Nonlinear coupled mode dynamics in hyperbolic and parabolic periodically structured spatially extended systems

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Abstract

Nonlinear coupled mode equations occur as universal modulation equations in various circumstances. It is the purpose of this paper to prove exact estimates between the approximations obtained via the nonlinear coupled mode equations and solutions of the original parabolic or hyperbolic systems. The models which we consider contain all difficulties which have to be overcome in the general case.

1 Introduction

Nonlinear partial differential equations on spatially extended domains are very often described by simpler models, so called amplitude or modulation equations. These equations can be derived by formal multiple scaling perturbation analysis. Typical examples of such universal modulation equations are the Ginzburg-Landau equation, the Nonlinear Schrödinger equation, or the Korteweg de Vries equation.

A very long time the formal derivation has been the only connection to the original systems. Only in the last years mathematicians started to analyze the question whether solutions of the original systems really can be described by the formally derived modulation equations, see [2, 7, 1, 16, 4, 10]. Interestingly it turned out that this is in general not true. There are counterexamples [11, 5], where the original system behaves differently than predicted by the formally derived modulation equations.

Motivated by [6], the subject of this paper is the mathematical justification of the so called nonlinear coupled mode equations (NLCME)

$$\begin{aligned} \partial_T A_+ &= +v_g \partial_X A_+ + \alpha A_+ + \kappa A_- + (\gamma_1 |A_+|^2 + \gamma_2 |A_-|^2) A_+, \\ \partial_T A_- &= -v_g \partial_X A_- + \alpha A_- + \kappa A_+ + (\gamma_2 |A_+|^2 + \gamma_1 |A_-|^2) A_-, \end{aligned} \tag{1.1}$$

in case of quadratic terms present in the original system. Herein, $A_\pm(X,T) \in \mathbb{C}$, $0 \neq v_g \in \mathbb{R}$, $T \geq 0$, $X \in \mathbb{R}$, and $\alpha, \kappa, \gamma_1, \gamma_2 \in \mathbb{C}$. Solutions u of the original systems can be described by

$$u(x,t) = \varepsilon A_+(\varepsilon^2 x, \varepsilon^2 t)\varphi_+ e^{\mathrm{i}k_0 x + \mathrm{i}\omega_0 t} + \varepsilon A_-(\varepsilon^2 x, \varepsilon^2 t)\varphi_- e^{-\mathrm{i}k_0 x + \mathrm{i}\omega_0 t} + \text{c.c.} + \mathcal{O}(\varepsilon^2)$$

with $0 \neq k_0 \in \mathbb{R}$, $0 \neq \omega_0 \in \mathbb{R}$, $x \in \mathbb{R}$, $t \geq 0$, with $\varphi_\pm$ in a Hilbert space, where c.c. means complex conjugate and where $0 < \varepsilon \ll 1$ is a small parameter.

The NLCME appear for instance in the nonlinear propagation of light in an optical fiber waveguide with an underlying spatially periodic structure [6]. In case of $\alpha = 0$ and $\kappa, \gamma_{1,2} \in \mathrm{i}\mathbb{R}$ they possess so called gap soliton solutions which play a big role in nonlinear optics.

As already said it is the purpose of this paper to prove exact estimates between the approximations obtained via the NLCME and solutions of the original systems. In order to keep the notations on a reasonable level we refrain from greatest generality and consider two special model problems as original systems which contain all difficulties which have to be solved in the general case. This has to be understood in the following sense. The NLCME are justified as a modulation equation if the dynamics in the original system can be predicted by the NLCME. This means that to solutions of the NLCME for $T \in [0, T_0]$ there should be solutions of the original system for $t \in [0, T_0/\varepsilon^2]$ which can be approximated via the NLCME. Since these solutions are of order $\mathcal{O}(\varepsilon)$, Gronwall's inequality only gives error estimates on time scales $\mathcal{O}(1/\varepsilon)$ if quadratic terms are present in the original system. If the nonlinearity starts with cubic terms Gronwall's inequality gives estimates on the right time scale $\mathcal{O}(1/\varepsilon^2)$. This has been used in [6], where the NLCME have already been justified as a modulation equation for the equations of nonlinear propagation of light in an optical fiber waveguide with an underlying spatially periodic structure.

From a perturbation theoretical point of view the handling of quadratic nonlinearities considered in this paper is the real challenge. In this sense our methods apply to the general case.

We have to distinguish hyperbolic and parabolic systems. In hyperbolic systems the required time scale can be obtained with the help of a normal form transform. See Section 2. In the parabolic case the required time scale can be obtained by using the smoothing properties of the linear semigroup and by controlling the nonlinear mode interactions. See Section 3.

By the results of this paper it is clear that the NLCME provide good approximations of the dynamics in the original systems and do not have to be added to the counter examples of not useful modulation equations.

Notation. Throughout this paper we assume $0 < \varepsilon \ll 1$ and denote many different constants uniformly by C if they can be chosen independent of $0 < \varepsilon \ll 1$.

2 The nonlinear wave equation

In this section we consider the hyperbolic case. The easiest model problem with a quadratic nonlinearity is a Klein-Fock equation with a spatially periodic potential

$$\partial_t^2 u = \partial_x^2 u - u + 2\varepsilon^2 \kappa \cos(2k_0 x) u - u^2 \tag{2.1}$$

with $x, t \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, $\kappa, k_0 \in \mathbb{R}$. This equation is a simplified purely phenomenological version of a system of nonlinear partial differential equations considered in [6] describing the nonlinear propagation of light in an optical fiber waveguide with an underlying spatially periodic structure. In [6] for the more complicated system, but without quadratic terms, it already has been shown that the NLCME provide good approximations of the solutions of the original system.

It is the purpose of this section to show that this is also true in the case of quadratic terms present in the nonlinearity, when as explained in Section 1, an application of Gronwall's inequality is not sufficient to obtain the required error estimates.

The method we use goes back to [7], where the validity question for the Nonlinear Schrödinger equation has been handled using averaging methods. A similar problem has been solved in [12] with the help of a normal form transform. By this method all quadratic terms can be eliminated such that after the transformation the proof for the cubic nonlinearities becomes applicable. In order to get rid of the quadratic terms the original system has to satisfy some non–resonance condition, see (2.13) below. In the same way we proceed for the validity question of the NLCME.

In the Nonlinear Schrödinger case there are no terms for the interaction of the counter propagating wave packets in lowest order. See [9]. For the NLCME the interaction term for the counter propagating spatially localized wave packets is of leading order and has to be included into the NLCME.

2.1 Derivation of the NLCME

In order to derive the NLCME we let $X = \varepsilon^2 x$, $T = \varepsilon^2 t$ and make the (preliminary) ansatz $u(x,t) = \varepsilon \Psi_1(x,t,\varepsilon)$ with

$$\begin{aligned} \varepsilon \Psi_1(x,t,\varepsilon) = {} & \varepsilon A_1(X,T)\mathbf{E}\mathbf{F} + \varepsilon^2 A_2(X,T)\mathbf{E}^2\mathbf{F}^2 + \tfrac{1}{2}\varepsilon^2 A_0(X,T)\mathbf{E}^0\mathbf{F}^0 \\ & + \varepsilon B_1(X,T)\mathbf{E}^{-1}\mathbf{F} + \varepsilon^2 B_2(X,T)\mathbf{E}^{-2}\mathbf{F}^2 \\ & + \varepsilon^2 E_2(X,T)\mathbf{E}^2 + \varepsilon^2 F_2(X,T)\mathbf{F}^2 + \text{c.c.}, \end{aligned} \tag{2.2}$$

where $\mathbf{E} = e^{\mathrm{i}k_0 x}$ such that $2\cos(2k_0 x) = \mathbf{E}^2 + \mathbf{E}^{-2}$, and $\mathbf{F} = e^{\mathrm{i}\omega_0 t}$ with $\omega_0 = \sqrt{k_0^2+1}$ determined by the linear dispersion relation for (2.1). Inserting (2.2) into (2.1) gives

$$
\begin{aligned}
0 = \; & \varepsilon^2[4(\omega_0^2 - k_0^2)A_2 - A_2 - A_1^2]\mathbf{E}^2\mathbf{F}^2 + \varepsilon^2[4(\omega_0^2 - k_0^2)B_2 - B_2 - B_1^2]\mathbf{E}^{-2}\mathbf{F}^2 \\
& + \varepsilon^2[-A_0 + 2(|A_1|^2 + |B_1|^2)]\mathbf{E}^0\mathbf{F}^0 \\
& + \varepsilon^2[-2A_1B_1 - (1-4\omega_0^2)F_2]\mathbf{F}^2 + \varepsilon^2[-2A_1B_{-1} - (1+4k_0^2)E_2]\mathbf{E}^2 \\
& + \varepsilon^3[-2\mathrm{i}\omega_0\partial_T A_1 + 2\mathrm{i}k_0\partial_X A_1 + \kappa B_1 \\
& \qquad -2A_0A_1 - 2A_2A_{-1} - 2B_1E_2 - 2B_{-1}F_2]\mathbf{E}\mathbf{F} \\
& + \varepsilon^3[-2\mathrm{i}\omega_0\partial_T B_1 - 2\mathrm{i}k_0\partial_X B_1 + \kappa A_1 \\
& \qquad -2A_0B_1 - 2B_2B_{-1} - 2A_1E_{-2} - 2A_{-1}F_2]\mathbf{E}^{-1}\mathbf{F} \\
& + \varepsilon^3[\kappa B_1 - 2A_{-1}B_2 - 2B_1E_{-2}]\mathbf{E}^{-3}\mathbf{F} + \varepsilon^3[\kappa A_1 - 2A_2B_{-1} - 2A_1E_2]\mathbf{E}^3\mathbf{F} \\
& + \varepsilon^3[-2A_1B_2 - 2B_1F_2]\mathbf{E}^{-1}\mathbf{F}^3 + \varepsilon^3[-2A_2B_1 - 2A_1F_2]\mathbf{E}\mathbf{F}^3 \\
& + \varepsilon^3[-2B_1B_2]\mathbf{E}^{-3}\mathbf{F}^3 + \varepsilon^3[-2A_1A_2]\mathbf{E}^3\mathbf{F}^3 + \text{c.c.} + \mathcal{O}(\varepsilon^4),
\end{aligned}
\tag{2.3}
$$

