\alpha_1}(\nu_0 - \eta\sqrt{\alpha_1^2 - \alpha_0^2})} \right),$$

and $W^{(2)} = -W^{(1)}, W^{(4)} = -W^{(3)}$. From the linearization of (C.1) around $W^{(j)}$ we obtain the eigenvalues

$$\lambda_{1,2} = -\eta\alpha_0 \pm \mathrm{i}\gamma_1 \text{ at } W^{(1,2)} \quad \text{and}$$
$$\lambda_{1,2} = \pm 2\eta\alpha_1 + \mathcal{O}(\eta^3) + \mathrm{i}\gamma_2 \text{ at } W^{(3,4)},$$

where $\gamma_{1,2} \in \mathbb{R}$ but we omit the lengthy formulas. Thus, we have two sinks $W^{(1,2)}$ and two saddles $W^{(3,4)}$ for (C.1). Since the nullclines in the fixed points intersect transversely these equilibria persist for small $c_{3r} \neq 0$, and moreover, for more general small perturbations of (C.1), that is, for the $\mathcal{O}(\varepsilon)$–perturbations in (4.11). Note however that the condition $\alpha_1 > \alpha_0 > 0$ is crucial for U_j, V_j to be real, i.e., for the existence of $W^{(j)}$, $j = 1..4$. Figure 10 shows the phase portrait for (C.1) with $\alpha_0 = 1, \eta = 0.1, \nu_0 = 1, c_{3i} = 4, c_{3r} = 0.1$ and $\alpha_1 = 0$ in a) and $\alpha_1 = 2$ in b).

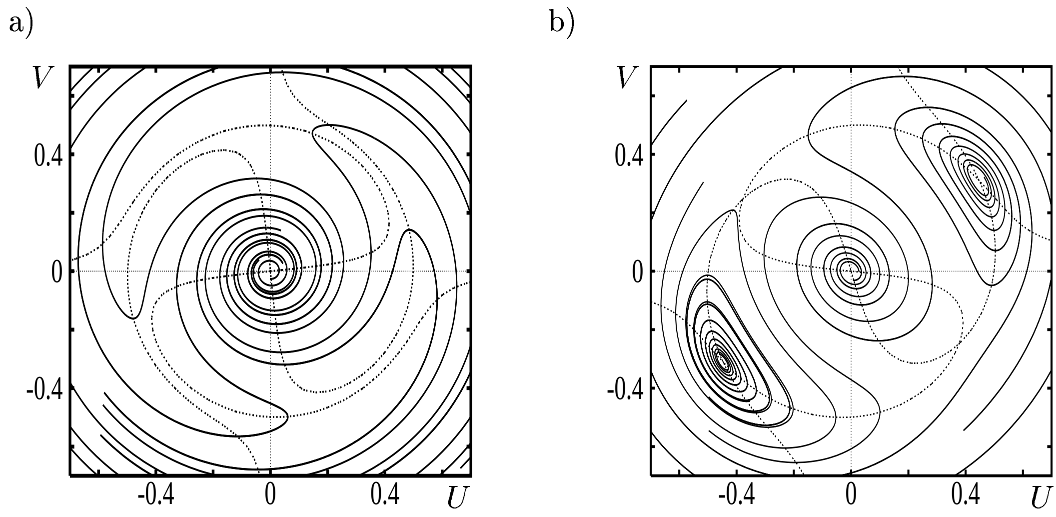

Figure 10: The phase portrait of (C.1) with $\alpha_0{=}1, \eta{=}0.1, \nu_0{=}1, c_{3i}{=}4, c_{3r}{=}0.1$ and $\alpha_1 = 0$ in a) and $\alpha_1 = 2$ in b). The nullclines of (C.1) are shown as dotted lines.

In summary, from each non–trivial fixed point $W^{(j)}$ of (C.1) we obtain a spatially 2π–periodic stationary solution $u_{\rm per}^{(j)}$ of (1.1), a so called roll, via

$$u_{\rm per}^{(j)}(x) = \varepsilon(U_j + {\rm i}V_j)e^{{\rm i}x} + {\rm c.c.} + \mathcal{O}(\varepsilon^2).$$

Due to standard center manifold theory the rolls $u_{\rm per}^{(1,2)}$ are stable in the space $H^1(\mathcal{T}_{2\pi})$ of 2π–periodic functions, while of course the rolls $u_{\rm per}^{(j)}$, $j = 3, 4$, corresponding to the saddle points $W^{(3,4)}$, are unstable.

