Stability and Diffusive Dynamics on Extended Domains

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Abstract. We consider dissipative systems on the real axis in situations when the evolution is dominated by a dynamics similar to the one of a linear diffusion equation. It is surprising that such a diffusive behavior occurs in relatively complicated systems.

After a discussion of the linear and nonlinear diffusion equation, we give a brief introduction into the methods which are available to describe diffusive behavior in nonlinear systems. These are L^1–L^∞ estimates, Lyapunov functions and discrete and continuous renormalization groups.

In the second part of the paper we show examples, where such a diffusive dynamics can be seen. For the Ginzburg–Landau equation we consider the nonlinear stability of Eckhaus–stable equilibria and the diffusive mixing of two different Eckhaus–stable equilibria. Diffusive dynamics also occurs in pattern forming systems as the Swift–Hohenberg equation or hydrodynamical stability problems as Bénard's problem. In such cases the method of reduced instability allows us to analyze the linearized problem.

We close with an outlook on situations, where diffusive behavior is expected, but where a proof is still missing.

1 Introduction

The dynamics of dissipative partial differential equations (PDEs) on extended domains differs significantly from that on bounded domains. Many new solution types appear, e.g., traveling waves, fronts and pulses, [DFKM96]. Besides studies of particular solution classes, an existence theory for attractors for PDEs on unbounded domains was developed in [BV90,Fei96,MS95,Mie97a,ES99b,EZ99]. The inherent lack of compactness enforces a parallel use of a uniform and a localized topology; the attractors can be characterized quantitatively by Kolmogorov's ε–entropy and the dimension per unit volume, [CE99b,CE99a,Zel99].

For PDEs on bounded domains the bifurcation theory is very well developed: the center–manifold theory and the Liapunov–Schmidt reduction allow for a finite–dimensional description. On unbounded domains a reduction may still be possible by the theory of modulation equations or, for special solution classes, by the Kirchg¨assner reduction (spatial center–manifold reduction). However, the reduced problem remains infinite dimensional and is given by a simple partial differential equation, e.g., the (complex) Ginzburg–Landau equation. Although the multiple scaling

ansatz of modulation theory has been used formally for more than 30 years, a mathematical justification was only obtained in [vH91,KSM92,Eck93,Sch94b,Sch94c,MS95,Sch9
for model problems and in [Sch94a,Sch99b] for the Navier–Stokes equation; for a survey see [Mie99].

Here we survey one particular subject in the theory of PDEs on extended domains, namely the stability of spatially homogeneous or spatially periodic steady states. Moreover, we address the phenomenon of diffusive mixing of such steady states.

A basic concept in stability theory is the stability induced by the linearization alone. This means that the nonlinear terms can be controlled if the linearized problem dissipates energy with an exponential rate. Then stability can be achieved by considering the linearized problem alone. For dissipative problems on unbounded domains the linearization possesses continuous spectrum up to the imaginary axis in the complex plane. For these problems the linearized problem shows a dynamics similar to the one of a linear diffusion equation and so by an interplay of norms very often polynomial decay rates of the linearized problem can be obtained. As a consequence of the polynomial decay not all nonlinear terms can be controlled by the linearized problem. But if the low order terms are absent, again stability can be established with the help of the linearized problem. Such nonlinearities are called irrelevant.

On unbounded domains diffusive behavior occurs as a new aspect in the theory of stability. In fact, it turned out that for many interesting problems the nonlinear terms are irrelevant, such that this method is widely applicable. For instance, it has been applied successfully for proving the nonlinear stability of Taylor vortices in infinite cylinders with respect to spatially localized perturbations.

After a discussion of the linear and nonlinear diffusion equation, we give a brief introduction into the methods which are available to describe the diffusive behavior when the basic state is spatially homogeneous. These are L^1–L^∞ estimates, Lyapunov functions and discrete and continuous renormalization groups.