where $A_{-k} = \overline{A_k}$, $B_{-k} = \overline{B_k}$, $E_{-k} = \overline{E_k}$, $F_{-k} = \overline{F_k}$, and where we used

$$
\partial_t^2(B(\varepsilon^2 x, \varepsilon^2 t)\mathbf{E}^p\mathbf{F}^q) = (\varepsilon^4\partial_T^2 B(X,T) + 2i\omega_0 q\varepsilon^2\partial_T B(X,T) - \omega_0^2 q^2 B(X,T))\mathbf{E}^p\mathbf{F}^q
$$

and similarly for ∂_x^2. Equating the coefficients of $\varepsilon^2\mathbf{E}^2\mathbf{F}^2, \varepsilon^2\mathbf{E}^{-2}\mathbf{F}^2$, $\varepsilon^2\mathbf{E}^0\mathbf{F}^0$, $\varepsilon^2\mathbf{F}^2$ and $\varepsilon^2\mathbf{E}^2$ to zero and using $\omega_0^2 - k_0^2 = 1$ gives

$$
\begin{aligned}
& A_2 = \tfrac{1}{3}A_1^2, \quad B_2 = \tfrac{1}{3}B_1^2, \quad A_0 = 2(|A_1|^2 + |B_1|^2), \\
& F_2 = \tfrac{-2}{1-4\omega_0^2}A_1B_1, \quad E_2 = \tfrac{-2}{1+4k_0^2}A_1B_{-1}.
\end{aligned}
\tag{2.4}
$$

Inserting this into the coefficients of $\varepsilon^3\mathbf{E}\mathbf{F}$ and $\varepsilon^3\mathbf{E}^{-1}\mathbf{F}$ we obtain

$$
\begin{aligned}
& -2\mathrm{i}\omega_0\partial_T A_1 + 2\mathrm{i}k_0\partial_x A_1 + \kappa B_1 \\
& \qquad -2(2|A_1|^2 + |B_1|^2)A_1 - \tfrac{2}{3}|A_1|^2A_1 + 4\left(\tfrac{1}{1+4k_0^2} + \tfrac{1}{1-4\omega_0^2}\right)|B_1|^2A_1 = 0, \\
& -2\mathrm{i}\omega_0\partial_T B_1 - 2\mathrm{i}k_0\partial_x B_1 + \kappa A_1 \\
& \qquad -2(2|B_1|^2 + |A_1|^2)B_1 - \tfrac{2}{3}|B_1|^2B_1 + 4\left(\tfrac{1}{1+4k_0^2} + \tfrac{1}{1-4\omega_0^2}\right)|A_1|^2B_1 = 0,
\end{aligned}
\tag{2.5}
$$

i.e, the NLCME with

$$
v_g = k_0/\omega_0, \quad \gamma_1 = \frac{7\mathrm{i}}{3\omega_0} \quad \text{and} \quad \gamma_2 = \frac{\mathrm{i}}{\omega_0}\Big(1 - 2(\frac{1}{1+4k_0^2} + \frac{1}{1-4\omega_0^2})\Big).
$$

2.2 The approximation result

As a number of counterexamples [11, 5] show, it is not obvious that the dynamics of (2.1) can be predicted by the formally derived system (2.5). We show the following result.

Theorem 2.1 *Let $(A_1, B_1) \in C([0, T_0], [H^3(\mathbb{R}, \mathbb{C})]^2)$ be solutions of the NLCME (2.5) for a fixed $T_0 > 0$. Then there exist ε_0, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (2.1) which can be approximated by*

$$\varepsilon\Psi_0(x, t, \varepsilon) = \varepsilon A_1(\varepsilon^2 x, \varepsilon^2 t)\mathbf{E}\mathbf{F} + \varepsilon B_1(\varepsilon^2 x, \varepsilon^2 t)\mathbf{E}^{-1}\mathbf{F} + \text{c.c.}$$

such that

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\cdot, t) - \varepsilon\Psi_0(\cdot, t, \varepsilon)\|_{C_b^0} \leq C\varepsilon^2.$$

Remark. The error of order $\mathcal{O}(\varepsilon^2)$ is small compared with the solution and the approximation which are both of order $\mathcal{O}(\varepsilon)$ for all $0 \leq t \leq T_0/\varepsilon^2$, i.e., the dynamics predicted by the NLCME (2.5) can be seen in (2.1).

Proof. We write the solution u as a sum of an approximation $\varepsilon\Psi$ and an error $\varepsilon^2 R$. Then the error is determined by the equation

$$\partial_t^2 R = \partial_x^2 R - R + 2\varepsilon^2\kappa\cos(2k_0 x)R - 2\varepsilon\Psi R - \varepsilon^2 R^2 + \varepsilon^{-2}\text{Res}(\varepsilon\Psi), \tag{2.6}$$

where the residual $\text{Res}(\varepsilon\Psi)$ is defined by all terms which do not cancel after inserting the ansatz into (2.1), i.e.

$$\text{Res}(u) = -\partial_t^2 u + \partial_x^2 u - u + 2\varepsilon^2\kappa\cos(2k_0 x)u - u^2.$$

It is easy to see that for $f = O(1)$ the inhomogeneous equation

$$\partial_t^2 R = \partial_x^2 R - R + \varepsilon^2 f$$

has $O(1)$-bounded solutions R for all t on a time scale of order $O(1/\varepsilon^2)$. Thus there are two difficulties which have to be overcome in order to prove the $\mathcal{O}(1)$-boundedness of the error R. We have to

a) prove that the residual $\text{Res}(\varepsilon\Psi)$ is of order $O(\varepsilon^4)$,

b) control the influence of the term $2\varepsilon\Psi R$.

We start with the goal in a) which can always be achieved by adding higher order terms to the approximation defined in (2.2). We define the final approximation

$$\begin{aligned}\varepsilon\Psi_2(x, t) = {} & \varepsilon A_1(X, T)\mathbf{E}\mathbf{F} + \varepsilon^2 A_2(X, T)\mathbf{E}^2\mathbf{F}^2 + \frac{1}{2}\varepsilon^2 A_0(X, T)\mathbf{E}^0\mathbf{F}^0 \\ & +\varepsilon B_1(X, T)\mathbf{E}^{-1}\mathbf{F} + \varepsilon^2 B_2(X, T)\mathbf{E}^{-2}\mathbf{F}^2 \\ & +\varepsilon^2 E_2(X, T)\mathbf{E}^2 + \varepsilon^2 F_2(X, T)\mathbf{F}^2 \\ & +\varepsilon^3 H_{-3,1}(X, T)\mathbf{E}^{-3}\mathbf{F} + \varepsilon^3 H_{3,1}(X, T)\mathbf{E}^3\mathbf{F} \\ & +\varepsilon^3 H_{-1,3}(X, T)\mathbf{E}^{-1}\mathbf{F}^3 + \varepsilon^3 H_{1,3}(X, T)\mathbf{E}\mathbf{F}^3 \\ & +\varepsilon^3 H_{-3,3}(X, T)\mathbf{E}^{-3}\mathbf{F}^3 + \varepsilon^3 H_{3,3}(X, T)\mathbf{E}^3\mathbf{F}^3 + \text{c.c.}\end{aligned} \tag{2.7}$$

where A_1, B_1 solve (2.5), where A_0, A_2, B_2 solve (2.4), and where $H_{-3,1}$, $H_{3,1}$, $H_{-1,3}$, $H_{1,3}$, $H_{-3,3}$, $H_{3,3}$ solve