For the stability of the rolls $u_{\rm per}^{(1,2)}$ in $H^1_{\rm lu}(\mathbb{R})$ we need to exclude the possibility of so called side–band instabilities. The spectrum of the linearization $\mathcal{L}^*$ of (1.1) around $u_{\rm per}^{(1,2)}$ can again be calculated by Bloch transform and Liapunov–Schmidt reduction as in Appendix A. However, here we rather give a heuristic discussion in terms of the Ginzburg–Landau formalism. Therefore we consider the stability of the equilibria $W^{(1,2)}$ by linearizing the full nsGLe (2.5), which gives

$$\partial_T W = (L^{\rm GL}(\partial_X) + DF(W^{(j)}))W. \qquad \text{(C.2)}$$

The constant matrix $DF(W^{(j)}) \in \mathbb{C}^{2\times 2}$ only perturbs the essential spectrum of $L^{\rm GL}(\partial_X)$ as given in (2.6) and does not generate eigenvalues. Here we do not present these algebraic calculations. Instead, in figure 11 we show the results obtained for the linearization of (C.2) around $W^{(1)}$, with fixed $\alpha_0 = 1, \alpha_1 = 2, \eta = 0.1, \nu_0 = 1, c_{3r} = 0.1, c_{3i} = 4, c_{1i} = 1$, and $c_{1r} = 0.1$ in a) and $c_{1r} = 2$ in b). This figure shows the real–parts of the eigenvalues

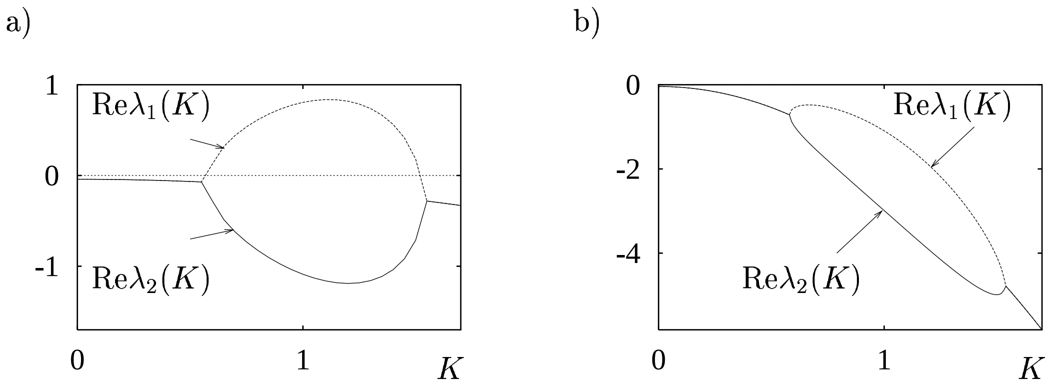

Figure 11: The real part of $\lambda_{1,2}(K)$ plotted over K: a) with small dissipation $c_{1r} = 0.1$, b) with large dissipation $c_{1r} = 2$. Note that in both cases we have $\text{Re}\lambda_{1,2}(0) = -\eta\alpha_0 < 0$.

$\lambda_{1,2}(K)$ of $L^{\text{GL}}(\text{i}K) + DF(W^{(1)})$ over the Fourier wavenumber K. For small $c_{1r} > 0$ the fixed point $W^{(1)}$ is unstable with respect to Fourier–modes with wave–number K in a band around $K = 1$. Note that small $c_{1r} > 0$ corresponds to the parameter region where stable n–pulse solutions to the nsGLe exist, see section 3.

For larger c_{1r}, where the analysis of section 3 for the pulse solutions is not valid, by dissipation $\lambda(K)$ gets pushed below the K–axes for all K. Thus, for c_{1r} sufficiently large we obtain stationary solutions $W^{(1,2)}$ to the nsGLe that are exponentially stable in H^1_{lu}.

In the critical parts $||k|-1| \leq \mathcal{O}(\varepsilon)$ for the Bloch wavenumber k, the largest eigenvalue $\mu_1(k)$ of the linearization $\mathcal{L}^*$ of (1.1) around the roll $u^{(1)}_{\text{per}}$ is given by $\mu_1(1+\varepsilon K) = \varepsilon^2\lambda_1(K) + \mathcal{O}(\varepsilon^4)$, $|K| \approx 1$. Thus, for small c_{1r} we have a so called detached sideband–instability, while for $c_{1r} = \mathcal{O}(1)$ and ε sufficiently small we obtain

$$\text{spec}\mathcal{L}^* \subset \{z \in \mathbb{C} : \text{Re}z \leq -\varepsilon^2\eta\alpha_0 + \mathcal{O}(\varepsilon^4) < 0\}.$$

Therefore, in the latter case the equilibria $W^{(1,2)}$ correspond to roll solutions of the original equation (1.1) which are exponentially stable in $H^1_{\text{lu}}(\mathbb{R})$. This concludes the discussion of Remark 1.8.

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References

[BvHS95] P. Bollermann, A. van Harten, and G. Schneider. On the justification of the Ginzburg–Landau approximation. In A. Doelman and A. van Harten, editors, *Nonlinear dynamics and pattern formation in the natural environment*, pages 20–37. Longman UK, 1995.

[CE90] P. Collet and J.-P. Eckmann. The time dependent amplitude equation for the Swift–Hohenberg problem. *Comm. Math. Phys.*, 132:139–152, 1990.

[CHQZ88] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods in Fluid Dynamics.* Springer, 1988.

[DS98] A. Doelman and R. Schielen. Modulation equations for spatially periodic systems: Derivation and solutions. *SIAM J. Appl Math.*, 58(6):1901–1930, 1998.