Next we provide typical examples, where such a diffusive dynamics can be seen.

For the Ginzburg–Landau equation we consider the nonlinear stability of Eckhaus–stable equilibria and the diffusive mixing of different steady state solutions.

We use Bloch's theory to generalize the spectral theory from spatially homogeneous steady states to spatially periodic ones. For steady states, which bifurcate from a homogeneous state, the linearized stability can be investigated by the theory of reduced instability (cf. [Mie95]). Nonlinear diffusive stability is obtained by showing the irrelevance of the nonlinear terms by proving appropriate convolution identities. For the Swift–Hohenberg equation the stability of the Eckhaus stable roll patterns is shown with respect to one– and two–dimensional perturbations. Applications in hydrodynamics include the roll solutions in Rayleigh–Bénard convection and the Taylor vortices in the Taylor–Couette experiment.

We close with an outlook on situations, where diffusive behavior is expected, but a proof is still missing.

Here, we restrict ourselves to the case of diffusive repair and diffusive mixing of equilibria. There are also results for diffusive stability for traveling front solutions [BK92,Gal94,EW94a,RK98,GR98,GR97]. The transfer of these last stability results to modulated front solutions ([HCS99]) will be found in [ES99a].

Solutions which are localized perturbations of an exponentially homogeneous state (traveling pulses or modulated pulses) [BL99b,KS98,Sch00,Uec00] may be exponentially stable. The stability question of localized perturbations of diffusively stable states was attacked in [SS98] where a spectral stability result was obtained.

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2 Diffusive repair and diffusive mixing in the linear case

2.1 Diffusive repair

In this section we consider the linear diffusion equation

$$\partial_t u = \partial_x^2 u, \quad u|_{t=0} = u_0 \tag{1}$$

with $x \in \mathbb{R}$, $t \geq 0$ and $u(t,x) \in \mathbb{R}$. The solution can be written explicitly as

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/(4t)} u_0(y)\,\mathrm{d}y = \int_{\mathbb{R}} G(x-y,t) u_0(y)\,\mathrm{d}y. \tag{2}$$

Spatially constant functions stay constant in time, but by Young's inequality for convolutions with $p > q$ we obtain

$$\|u(t)\|_{L^p} \leq C t^{-1/(2r)} \|u_0\|_{L^q}, \quad \text{where} \quad 1/p = 1/q - 1/r,\ 1 \leq p, r \leq \infty, \tag{3}$$

for some constant C independent of time. Thus, spatially localized initial conditions give rise to solutions with polynomial decay rates. Moreover, the solutions become flatter and flatter, since we have for instance

$$\|\partial_x^n u\|_{L^\infty} \leq \|\partial_x^n G\|_{L^\infty} \|u_0\|_{L^1} \leq C t^{-(n+1)/2} \|u_0\|_{L^1}.$$

There is some additional structure which can be seen by looking at the Fourier transform

$$\hat{u}(k) = (Fu)(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

of (1). The equation $\partial_t \hat{u} = -k^2 \hat{u}$ possesses the solution $\hat{u}(t,k) = \mathrm{e}^{-k^2 t} \hat{u}_0(k)$. Renormalizing this solution gives

$$\hat{u}(t, k/\sqrt{t}) = \mathrm{e}^{-k^2} \hat{u}_0(k/\sqrt{t}) = \mathrm{e}^{-k^2} \left(\sum_{j=0}^{n} t^{-j/2} k^j \hat{u}_0^{(j)}(0) + o(t^{-n/2}) \right) \tag{4}$$

if $\hat{u}_0$ is n–times differentiable. Since smoothness in Fourier space corresponds to decay properties in x–space, solutions to spatially localized initial conditions decay in a universal way to 0. Loosely speaking, if the initial conditions spatially decay like $|x|^{-n}$, we obtain

$$u(t,x) = \sum_{j=0}^{n-1} t^{-(j+1)/2} \hat{u}_0^{(j)}(0) H_j(x/\sqrt{t}) + O(t^{-n/2})$$

for $t \to \infty$, where H_j is a multiple of the jth Hermite polynomial.