$$
\begin{aligned}
\varepsilon^3\mathbf{E}^{-3}\mathbf{F}: &\ (\omega_0^2-9k_0^2-1)H_{-3,1} + \kappa B_1 - 2A_{-1}B_2 - 2B_1E_{-2} \\
&\qquad = -8k_0^2 H_{-3,1} + \kappa B_1 - 2A_{-1}B_2 - 2B_1E_{-2} = 0, \\
\varepsilon^3\mathbf{E}^{3}\mathbf{F}: &\ (\omega_0^2-9k_0^2-1)H_{3,1} + \kappa A_1 - 2A_2B_{-1} - 2A_1E_2 \\
&\qquad = -8k_0^2 H_{3,1} + \kappa A_1 - 2A_2B_{-1} - 2A_1E_2 = 0, \\
\varepsilon^3\mathbf{E}^{-1}\mathbf{F}^3: &\ (9\omega_0^2-k_0^2-1)H_{-1,3} - 2A_1B_2 - 2B_1F_2 \\
&\qquad = 8\omega_0^2 H_{-1,3} - 2A_1B_2 - 2B_1F_2 = 0, \\
\varepsilon^3\mathbf{E}^{1}\mathbf{F}^3: &\ (9\omega_0^2-k_0^2-1)H_{1,3} - 2A_2B_1 - 2A_1F_2 \\
&\qquad = 8\omega_0^2 H_{1,3} - 2A_2B_1 - 2A_1F_2 = 0, \\
\varepsilon^3\mathbf{E}^{-3}\mathbf{F}: &\ (9\omega_0^2 - 9k_0^2 - 1)H_{-3,3} - 2B_1B_2 = 8H_{-3,3} - 2B_1B_2 = 0, \\
\varepsilon^3\mathbf{E}^{3}\mathbf{F}^3: &\ (9\omega_0^2 - 9k_0^2 - 1)H_{3,3} - 2A_1A_2 = 8H_{3,3} - 2A_1A_2 = 0,
\end{aligned} \tag{2.8}
$$

with $A_{-j} = \overline{A}_j, B_{-j} = \overline{B}_j$. Thus, by adding the terms $H_{i,j}$ to the original ansatz all terms of order $O(\varepsilon^3)$ in the original residual can be eliminated, and so the final residual will be of order $O(\varepsilon^4)$.

To formulate and prove this rigorously we have to choose some function spaces. We consider (2.1) in Fourier-space and look for solutions in $L^1(\mathbb{R}, \mathbb{C})$. In the following we usually simply write $L^1(\mathbb{R})$ for both real and complex valued functions, and similar for the other spaces appearing below. The choice of $L^1(\mathbb{R})$ is mainly motivated by the invariance of $L^1(\mathbb{R})$ with respect to scaling in the following sense; defining the scaling operator S_ε by $(S_\varepsilon u)(x) = u(\varepsilon x)$, for the Fourier transform

$$\hat{u}(k) = (\mathcal{F}u)(k) = \frac{1}{2\pi}\int u(x)e^{-ikx}\mathrm{d}x$$

of a scaled function we obtain

$$\mathcal{F}(S_\varepsilon u) = \frac{1}{\varepsilon}S_{\frac{1}{\varepsilon}}(\mathcal{F}u) \quad \text{and} \quad \|\hat{u}\|_{L^1} = \|\frac{1}{\varepsilon}S_{\frac{1}{\varepsilon}}\hat{u}\|_{L^1}.$$

Estimates for $\hat{u}$ in $L^1(\mathbb{R})$ transfer easily into physical space, since $\mathcal{F}^{-1}$ is continuous from $L^1(\mathbb{R})$ into $C_b^0(\mathbb{R})$ equipped with the sup-norm, i.e., $\|u\|_{C_b^0} \leq \|\hat{u}\|_{L^1}$, but not vice versa. We remark that in classical Sobolev spaces $H^m(\mathbb{R})$ we would loose too many powers in ε since $\|S_{\varepsilon^2}A\|_{H^m} \leq C\varepsilon^{-1}\|A\|_{H^m}$.

Before we go on with the proof we recall some basic facts. Let

$$\|\hat{u}\|_{L^p(m)}^p = \int |\hat{u}(k)|^p(1+|k|^2)^{pm}\mathrm{d}k, \quad L^p(m) := \{\hat{u} \in L^p(\mathbb{R}) : \|\hat{u}\|_{L^p(m)} < \infty\}.$$

Fourier transform is an isomorphism between $H^m(\mathbb{R})$ and $L^2(m)$, i.e., there exist constants $C_1, C_2 > 0$ such that $\|u\|_{H^m} \leq C_1\|\hat{u}\|_{L^2(m)} \leq C_2\|u\|_{H^m}$. Sobolev's inequality is given by $\|\hat{u}\|_{L^1(m)} \leq C\|\hat{u}\|_{L^2(m+s)}$ for $s > \frac{1}{2}$ and a constant $C = C(s)$. Multiplication in physical space corresponds in Fourier space to convolution

$$(\hat{u} * \hat{v})(k) = \int \hat{u}(k-\ell)\hat{v}(\ell)\, d\ell \quad \text{with} \quad \|\hat{u} * \hat{v}\|_{L^1} \leq \|\hat{v}\|_{L^1}\|\hat{u}\|_{L^1}.$$

With these estimates and the formal computation from above we may estimate the residual as follows.

Lemma 2.2 *Let $A_1, B_1 \in C([0,T_0],[H^3(\mathbb{R},\mathbb{C})]^2)$ be solutions of (1.1). Define A_0, A_2, B_2 by (2.4) and $H_{i,j}$ by (2.8). Then there exist positive constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the approximation Ψ defined in (2.7) satisfies*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathcal{F}\varepsilon\Psi_2(\cdot,t)\|_{L^1} < C\varepsilon \quad \text{and} \quad \sup_{t\in[0,T_0/\varepsilon^2]} \|\mathcal{F}(\mathrm{Res}(\varepsilon\Psi_2(\cdot,t)))\|_{L^1} < C\varepsilon^4.$$

Thus we have shown a) of the above program. In order to establish b) we consider (2.1) in Fourier space, i.e.

$$\partial_t^2\hat{u}(k) = (-k^2-1)\hat{u}(k) + \kappa[\varepsilon^2\hat{u}(k+2) + \varepsilon^2\hat{u}(k-2)] - \hat{u} * \hat{u}(k). \tag{2.9}$$

To eliminate the term $-2\varepsilon\Psi R$ in (2.6) we use a normal form transform. Due to the special structure of (2.1) it is possible to eliminate the quadratic terms in (2.1) completely. This has been observed in [14] and used since this time for different purposes. However, for the validity proof there is no need for the complete elimination of all quadratic terms, see [12].

We formulate (2.9) as a first order system

$$\partial_t\underline{\hat{u}} = L\underline{\hat{u}} + \tilde{N}(\underline{\hat{u}}), \tag{2.10}$$

$$\underline{\hat{u}} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} = \begin{pmatrix} \hat{u} \\ \frac{1}{\sqrt{k^2+1}}\partial_t\hat{u} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \sqrt{k^2+1} \\ -\sqrt{k^2+1} & 0 \end{pmatrix},$$

$$\tilde{N}(\underline{\hat{u}}) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{k^2+1}}(\kappa(\varepsilon^2\hat{u}_1(k-2)+\varepsilon^2\hat{u}_1(k+2)) - (\hat{u}_1 * \hat{u}_1)(k)) \end{pmatrix}.$$

Diagonalizing (2.10) by $\underline{\hat{u}} = S\hat{v}$ with $S = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$ we obtain

$$\partial_t\hat{v} = M\hat{v} + N(\hat{v}), \tag{2.11}$$

with

$$M = S^{-1}LS = \begin{pmatrix} \mathrm{i}\sqrt{k^2+1} & 0 \\ 0 & -\mathrm{i}\sqrt{k^2+1} \end{pmatrix}, \quad N(\hat{v}) = S^{-1}\tilde{N}(S\hat{v}).$$

We write

$$N(\hat{v}) = \varepsilon^2 N_1(\hat{v}) + N_2(\hat{v}, \hat{v})$$

with N_1 linear in $\hat{v}$ and N_2 containing the bilinear terms of N and make a near identity change of variables

$$\hat{v} = \hat{w} + B(\hat{w}, \hat{w}) = T(\hat{w}) \tag{2.12}$$

with B an autonomous bilinear mapping. This gives

$$\begin{aligned}&\partial_t \hat{w} + B(\partial_t \hat{w}, \hat{w}) + B(\hat{w}, \partial_t \hat{w}) \\ &= M\hat{w} + MB(\hat{w}, \hat{w}) + \varepsilon^2 N_1(\hat{w} + B(\hat{w}, \hat{w})) + N_2(\hat{w}, \hat{w}) + \mathcal{O}(\|\hat{w}\|^3)\end{aligned}$$

and so

$$\begin{aligned}\partial_t \hat{w} = &\; M\hat{w} + MB(\hat{w}, \hat{w}) - B(M\hat{w}, \hat{w}) - B(\hat{w}, M\hat{w}) + N_2(\hat{w}, \hat{w}) \\ &+ \varepsilon^2 N_1(\hat{w} + B(\hat{w}, \hat{w})) + \mathcal{O}(\|\hat{w}\|^3).\end{aligned}$$

In order to eliminate the quadratic terms $N_2(\hat{w}, \hat{w})$ we have to find a B such that

$$MB(\hat{w}, \hat{w}) - B(M\hat{w}, \hat{w}) - B(\hat{w}, M\hat{w}) + N_2(\hat{w}, \hat{w}) = 0.$$