[Eck93] W. Eckhaus. The Ginzburg–Landau equation is an attractor. *J. Nonlinear Science*, 3:329–348, 1993.

[EK97] W. Eckhaus and R. Kuske. Pattern formation in systems with slowly varying geometry. *SIAM J. Appl Math.*, 57:112–152, 1997.

[EW91] J.-P. Eckmann and C.E. Wayne. Propagating fronts and the center manifold theorem. *Comm. Math. Phys.*, 136:285–307, 1991.

[FV92] B. Fiedler and A. Vanderbauwhede. Homoclinic period blow–up in reversible and conservative systems. *ZAMP*, 43:292–318, 1992.

[HCS98] M. Haragus-Courcelle and G. Schneider. Bifurcating fronts for the Taylor Couette problem in infinite cylinders. *ZAMP*, 50(1):397–422, 1998.

[Hen81] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer Lecture Notes in Mathematics, Vol. 840, 1981.

[IM91] G. Iooss and A. Mielke. Bifurcating time–periodic solutions of Navier–Stokes equations in infinite cylinders. *J. Nonlinear Science*, 1:147–229, 1991.

[IV92] G. Iooss and A. Vanderbauwhede. Center manifolds in infinite dimensions. *Springer, Dynamics reported*, 1:125–163, 1992.

[Kir82] K. Kirchgässner. Wave solutions of reversible systems and applications. *J. Diff. Eq.*, 45:113–127, 1982.

[KK96] J. N. Kutz and W. L. Kath. Stability of pulses in nonlinear optical fibers using phase sensitive amplifiers. *SIAM J. Appl. Math.*, 56(2):611–626, 1996.

[KMS92] P. Kirrmann, A. Mielke, and G. Schneider. The validity of modulation equations for extended systems with cubic nonlinearities. *Proc. of the Royal Society of Edinburgh*, 122A:85–91, 1992.

[KS98] T. Kapitula and B. Sandstede. Stability of bright solitary–wave solutions to perturbed nonlinear Schrödinger equations. *Physica D*, 124:58–103, 1998.

[LMN95] J. Lega, J.V. Moloney, and A.C. Newell. Universal description of laser dynamics near threshold. *Physica D*, 83:478–498, 1995.

[LPSS99] G. Lord, D. Peterhof, B. Sandstede, and A. Scheel. Numerical calculation of solitary waves in infinite cylindrical domains, Preprint. 1999.

[Man92] P. Manneville. *Dissipative Structures and Weak Turbulence.* Academic Press, 1992.

[Mie97a] A. Mielke. The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors. *Nonlinearity*, 10:199–222, 1997.

[Mie97b] A. Mielke. Instability and stability of rolls in the Swift–Hohenberg equation. *Comm. Math. Phys.*, 189:829–853, 1997.

[MS95] A. Mielke and G. Schneider. Attractors for modulation equations on unbounded domains–existence and comparison. *Nonlinearity*, 8(5):743–768, 1995.

[MS96] A. Mielke and G. Schneider. Derivation and justification of the complex Ginzburg–Landau equation as a modulation equation. In [LO96], pages 191–216, 1996.

[MS99] B. Malomed and A. Shipulin. Stabilization of optical pulse propagation by a supporting periodic structure. *Opt. Commun.*, 162:140–147, 1999.

[Sch94a] G. Schneider. Error estimates for the Ginzburg–Landau approximation. *J. Appl. Math. Phys. (ZAMP)*, 45:433–457, 1994.

[Sch94b] G. Schneider. Global existence via Ginzburg–Landau formalism and pseudo–orbits of Ginzburg–Landau approximations. *Comm. Math. Phys.*, 164:157–179, 1994.

[Sch94c] G. Schneider. A new estimate for the Ginzburg–Landau approximation on the real axis. *J. Nonlinear Science*, 4:23–34, 1994.

[Sch95] G. Schneider. Analyticity of Ginzburg–Landau modes. *J. Diff. Eqns.*, 121:233–257, 1995.

[Sch96] G. Schneider. Diffusive stability of spatial periodic solutions of the Swift–Hohenberg equation. *Comm. Math. Phys.*, 178:679–702, 1996.

[Sch99] G. Schneider. Existence and stability of modulating pulse–solutions in dissipative systems. Preprint, 1999.

[SH77] J. Swift and P.C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Physical Review A*, 15, Nr.1:319–328, 1977.

[SJA97] B. Sandstede, C.K.R.T. Jones, and J. Alexander. Existence and stability of n–pulses on optical fibers with phase sensitive amplifiers. *Physica D*, 106:167–206, 1997.

[Uec99] H. Uecker. Diffusive stability of rolls in the two–dimensional real and complex Swift–Hohenberg equation. *Comm. PDE*, 24(11&12):2109–2146, 1999.

[vH91] A. van Harten. On the validity of Ginzburg–Landau's equation. *J. Nonlinear Science*, 1:397–422, 1991.

[Wal97] D. Walgraef. *Spatio–Temporal Pattern Formation.* Springer Verlag New York, 1997.