To be more precise we use the fact that Fourier transform is an isomorphism from $H^n(m)$ to $H^m(n)$, where

$$H^m(n) = \{u : \mathbb{R} \to \mathbb{C} \mid \|u\|_{H^m(n)} = \|u\rho^n\|_{H^m} < \infty\} \quad \text{and} \quad \rho(x) = (1+x^2)^{1/2}. \tag{5}$$

For the lowest order terms we obtain $\|\hat{u}(t, k/\sqrt{t}) - \mathrm{e}^{-k^2} \hat{u}_0(0)\|_{H^2(2)} \leq Ct^{-1/2}$, i.e.,

$$\|\sqrt{t}\, u(t, \sqrt{t}\, x) - \sqrt{\pi}\, \hat{u}_0(0) \mathrm{e}^{-x^2/4}\|_{H^2(2)} \leq Ct^{-1/2}. \tag{6}$$

The result is based on the fact that the linear evolution operator $\mathrm{e}^{-k^2 t}$ concentrates the Fourier modes at the wave number $k = 0$. Depending on the differentiability of the initial conditions $\hat{u}_0$ the local behavior of $\hat{u}_0$ at the wavenumber $k = 0$ is extracted by the linear evolution operator $\mathrm{e}^{-k^2 t}$ for $t \to \infty$.

Remark 1. Most of the above theory also holds if we have a linear evolution operator $\mathrm{e}^{t\lambda(k)}$ with eigenvalues $\lambda(k) \sim -k^2$ for $k \to 0$. This is the reason why diffusive behavior can be observed in a big variety of problems.

From (6), i.e., $u(t,x) = \sqrt{\pi/t}\, \hat{u}_0 \mathrm{e}^{-x^2/(4t)} + O(t^{-1})$, it is obvious that stability of $u = 0$ in $H^2(2)$ does not hold, but again we have

$$\sup_{x\in\mathbb{R}} |u(t,x)| \leq Ct^{-1/2}$$

for initial conditions $u|_{t=0} = u_0 \in H^2(2)$. Thus, it makes sense to introduce the following definition.

Definition 2. Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and let S_t be an evolution operator. A fixed point $u_0 = S_t u_0$ is called $(\mathcal{B}_1, \mathcal{B}_2)$–stable under S_t if the following holds: For all $\varepsilon > 0$ there exists a $\delta > 0$ such that from $\|v - u_0\|_{\mathcal{B}_1} < \delta$ it follows that $\|S_t v - u_0\|_{\mathcal{B}_2} < \varepsilon$ for all $t \geq 0$. The point u_0 is called asymptotically $(\mathcal{B}_1, \mathcal{B}_2)$–stable if additionally $\lim_{t\to\infty} S_t v = u_0$ in $\mathcal{B}_2$.

The proof of (6) for some nonlinear problem also establishes asymptotic $(H^2(2), L^\infty)$–stability of $u = 0$. The usage of two different norms is very common for problems posed on unbounded domains, cf. [BV90,MS95].

2.2 Diffusive mixing

Above we have shown that spatially localized perturbations decay diffusively to 0. Another interesting question is the behavior of solutions to initial conditions

$$u_0(x) = \begin{cases} a \text{ for } x < -\ell, \\ b \text{ for } x \geq \ell, \end{cases}$$

for some $\ell > 0$, i.e., we prescribe two different constants on the left and on the right. Since $\partial_x u_0$ is again spatially localized and since $\partial_x u$ also satisfies (1) we obtain the results from above for $\partial_x u$. Integration with respect to x leads to

$$\sup_{x\in\mathbb{R}} |u(t,x) - [a+(b-a)\mathrm{Erf}(x/\sqrt{t})]| \leq C/\sqrt{t},$$

where $\mathrm{Erf}(x) = 1/\sqrt{4\pi} \int_{-\infty}^{x} \mathrm{e}^{-y^2/4}\, \mathrm{d}y$. For the linear diffusion equation all spatially constant functions are stable equilibria. Thus, we have some diffusive mixing of the stable states.