With

$$\begin{aligned}(N_2(\hat{w}, \hat{w}))_h &= \sum_{i,j=1}^{2} \int n_{hpq}(k, k-\ell, \ell) \hat{w}_p(k-\ell) \hat{w}_q(\ell) \, \mathrm{d}\ell, \\ (B(\hat{w}, \hat{w}))_h &= \sum_{i,j=1}^{2} \int b_{hpq}(k, k-\ell, \ell) \hat{w}_p(k-\ell) \hat{w}_q(\ell) \, \mathrm{d}\ell\end{aligned}$$

for the h-th component of N_2 and B, $\lambda_1(k) = -\lambda_2(k) = i\sqrt{k^2+1}$ and n_{hpq}, b_{hpq} some coefficients we obtain the well known relation

$$(\lambda_h(k) - \lambda_p(k-\ell) - \lambda_q(\ell)) b_{hpq}(k, k-\ell, \ell) = n_{hpq}(k, k-\ell, \ell).$$

This can be resolved with respect to b_{hpq} due to the non resonance property

$$\inf_{\substack{h,p,q \in \{1,2\} \\ k,\ell \in \mathbb{R}}} |\lambda_h(k) - \lambda_p(k-\ell) - \lambda_q(\ell)| \geq 1 \tag{2.13}$$

of the dispersion relation of (2.1). Since $\sup_{k,\ell \in \mathbb{R}, p,q=1,2} |n_{hpq}(k, k-\ell, \ell)| < C < \infty$ we obtain

$$\begin{aligned}\|(B(\hat{w}, \hat{w}))_h\|_{L^1} &= \int\int \sum_{p,q=1}^{2} |b_{hpq}(k, k-\ell, \ell) \hat{w}_p(k-\ell) \hat{w}_q(\ell)| \, \mathrm{d}\ell \, \mathrm{d}k \\ &\leq \sup_{k,\ell \in \mathbb{R}, p,q=1,2} |n_{hpq}(k, k-\ell, \ell)| \int\int \sum_{p,q=1}^{2} |\hat{w}_p(k-\ell) \hat{w}_q(\ell)| \, \mathrm{d}\ell \, \mathrm{d}k \\ &\leq C \|\hat{w}\|_{L^1} \|\hat{w}\|_{L^1}.\end{aligned}$$

Thus the transformation (2.12) can be resolved with respect to $\hat{w}$ for $\|\hat{w}\|_{L^1}$ sufficiently small. Therefore (2.11) transforms into

$$\partial_t \hat{w} = M\hat{w} + N_3(\hat{w}) \tag{2.14}$$

with a nonlinearity N_3 which satisfies

$$\|N_3(\hat{w})\|_{L^1} \leq C\varepsilon^2 \|\hat{w}\|_{L^1} + C\|\hat{w}\|_{L^1}^3$$

for a constant C if $\|\hat{w}\|_{L^1} < \delta$ for a fixed $\delta > 0$ sufficiently small independent of $\varepsilon > 0$.

We write a solution $\hat{w}$ of (2.14) as a sum of an approximation $\varepsilon\hat{\Psi}$ and an error $\varepsilon^2 \hat{R}$. The approximation $\varepsilon\hat{\Psi}$ is defined by

$$\varepsilon\hat{\Psi} = T^{-1}S^{-1} \begin{pmatrix} \mathcal{F}\Psi_2 \\ \mathcal{F}\Psi_2/\sqrt{k^2+1} \end{pmatrix}$$

and satisfies the assertions of Lemma 2.2. Therefore the error R satisfies a differential equation of the form

$$\partial_t R = MR + h(\varepsilon\hat{\Psi}, R)$$

with

$$\|h(\varepsilon\hat{\Psi}, R)\|_{L^1} \leq C_1 \varepsilon^2 \|R\|_{L^1} + C_2 \varepsilon^3 \|R\|_{L^1}^2 + C_{\text{Res}}\varepsilon^2,$$

where the constants C_1 and C_{Res} (coming from the residual) are independent of $\varepsilon \in (0,1)$ and $\|R\|_{L^1}$ and where C_2 is independent of $\varepsilon \in (0,1)$ but depends on $C_{\max} = \sup_{t\in[0,T_0/\varepsilon^2]} \|R(t)\|_{L^1}$. Choosing $\varepsilon > 0$ so small that $C_2(C_{\max})\varepsilon < 1$ we obtain

$$\|R(t)\|_{L^1} \leq C_{\text{Res}} e^{(C_1+1)T_0} =: C_{\max}$$

with the help Gronwall's inequality. Doing back all transformations and using $\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\Psi_2 - \varepsilon\Psi_0\| = \mathcal{O}(\varepsilon^2)$ shows the assertion of Theorem 2.1. □

3 The Swift-Hohenberg model

In this section we consider the parabolic case. The easiest model problem with a quadratic nonlinearity is a system of coupled Swift-Hohenberg type equations

$$\partial_t U = \Lambda U + c \begin{pmatrix} \partial_x u \\ -\partial_x v \end{pmatrix} + \varepsilon^2 \alpha_0 U + 2\varepsilon^2 \kappa \sin(2x) \begin{pmatrix} u - v \\ v - u \end{pmatrix} + N(U, \partial_x U), \tag{3.1}$$

$$\text{where} \quad U = U(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} \in \mathbb{R}^2, \quad \Lambda U = \begin{pmatrix} -(1+\partial_x^2)^2 u \\ -(1+\partial_x^2)^2 v \end{pmatrix},$$

$c > 0$, $\alpha_0, \kappa \in \mathbb{R}$, and where N is some quadratic nonlinearity with $(N_1, N_2) \mapsto (N_2, N_1)$ as $x \mapsto -x$, such that $(u, v) \mapsto (v, u)$ as $x \mapsto -x$ in (3.1). This system is a phenomenological model of a reflection symmetric, pattern forming system in a spatially periodic domain undergoing a Hopf-bifurcation at a non-zero wavenumber $k_0 = 1$, where $0 < \varepsilon \ll 1$ is the small bifurcation parameter.

Setting

$$\mathrm{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathrm{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the constant coefficient linear part L_1 of (3.1) with

$$L_1 U = \Lambda U + \varepsilon^2 \alpha_0 U + c \begin{pmatrix} \partial_x u \\ -\partial_x v \end{pmatrix},$$

fulfills $L_1 e^{\mathrm{i}kx} \mathrm{e}_j = \lambda_j(k) \mathrm{e}_j$, where

$$\lambda_1(k) = -(1-k^2)^2 + \mathrm{i}ck + \alpha_0 \varepsilon^2, \quad \lambda_2(k) = -(1-k^2)^2 - \mathrm{i}c_k + \alpha_0 \varepsilon^2.$$

Therefore, the linear semigroup e^{tL_1} damps all modes $e^{\mathrm{i}kx}\hat{U}(k)$ except of the so called critical modes with $|k|$ in a small neighborhood of the critical wavenumber k_0. The basic idea used for proving the approximation result for the NLCME below is to separate the critical modes from the stable modes with $||k| - 1| \geq 1/4$, and to control the quadratic mode interaction. This idea goes back to [10], where the validity question for the Ginzburg-Landau equation for quadratic nonlinearities has been handled. In order so see interesting effects of the spatially periodic coefficient $2\varepsilon^2 \kappa \sin(k_p x)$, its wavenumber k_p has to be in resonance with the critical wavenumber $k_0 = 1$, i.e. $k_p = nk_0$ for some $n \in \mathbb{N}$. See, e.g., [3, 15]. Choosing $n = 2$ is the simplest possibility to obtain the NLCME.

A particular simple choice for $N(U, \partial_x U)$ that leads to NLCME in the regime of gap solitons is

$$N(U, \partial_x U) = f_1(u^2 + v^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + f_2(u\partial_x u + v\partial_x v) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.2}$$

with suitable coefficients $f_1, f_2 \in \mathbb{R}$. For notational simplicity we will use this non-linearity throughout.

Remark. For systems like (3.1) the long time dynamics of the bifurcating solutions can be described by a system of singularly coupled Ginzburg-Landau equations in which space scales as $X = \varepsilon x$, see, e.g., [13]. If we consider the special class of very long wave modulations $X = \varepsilon^2 x$ of the most unstable pattern we arrive at the NLCME which describe some interesting transient dynamics in the original system before the Ginzburg–Landau equations take their role in the description of the solutions. This is sketched in Figure 1.

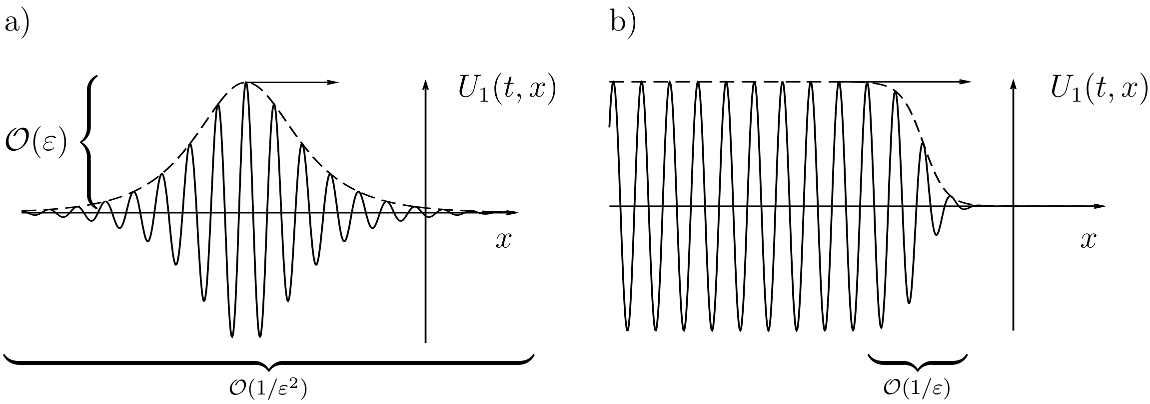

Figure 1: Sketches of the dynamics of (3.1). a) transient dynamics, envelope described by NLCME; b) example of asymptotic dynamics described by coupled Ginzburg–Landau equations.

3.1 Derivation of the NLCME

We let again $\mathbf{E} = e^{ix}$, $\mathbf{F} = e^{i\omega_0 t}$, $X = \varepsilon^2 x$, $T = \varepsilon^2 t$. As before we start with a preliminary approximation

$$\begin{aligned}\varepsilon\Psi_1(x,t) = {} & [\varepsilon A_1(X,T)\mathbf{E}\mathbf{F} + \varepsilon^2 A_2(X,T)\mathbf{E}^2\mathbf{F}^2 + \frac{\varepsilon^2}{2}A_0(X,T)]\mathrm{e}_1 \\ & +[\varepsilon B_1(X,T)\mathbf{E}^{-1}\mathbf{F} + \varepsilon^2 B_2(X,T)\mathbf{E}^{-2}\mathbf{F}^2 + \frac{\varepsilon^2}{2}B_0(X,T)]\mathrm{e}_2 \\ & +\varepsilon^2 C_{-2,2}\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_1 + \varepsilon^2 D_{2,2}\mathbf{E}^2\mathbf{F}^2\mathrm{e}_2 + \mathrm{c.c.}.\end{aligned} \tag{3.3}$$

Inserting (3.3) into (3.1) gives

$$\begin{aligned}0 = {} & \frac{\varepsilon^2}{2}[-A_0+2f_1(|A|^2+|B|^2)]\mathbf{E}^0\mathbf{F}^0\mathrm{e}_1+\frac{\varepsilon^2}{2}[-B_0+2f_1(|A|^2+|B|^2)]\mathbf{E}^0\mathbf{F}^0\mathrm{e}_2 \\ & +\varepsilon^2[-9A_2+(f_1+\mathrm{i}f_2)A_1^2]\mathbf{E}^2\mathbf{F}^2\mathrm{e}_1+\varepsilon^2[-9B_2+(f_1+\mathrm{i}f_2)B_1^2]\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_2 \\ & +\varepsilon^2[(-9-2\mathrm{i}c)C_{-2,2}+(f_1-\mathrm{i}f_2)B^2]\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_1 \\ & +\varepsilon^2[(-9+2\mathrm{i}c)D_{2,2}+(f_1-\mathrm{i}f_2)A^2]\mathbf{E}^2\mathbf{F}^2\mathrm{e}_2 \\ & +\varepsilon^3[-\partial_T A_1+c\partial_X A_1+\alpha_0 A_1+\mathrm{i}\kappa B_1+(2f_1+\mathrm{i}f_2)(A_0A_1+A_2A_{-1})]\mathbf{E}\mathbf{F}\mathrm{e}_1\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&+\varepsilon^3[-\partial_T B_1 - c\partial_X B_1 + \alpha_0 B_1 + \mathrm{i}\kappa A_1 + (2f_1+\mathrm{i}f_2)(B_0B_1+B_2B_{-1})]\mathbf{E}^{-1}\mathbf{F}\mathrm{e}_2\\
&+\varepsilon^3[\mathrm{i}\kappa A_{-1} + (2f_1-\mathrm{i}f_2)(B_0B_1+B_2B_{-1})]\mathbf{E}^{-1}\mathbf{F}\mathrm{e}_1\\
&+\varepsilon^3[\mathrm{i}\kappa B_1 + (2f_1-\mathrm{i}f_2)(A_0A_1+A_{-2}A_1)]\mathbf{E}\mathbf{F}\mathrm{e}_2\\
&+\varepsilon^3[(2f_1-\mathrm{i}f_2)C_{-2,2}A_1]\mathbf{E}^{-1}\mathbf{F}^3\mathrm{e}_1 + \varepsilon^3[(2f_1+\mathrm{i}f_2)D_{2,2}B_1]\mathbf{E}\mathbf{F}^3\mathrm{e}_1\\
&+\varepsilon^3[(2f_1+\mathrm{i}f_2)C_{-2,2}A_1]\mathbf{E}^{-1}\mathbf{F}^3\mathrm{e}_2 + \varepsilon^3[(2f_1-\mathrm{i}f_2)D_{2,2}B_1]\mathbf{E}\mathbf{F}^3\mathrm{e}_2\\
&+\varepsilon^3[(2f_1-2\mathrm{i}f_2)C_{-2,2}A_0]\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_1 + \varepsilon^3[(2f_1+2\mathrm{i}f_2)D_{2,2}B_0]\mathbf{E}^2\mathbf{F}^2\mathrm{e}_1\\
&+\varepsilon^3[(2f_1+2\mathrm{i}f_2)C_{-2,2}A_0]\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_2 + \varepsilon^3[(2f_1-2\mathrm{i}f_2)D_{2,2}B_0]\mathbf{E}^2\mathbf{F}^2\mathrm{e}_2\\
&+\varepsilon^3[(2f_1+3\mathrm{i}f_2)A_2A_1]\mathbf{E}^3\mathbf{F}^3\mathrm{e}_1 + \varepsilon^3[(2f_1-3\mathrm{i}f_2)B_2B_1]\mathbf{E}^{-3}\mathbf{F}^3\mathrm{e}_1\\
&+\varepsilon^3[(2f_1-3\mathrm{i}f_2)A_1A_2]\mathbf{E}^3\mathbf{F}^3\mathrm{e}_2 + \varepsilon^3[(2f_1+3\mathrm{i}f_2)B_2B_1]\mathbf{E}^{-3}\mathbf{F}^3\mathrm{e}_2\\
&+\varepsilon^3[(2f_1+3\mathrm{i}f_2)D_{2,2}B_{-1} - \mathrm{i}\kappa A_1]\mathbf{E}^3\mathbf{F}\mathrm{e}_1 + \varepsilon^3[(2f_1-3\mathrm{i}f_2)C_{-2,2}A_{-1} - \mathrm{i}\kappa B_1]\mathbf{E}^{-3}\mathbf{F}\mathrm{e}_1\\
&+\varepsilon^3[(2f_1-3\mathrm{i}f_2)D_{2,2}B_{-1} - \mathrm{i}\kappa A_1]\mathbf{E}^3\mathbf{F}\mathrm{e}_2 + \varepsilon^3[(2f_1+3\mathrm{i}f_2)C_{-2,2}A_{-1} - \mathrm{i}\kappa B_1]\mathbf{E}^{-3}\mathbf{F}\mathrm{e}_2\\
&+\text{c.c.} + \mathcal{O}(\varepsilon^4),
\end{aligned}
$$

where $A_{-j} = \overline{A_j}$, $B_{-j} = \overline{B_j}$. In the seventh line the (preliminary) residual starts, see below. Solving for A_0 at $\varepsilon^2\mathrm{e}_1$, for A_2 at $\varepsilon^2\mathbf{E}^2\mathbf{F}^2\mathrm{e}_1$, for B_0 at $\varepsilon^2\mathrm{e}_2$, for B_2 at $\varepsilon^2\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_2$, for $C_{-2,2}$ at $\varepsilon^2\mathbf{E}^{-2}\mathbf{F}^2$ and for $D_{2,2}$ at $\varepsilon^2\mathbf{E}^2\mathbf{F}^2\mathrm{e}_2$, i.e.

$$
A_0 = 2f_1(|A_1|^2+|B_1|^2), \quad B_0 = 2f_1(|A_1|^2+|B_1|^2), \quad A_2 = \tfrac{1}{9}(f_1+\mathrm{i}f_2)A_1^2 \quad \text{etc.,} \tag{3.5}
$$

the order $\mathcal{O}(\varepsilon^2)$ terms vanish. Inserting the results into the equations at $\varepsilon^3\mathbf{EF}$ and $\varepsilon^3\mathbf{E}^{-1}\mathbf{F}$ we obtain the NLCME for A_1, B_1, i.e.,

$$
\begin{aligned}
\partial_T A_1 &= c\partial_X A_1 + \alpha_0 A_1 + \mathrm{i}\kappa B_1 + (\gamma_1|A_1|^2 + \gamma_2|B_1|^2)A_1,\\
\partial_T B_1 &= -c\partial_X B_1 + \alpha_0 B_1 + \mathrm{i}\kappa A_1 + (\gamma_1|B_1|^2 + \gamma_2|A_1|^2)B_1,
\end{aligned} \tag{3.6}
$$

with

$$
\gamma_1 = (2f_1+\mathrm{i}f_2)(2f_1 + \frac{1}{9}(f_1+\mathrm{i}f_2)), \quad \gamma_2 = 2f_1(2f_1+\mathrm{i}f_2).
$$