The question arises whether such a behavior also occurs in more complicated systems if we have different diffusively stable equilibria for $x \to \pm\infty$.

3 Irrelevant nonlinearities and nonlinear diffusive stability

In this section we explain how the polynomial decay rates for the linearized problem can be used to control the nonlinear terms. As an example we consider a nonlinear diffusion equation

$$\partial_t u = \partial_x^2 u + c\, u^p, \quad u|_{t=0} = u_0 \tag{7}$$

with $t \geq 0$, $x \in \mathbb{R}$, $p \in \mathbb{N}$ and $c \in \mathbb{R} \setminus \{0\}$.

In case $p = 2$ we have blowup of the solution for most initial conditions, in case $p = 3$ the sign of c decides about stability ($c < 0$) and instability ($c > 0$), but for $p \geq 4$ the sign of c doesn't play any role and small spatially localized perturbations vanish for $t \to \infty$ with the same polynomial decay rate as in the linear case.

There are essentially three methods to prove the last assertion, namely a) L^1–L^∞ estimates, b) the construction of Lyapunov functions, and c) the discrete and continuous renormalization approach.

3.1 L^1–L^∞ estimates

This method relies on the L^q–L^p estimate (3), the variation–of–constants formula and suitable estimates of the nonlinearity.

Lemma 3. *Let $p > 3$. For all $C > 0$ there exists $\varepsilon > 0$ such that solutions u of (7) with $\|u_0\|_{L^1} + \|u_0\|_{L^\infty} \leq \varepsilon$ satisfy*

$$\|u(t)\|_{L^1} \leq C \;\; \textit{and} \;\; \|u(t)\|_{L^\infty} \leq C/(1+t)^{\frac{1}{2}}$$

for all $t \geq 0$.

Proof. We consider the variation–of–constants formula

$$u(t) = \mathrm{e}^{t\partial_x^2} u_0 + c\int_0^t \mathrm{e}^{(t-s)\partial_x^2} u^p(s)\,\mathrm{d}s$$

for (7). With $\|u^p\|_{L^\infty} \leq \|u\|_{L^\infty}^p$ and $\|u\|_{L^1}^p \leq \|u\|_{L^\infty}^{p-1}\|u\|_{L^1}$, the abbreviations

$$a(t) = \sup_{0\leq s\leq t} \|u(s)\|_{L^1} \quad \text{and} \quad b(t) = \sup_{0\leq s\leq t} \|(1{+}s)^{1/2} u(s)\|_{L^\infty}$$

and the estimates of Section 2 we obtain

$$(1{+}t)^{1/2}\left\|\int_0^t \mathrm{e}^{(t-s)\partial_x^2} u^p(s)\,\mathrm{d}s\right\|_{L^\infty} \leq (1{+}t)^{1/2}\int_0^t \|\mathrm{e}^{(t-s)\partial_x^2}\|_{L^1\to L^\infty}\|u^p\|_{L^1}\,\mathrm{d}s$$
$$\leq (1{+}t)^{1/2}\int_0^t (t{-}s)^{-1/2}(1{+}s)^{-(p-1)/2}\,\mathrm{d}s\cdot b(t)^{p-1}a(t) \leq C_1\, b(t)^{p-1}a(t)$$

with a constant C_1 independent of t for $p > 3$. Furthermore, we have

$$\left\|\int_0^t \mathrm{e}^{(t-s)\partial_x^2} u^p\,\mathrm{d}s\right\|_{L^1} \leq \int_0^t \|e^{(t-s)\partial_x^2}\|_{L^1\to L^1}\|u^p\|_{L^1}\,\mathrm{d}s$$
$$\leq \int_0^t (1{+}s)^{-(p-1)/2}\,\mathrm{d}s\cdot b(t)^{p-1}a(t) \leq C_2 b(t)^{p-1}a(t).$$