Remark. Note that for the derivation of the NLCME (3.6) the terms $\varepsilon^2 C_{-2,2}\mathbf{E}^{-2}\mathbf{F}^2\mathrm{e}_1$ and $\varepsilon^2 D_{2,2}\mathbf{E}^2\mathbf{F}^2\mathrm{e}_2$ are not needed. However, they are needed to produce a formally consistent approximation, i.e., to remove the $\mathcal{O}(\varepsilon^2)$ terms from the residual in (3.4). In general one would also need

$$
\varepsilon^2 C_{2,0}\mathbf{E}^2\mathrm{e}_1, \quad \varepsilon^2 C_{0,2}\mathbf{F}^2\mathrm{e}_1, \quad \varepsilon^2 D_{-2,0}\mathbf{E}^{-2}\mathrm{e}_2, \quad \varepsilon^2 D_{0,2}\mathbf{F}^2\mathrm{e}_2 \tag{3.7}
$$

in (3.3) to balance terms at $\varepsilon^2\mathbf{E}^2\mathrm{e}_1, \varepsilon^2\mathbf{F}^2\mathrm{e}_1, \varepsilon^2\mathbf{E}^{-2}\mathrm{e}_2, \varepsilon^2\mathbf{F}^2\mathrm{e}_2$ in (3.4) generated by the quadratic nonlinearity. However, due to our special nonlinearity, i.e., since there is no uv in (3.2), these terms are not generated. Thus we may omit the terms from (3.7) in (3.3).

3.2 The approximation result

In contrast to the hyperbolic equation (2.1) system (3.1) is dissipative and allows for bigger spaces concerning the local existence and uniqueness of solutions. Therefore, we consider (3.1) in uniformly local Sobolev spaces $H^m_{\rm ul}(\mathbb{R})$, where again in the notation we usually do not distinguish between real or complex or vector valued functions. The Banach spaces $H^m_{\rm ul}(\mathbb{R})$ contain all kinds of bounded functions and are defined as follows: fix the weight function $\rho(x) = 1/\cosh(x)$ and let

$$\|u\|^2_{L^2_{\rm ul}} = \sup_{y\in\mathbb{R}} \int |u(x)|^2 \rho(x+y) \mathrm{d}x, \quad \tilde{L}^2_{\rm ul}(\mathbb{R}) = \{u \in L^2_{\rm loc}(\mathbb{R}) : \|u\|_{L^2_{\rm ul}} < \infty\},$$

$$L^2_{\rm ul}(\mathbb{R}) = \{u \in \tilde{L}^2_{\rm ul}(\mathbb{R}) : \|T_y u - u\|_{L^2_{\rm ul}} \to 0 \quad \text{as} \quad y \to 0\},$$

where $(T_y u)(x) = u(x-y)$. Then

$$H^m_{\rm ul}(\mathbb{R}) := \{u \in L^2_{\rm ul} : \partial^j_x u \in L^2_{\rm ul} \quad \text{for} \quad 0 \leq j \leq k\}. \tag{3.8}$$

Since the spaces $H^m_{\rm ul}(\mathbb{R})$ are based on $L^2(\mathbb{R})$ the global existence of solutions for typical dissipative systems can be shown via Fourier transform methods and weighted energy estimates, see [10, 8]. Moreover we have the estimates

$$\|S_\delta A\|_{L^2_{lu}} \leq \delta^{-1/2} \|A\|_{L^2_{\rm ul}} \quad \text{and} \quad \|S_\delta A\|_{H^1_{\rm ul}} \leq C \|A\|_{H^1_{\rm ul}}, \tag{3.9}$$

where as before $(S_\delta A)(x) = A(\delta x)$. In the first estimate the factor $\delta^{-1/2}$ is due to scaling, and the second estimates holds due to $\|S_\delta A\|_{L^2_{ul}} \leq C\|S_\delta A\|_{L^\infty} \leq C\|A\|_{L^\infty} \leq C\|A\|_{H^1_{\rm ul}}$ and the scaling properties of the derivative. With these preparations our approximation result reads as follows.

Theorem 3.1 *Let $(A_1, B_1) \in C([0, T_0], [H^5_{\rm ul}(\mathbb{R}, \mathbb{C})]^2)$ be solutions of the NLCME (2.5) for a fixed $T_0 > 0$. Then there exist ε_0, $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (2.1) which can be approximated by*

$$\varepsilon\Psi_0(x, t, \varepsilon) = \varepsilon A_1(\varepsilon^2 x, \varepsilon^2 t)\mathbf{E}\mathbf{F}\mathrm{e}_1 + \varepsilon B_1(\varepsilon^2 x, \varepsilon^2 t)\mathbf{E}^{-1}\mathbf{F}\mathrm{e}_2 + \text{c.c.}$$

such that

$$\sup_{[t\in 0, T_0/\varepsilon^2]} \|U(\cdot, t) - \varepsilon\Psi_0(\cdot, t, \varepsilon)\|_{H^1_{\rm ul}} \leq C\varepsilon^2.$$

Proof. As in section 2 we start with a brief formal discussion. We set

$$L_2 U = 2\kappa \sin(2x) \begin{pmatrix} u - v \\ v - u \end{pmatrix},$$

and write the nonlinearity as a symmetric bilinear form $N_1(U,U)$, i.e.,

$$N_1(U,V) = \frac{1}{2}\left[N(U,\partial_x V) + N(V,\partial_x U)\right].$$

Letting $U = \varepsilon\Psi + \varepsilon^2 R$, where Ψ will be close to Ψ_1, the error R fulfills

$$\partial_t R = L_1 R + \varepsilon^2 L_2 R + 2\varepsilon N_1(\Psi, R) + \varepsilon^2 N_1(R,R) + \varepsilon^{-2}\mathrm{Res}(\varepsilon\Psi), \tag{3.10}$$

where

$$\mathrm{Res}(U) = -\partial_t U + L_1 U + \varepsilon^2 L_2 U + N_1(U,U). \tag{3.11}$$

From the properties of the semigroup generated by L_1, see below, it follows that for $\|f\|_{L^2_{\mathrm{ul}}} = \mathcal{O}(1)$ the inhomogenous equation $\partial_t R = L_1 R + \varepsilon^2 L_2 R + \varepsilon^2 f$ has $O(1)$-bounded solutions R in $H^1_{\mathrm{ul}}(\mathbb{R})$ for all t on a time scale of order $O(1/\varepsilon^2)$. Therefore, roughly speaking, we again have to

a) show that $\|\mathrm{Res}(\varepsilon\Psi)\|_{L^2_{\mathrm{ul}}}$ is sufficiently small,

b) control the influence of the term $2\varepsilon N_1(\Psi, R)$.

In order to achieve a) and b) we use the method from [10] and introduce so called mode-filters to separate the critical from the stable modes.

However, first we modify our approximation Ψ_1 in such a way that in the residual of (3.4) all coefficients of $\varepsilon^3\mathbf{E}^{\pm1}\mathbf{F}^j$, $j = 1,3$ vanish, since these terms are critical modes. The remaining $\mathcal{O}(\varepsilon^3)$ terms in the residual belong to damped modes and will be controlled using the mode filters. By a simple calculation we find that the coefficients of $\varepsilon^3\mathbf{E}^{\pm1}\mathbf{F}^j$, $j = 1,3$ vanish if we modify Ψ_1 to

$$\begin{aligned}\varepsilon\Psi_2(x,t) = \varepsilon\Psi_1(x,t) &+\varepsilon^3[C_{-1,1}\mathbf{E}^{-1}\mathbf{F} + C_{-1,3}\mathbf{E}^{-1}\mathbf{F}^3 + C_{1,3}\mathbf{F}\mathbf{E}^3]\mathrm{e}_1 \\ &+\varepsilon^3[D_{1,1}\mathbf{E}\mathbf{F} + D_{-1,3}\mathbf{E}^{-1}\mathbf{F}^3 + D_{1,3}\mathbf{F}\mathbf{E}^3]\mathrm{e}_2\end{aligned} \tag{3.12}$$

where $C_{i,j}, D_{i,j}$ satisfy

$$\begin{aligned}&-2\mathrm{i}cC_{-1,1} + \mathrm{i}\kappa A_{-1} + (2f_1 - \mathrm{i}f_2)(B_0B_1 + B_2B_{-1}) = 0,\\ &-2\mathrm{i}cD_{1,1} + \mathrm{i}\kappa B_1 + (2f_1 - \mathrm{i}f_2)(A_0A_1 + A_{-2}A_1) = 0,\\ &-4\mathrm{i}cC_{-1,3} + (2f_1 - \mathrm{i}f_2)C_{-2,2}A_1 = 0, \quad -2\mathrm{i}cC_{1,3} + (2f_1 + \mathrm{i}f_2)D_{2,2}B_1 = 0,\\ &-2\mathrm{i}cD_{-1,3} + (2f_1 - \mathrm{i}f_2)C_{-2,2}A_1 = 0, \quad -4\mathrm{i}cD_{1,3} + (2f_1 + \mathrm{i}f_2)C_{-2,2}A_1 = 0.\end{aligned} \tag{3.13}$$

Note that adding the $\mathcal{O}(\varepsilon^3)$ terms to Ψ_1 does not change the remaining $\mathcal{O}(\varepsilon^3)$ terms in (3.4).

For the mode filters we use multipliers: given a function $\hat{M} \in L^\infty(\mathbb{R}, \mathbb{C})$, we define a multiplier $M : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R})$ by multiplying $\hat{u}$ with $\hat{M}$, i.e.,

$$Mu = \mathcal{F}^{-1}(\hat{M}\hat{u}).$$

In typical examples $\hat{M}$ is a characteristic function, or the symbol of a differential operator, see below. The following lemma shows that such multipliers extend to bounded operators in uniformly local Sobolev spaces.

Lemma 3.2 *[10, Lemma 5] Let $m \in \mathbb{Z}$, $(1+|\cdot|^2)^{m/2}\hat{M}(\,\cdot\,) \in C^2(\mathbb{R}, \mathbb{C})$ and $q \in \mathbb{N}$. Then M extends to a bounded operator (denoted by the same symbol) $M : H^q_{\mathrm{ul}}(\mathbb{R}) \to H^{q+m}_{\mathrm{ul}}(\mathbb{R})$ with*

$$\|Mu\|_{H^{q+m}_{\mathrm{ul}}} \le C(q,m)\|(1+|\cdot|)^{m/2}\hat{M}(\,\cdot\,)\|_{C^2_b}\|u\|_{H^q_{\mathrm{ul}}}$$

with $C(q,m)$ independent of $\hat{M}$.

Now the mode filters are defined as follows. Let $\chi_0 : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with

$$\chi_0(k) = \begin{cases} 1, & |k| \le 1/8 \\ \in [0,1], & 1/8 \le |k| \le 1/4 \\ 0, & 1/4 \le |k| \end{cases}$$

and let E_0 be the multiplier formally defined by χ_0. Similarly, define E_c by

$$\chi_c(k) = \chi_0(1+k) + \chi_0(-1+k),$$

and let $E_s = \mathrm{Id} - E_c$. We call E_c, E_s mode filters since they separate critical and stable modes. Since they are not projections it will be helpful to define auxiliary mode filters E^h_c and E^h_s by

$$\chi^h_c = \chi_0((1+k)/2) + \chi_0((-1+k)/2), \quad \chi^h_s = 1 - \chi_0(2(1+k)) - \chi_0(2(-1+k)).$$

Note that $E^h_c E_c = E_c$ and $E^h_s E_s = E_s$. Finally, for $U = (u,v)^T$ let

$$\mathcal{E}_c U = \begin{pmatrix} E_c u \\ E_c v \end{pmatrix}, \quad \mathcal{E}_s = \mathrm{Id} - E_c, \quad \mathcal{E}^h_c U = \begin{pmatrix} E^h_c u \\ E^h_c v \end{pmatrix}, \quad \mathcal{E}^h_s = \begin{pmatrix} E^h_s u \\ E^h_s v \end{pmatrix}.$$

We define our final approximation Ψ by applying E_0 to all $A_j, B_j, C_{i,j}, D_{i,j}$ in Ψ_2, i.e.,

$$\varepsilon\Psi(x,t) = \varepsilon[(E_0 S_{\varepsilon^2} A_1)(T)\mathbf{E}\mathbf{F} + \varepsilon^2(E_0 S_{\varepsilon^2} A_2)(T)\mathbf{E}^2\mathbf{F}^2 + \ldots]\mathrm{e}_1 + \ldots \tag{3.14}$$

and assume that (A_1, B_1) fulfill the NLCME (3.6) and $A_0, A_2, B_0, B_2, C_{-2,2}, D_{2,2}, \ldots$ fulfill the auxiliary equations (3.5), (3.13) as before. In order to show that the filtered approximation Ψ is close to Ψ_2 and to control $\mathrm{Res}(\varepsilon\Psi)$ we need the following lemma that estimates multipliers acting on scaled functions.

Lemma 3.3 *[10, Lemma 6] Let $m \in \mathbb{N}$ and $(1+|\cdot|^2)^{-m/2}\hat{M}(\cdot) \in C^2(\mathbb{R},\mathbb{C})$. Then* $\|M[(S_\delta U)e^{ik_0x}]\|_{H^{q-r}_{\rm ul}}$

$$\leq C(q,r,m)\|(1+|\cdot|)^{-m/2}\hat{M}(\delta(\cdot+k_0))\|_{C_b^2}\|S_\delta\|_{\mathcal{L}(H^{q-m}_{\rm ul},H^{q-r}_{\rm ul})}\|U\|_{H^q_{\rm ul}}.$$

for all $q \geq r \geq m$, with $C(q,r,m)$ independent of $\hat{M}$.

In particular, in the situation of Lemma 3.3 assume that $\hat{M}(k_0+k) = \mathcal{O}(|k|^s)$ as $k \to 0$ with $s \leq m$. Then [10, Lemma 7]

$$\|(1+|\cdot|^2)^{-m/2}\hat{M}(\delta(k_0+\cdot))\|_{C_b^2} = \mathcal{O}(\delta^s). \tag{3.15}$$

Using Lemma 3.2 with $q=4$, $r,m=3$ and (3.15) we obtain

$$\begin{aligned}\|(\mathrm{Id}-E_0)[(S_{\varepsilon^2}A)\mathbf{E}^j\mathbf{F}^l]\|_{H^1_{\rm ul}} &\leq C\|(\mathrm{Id}-E_0)[(S_{\varepsilon^2}A)\mathbf{E}^j]\|_{H^1_{\rm ul}} \\ \leq \|(1+|\cdot|^2)^{-3/2}&(1-\chi_0(\varepsilon^2(j+\cdot)))\|_{C_b^2}\|S_{\varepsilon^2}\|_{\mathcal{L}(H^1_{\rm ul},H^1_{\rm ul})}\|A\|_{H^4_{\rm ul}} \leq C\varepsilon^6.\end{aligned} \tag{3.16}$$

This holds since $1-\chi_0(\varepsilon^2(j+k)) \equiv 0$ for $|k+j| \leq 1/(8\varepsilon^2)$, for ε sufficiently small. By (3.16) Ψ is close to Ψ_2, i.e.

$$\|\Psi_2 - \Psi\|_{H^1_{\rm ul}} = \mathcal{O}(\varepsilon^6). \tag{3.17}$$

To estimate $\|\mathrm{Res}(\varepsilon\Psi)\|_{H^1_{\rm ul}}$ we also need estimates of $(\mathrm{Id}-E_0)[\partial_x^l(S_\varepsilon^2 A)\mathbf{E}^j\mathbf{F}^l]$. Using Lemma 3.3 with $q=3$, $r,m=2$ we obtain

$$\begin{aligned}\|(\mathrm{Id}-E_0)[\partial_x(S_{\varepsilon^2}A)\mathbf{E}^j\mathbf{F}^l]\|_{H^1_{\rm ul}} &= \varepsilon^2\|(\mathrm{Id}-E_0)[(S_{\varepsilon^2}\partial_X A)\mathbf{E}^j\mathbf{F}^l]\|_{H^1_{\rm ul}} \\ \leq \|(1+|\cdot|^2)^{-2}&(1-\chi_0(\varepsilon^2(j+\cdot)))\|_{C_b^2}\|S_{\varepsilon^2}\|_{\mathcal{L}(H^1_{\rm ul},H^1_{\rm ul})}\|\partial_X A\|_{H^3_{\rm ul}} \leq C\varepsilon^6,\end{aligned} \tag{3.18}$$

and similarly for higher order derivatives of $S_{\varepsilon^2}A$.

In order to achieve b) we now split the error into a critical part $\varepsilon^2 R_c$ and a stable part $\varepsilon^3 R_s$, i.e., we let

$$\varepsilon^2 R = \varepsilon^2 R_c + \varepsilon^3 R_s \quad \text{with} \quad \mathcal{E}_c^h R_c = R_c \quad \text{and} \quad \mathcal{E}_s^h R_s = R_s.$$

Moreover, we introduce

$$\begin{aligned}\varepsilon\Psi_c &= \varepsilon(E_0 S_{\varepsilon^2}A_1)(T)\mathbf{E}\mathbf{F}\mathrm{e}_1 + \varepsilon(E_0 S_{\varepsilon^2}B_1)(T)\mathbf{E}^{-1}\mathbf{F}\mathrm{e}_2 + \mathrm{c.c.}, \\ \varepsilon^2\Psi_s &= \varepsilon\Psi - \varepsilon\Psi_c,\end{aligned}$$

such that $\varepsilon\Psi_c$ contains the lowest order critical modes and $\varepsilon^2\Psi_s$ contains the stable modes and the $\mathcal{O}(\varepsilon^3)$ corrections of the critical modes. In fact, by (3.16) we have $\|\Psi_s\|_{H^1_{\rm ul}} = \mathcal{O}(1)$. Inserting $U = \varepsilon\Psi_c + \varepsilon^2\Psi_s + \varepsilon^2 R_c + \varepsilon^3 R_s$ into (3.1) gives

$$\begin{aligned}\partial_t R_c + \varepsilon\partial_t R_s = &\ L_1 R_c + \varepsilon L_1 R_s + \varepsilon^2 L_2 R_c + \varepsilon^3 L_2 R_s + 2\varepsilon N_2(\Psi_c, R_c) \\ &+ \varepsilon^2 N_2(R_c,R_c) + \varepsilon^2 G_1 R + \varepsilon^3 G_2(R) + \varepsilon^{-2}\mathrm{Res}(\varepsilon\Psi),\end{aligned} \tag{3.19}$$

where

$$G_1R = 2[N_2(\Psi_c, R_s) + N_2(\Psi_s, R_c)], \quad G_2(R) = N_2(R_s, 2\Psi_s + 2R_c + \varepsilon R_s).$$

The crucial feature of quadratic nonlinearities is that the quadratic interaction of critical modes generates only stable modes, i.e., for $u, v \in H^1_{\text{ul}}(\mathbb{R})$ we have

$$E_c N(E_c^h u, E_c^h v) = 0, \tag{3.20}$$

where $N : H^1_{\text{ul}}(\mathbb{R}) \times H^1_{\text{ul}}(\mathbb{R}) \to L^2_{\text{ul}}(\mathbb{R})$ is some bilinear mapping. This follows from looking at the support of $\mathcal{F}E_c^h u$, see [10, Lemma 9].

We define R_c and R_s to be the solutions of the system

$$\begin{aligned} \partial_t R_c &= L_1 R_c + \varepsilon^2 \mathcal{E}_c(L_2 R_c + G_1 R) + \varepsilon^3 N_c(R) + \varepsilon^2 \delta_c, \\ \partial_t R_s &= L_1 R_s + L_s R_c + \varepsilon N_s(R) + \delta_s, \\ (R_c, R_s)|_{t=0} &= 0, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} N_c(R) &= \mathcal{E}_c(L_2 R_s + G_2(R)), \\ \delta_c &= \varepsilon^{-4} \mathcal{E}_c \text{Res}(\varepsilon\Psi), \\ L_s R_c &= 2\mathcal{E}_s(N_2(\Psi_c, R_c)), \\ N_s(R) &= \mathcal{E}_s(L_2 R_c + G_1 R + N_2(R_c, R_c) + \varepsilon L_2 R_s + \varepsilon G_2(R)), \\ \delta_s &= \varepsilon^{-3} \mathcal{E}_s \text{Res}(\varepsilon\Psi). \end{aligned}$$

If (R_c, R_s) is a solution of (3.21), then $R_c + \varepsilon R_s$ is a solution of (3.19). In the derivation of (3.21) we used

$$E_c N_2(\Psi_c, R_c) = 0 \quad \text{and} \quad E_c N_2(R_c, R_c) = 0$$

due to (3.20). Thus there are no order $\mathcal{O}(\varepsilon)$ terms linear in R_c and no order $\mathcal{O}(\varepsilon^2)$ terms nonlinear in R_c in the equation for the critical part R_c of the error. This corresponds to b) on page 14.

For the approximation $\varepsilon\Psi_2$ defined in (3.12) all terms at orders $\varepsilon^3 \mathbf{E}^{\pm 1}$ in (3.4) vanish by construction, i.e., formally we have $\mathcal{E}_c \text{Res}(\varepsilon\Psi_2) = \mathcal{O}(\varepsilon^4)$. For the filtered approximation $\varepsilon\Psi$ we obtain rigorously

$$\|\mathcal{E}_c \text{Res}(\varepsilon\Psi)\|_{H^1_{\text{ul}}} = \mathcal{O}(\varepsilon^4), \quad \|\mathcal{E}_s \text{Res}(\varepsilon\Psi)\|_{H^1_{\text{ul}}} = \mathcal{O}(\varepsilon^3), \tag{3.22}$$

using (3.16) and (3.18). See also [10, Lemma 8] for a detailed proof of this result in a related problem. Basically, (3.22) corresponds to a) on page 14. The nonlinearity $\widetilde{N}(R) : H^1_{\text{ul}}(\mathbb{R}) \to L^2_{\text{ul}}(\mathbb{R})$ with

$$\widetilde{N}(R) = (N_c(R), N_s(R)).$$

is a sum of linear and bilinear terms and therefore locally Lipschitz. In order to solve (3.21), for $j = 0, 1$ we introduce the spaces

$$\mathcal{H}^j = C([0, T_0/\varepsilon^2], [H^j_{\rm ul}(\mathbb{R})]^2) \quad \text{with} \quad \|(R_c, R_s)\|_{\mathcal{H}^j} = \sup_{0\leq t\leq T_0/\varepsilon^2} (\|R_c\|_{H^j_{\rm ul}} + \|R_s\|_{H^j_{\rm ul}}),$$

invert the linear part of (3.21) and apply a fixed point argument. By (3.22) we have $\|(\delta_c, \delta_s)\|_{\mathcal{H}^1} = \mathcal{O}(1)$. Note that $\mathcal{E}^h_c$ (resp. $\mathcal{E}^h_s$) leaves the equation for R_c (resp. R_s) invariant. Therefore we first estimate the solutions of

$$\begin{aligned} \partial_t R_c &= L_1 R_c + \varepsilon^2 \mathcal{E}_c(L_2 R_c + G_1 R) + \varepsilon^2 f_c, \\ \partial_t R_s &= L_1 R_s + L_s R_c + f_s \end{aligned} \tag{3.23}$$

with $\mathcal{E}^h_c f_c = f_c$ and $\mathcal{E}^h_s f_s = f_s$. Using a modification of Lemma 3.2, we obtain for the linear semigroup generated by L_1 the estimate

$$\|e^{tL_1}\mathcal{E}^h_c\|_{\mathcal{L}(L^2_{\rm ul}, H^1_{\rm ul})} \leq C e^{\varepsilon^2 t}, \tag{3.24}$$

due to the compact support of χ^h_c and since $\mathrm{Re}\lambda_{1,2}(k) \leq \alpha_0\varepsilon^2$ for $|k| \in [3/4, 5/4]$. Similarly, for some $\sigma_0 > 0$ we have

$$\|e^{tL_1}\mathcal{E}^h_c\|_{\mathcal{L}(L^2_{\rm ul}, H^1_{\rm ul})} \leq \|(1+|\cdot|^2)^{1/2} \begin{pmatrix} e^{\lambda_1(\cdot)t} \\ e^{\lambda_2(\cdot)t} \end{pmatrix} \chi^h_s(\cdot)\|_{C^2(\mathbb{R},\mathbb{C})} \leq C e^{-\sigma_0 t}(1+t^{-1/4}), \tag{3.25}$$

since $\mathrm{Re}\lambda_{1,2}(k) \leq -2\sigma_0$ for $|k| \notin [3/4, 5/4]$ and since $\lambda_{1,2}(k) \sim -k^4$ as $|k| \to \infty$. From (3.24),(3.25) follows the local solvability of (3.23) in $H^1_{\rm ul}(\mathbb{R})$. Writing the solutions R_c, R_s as

$$\begin{aligned} R_c(t) &= \varepsilon^2 \int_0^t e^{(t-\tau)L_1}(\mathcal{E}_c(L_2 R_c + G_1 R + f_c) \mathrm{d}\tau, \\ R_s(t) &= \int_0^t e^{(t-\tau)L_1}(L_s R_c + f_s) \mathrm{d}\tau \end{aligned}$$

and introducing $S_i(s) = \sup_{\tau\leq s} \|R_i(\cdot, \tau)\|_{L^2_{\rm ul}}$ we obtain

$$\begin{aligned} S_s(t) &\leq \textstyle\int_0^t C(1 + (t-\tau)^{-1/4})e^{-\sigma_0(t-\tau)} \mathrm{d}\tau (S_c(t) + \|f\|_{\mathcal{H}^0}) \\ &\leq C S_c(t) + C\|f\|_{\mathcal{H}^0}. \end{aligned} \tag{3.26}$$

Inserting this into the equation for S_c shows that

$$S_c(t) \leq \varepsilon^2 \textstyle\int_0^t C e^{C\varepsilon^2(t-\tau)}(C S_c(\tau) + \|f\|_{\mathcal{H}^0}) \mathrm{d}\tau. \tag{3.27}$$

Gronwall's inequality gives $S_c(T_0/\varepsilon^2) = \mathcal{O}(1)$. Thus we have $J \in \mathcal{L}(\mathcal{H}^0, \mathcal{H}^1)$ for the solution operator J of (3.23).

Applying J to (3.21) we obtain $R = J\delta + \varepsilon J\widetilde{N}(R) =: F(R)$. For $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ the function $F : \mathcal{H}^1 \to \mathcal{H}^1$ is a contraction on a ball in $\mathcal{H}^1$ with radius $\mathcal{O}(1)$ and center $J\delta$ due to the Lipschitz continuity of $\widetilde{N}$ and the factor ε in front $J\widetilde{N}$. Thus there exists a unique fixed point which is a solution of order $\mathcal{O}(1)$ of (3.21). Using (3.17) and $\|\varepsilon\Psi_2 - \varepsilon\Psi_0\|_{\mathcal{H}^1} = \mathcal{O}(\varepsilon^2)$ the proof of Theorem 3.1 is complete. □

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