Together we obtain

$$a(t) \leq a(0) + |c|C_1 b(t)^{p-1}a(t) \quad \text{and} \quad b(t) \leq a(0) + |c|C_2 b(t)^{p-1}a(t).$$

If $a(0) + b(0) < \varepsilon$ with $\varepsilon > 0$ sufficiently small we have the existence of $C > 0$ such that $a(t), b(t) \leq C$ for all $t \geq 0$.

3.2 Lyapunov functions

The usage of Lyapunov functions is well established in nonlinear stability problems if the nonlinearity has some sign as in $\partial_t u = \partial_x^2 u - u^3$. This has been used for diffusive stability problems in [EW94b] and [GR97] and in [GM98] for the proof of diffusive mixing. But also if the nonlinearity has the wrong sign, Lyapunov functions can be used. We do not obtain the optimal power p of the irrelevant nonlinearities, but the method is also applicable on unbounded domains $\Omega \subset \mathbb{R}^d$ with $\Omega \neq \mathbb{R}^d$, cf. [ES98]. As an example we consider again

$$\partial_t u = \partial_x^2 u + c\,u^p, \quad u|_{t=0} = u_0.$$

We introduce the functionals $I(u) = \int_{\mathbb{R}} u^2\,\mathrm{d}x$, $J(u) = \int_{\mathbb{R}} (\partial_x u)^2\,\mathrm{d}x$ and $K(u) = \int_{\mathbb{R}} (\partial_x^2 u)^2\,\mathrm{d}x$. This fixes the L^q–power to $q = 2$, and due to the L^2–L^∞ estimate (3) we can only handle nonlinearities for $p \geq 5$.

With $\|u\|_{L^\infty}^2 \leq I^{1/2}J^{1/2}$ and $J^2 \leq I\,K$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} I = \int u\partial_t u\,\mathrm{d}x = \int u\partial_x^2 u + cu^{p+1}\,\mathrm{d}x$$
$$= \int -(\partial_x u)^2 + cu^{p+1}\,\mathrm{d}x \leq -J + |c|\|u(x)\|_{L^\infty}^{p-1}\int u^2\,\mathrm{d}x$$
$$\leq -J + |c| I^{(p+3)/4} J^{(p-1)/4} = -J\,[1 - |c| I^{(p+3)/4} J^{(p-5)/4}]$$

and

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}J &= \int(\partial_x u)(\partial_t\partial_x u)\,\mathrm{d}x \\ &= \int(\partial_x u)(\partial_x^3 u) + cpu^{p-1}(\partial_x u)^2\,\mathrm{d}x \;\le\; -\int(\partial_x^2 u)^2\,\mathrm{d}x + |c|p\|u\|_{L^\infty}^{p-1}J \\ &\le -K + |c|pI^{(p-1)/4}J^{(p+3)/4} \;\le\; -K\,[1-|c|pI^{(p+3)/4}J^{(p-5)/4}].\end{aligned}$$

Hence, we have $\dot I \le 0$ and $\dot J \le 0$ if $I(u_0)^{(p+3)/4}J(u_0)^{(p-5)/4}$ is sufficiently small. Thus, we have proved the following result.

Lemma 4. *Let* $p \ge 5$. *Then, there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and all solutions* u *of (7) with* $\|u_0\|_{H^1} \le \varepsilon$ *we have* $\sup_{t\ge 0}\|u(t)\|_{H^1} \le \varepsilon$.

3.3 The discrete and continuous renormalization process

In addition to some stability result this method gives the asymptotics of the decay to 0. It relies on formula (6). By a fixed point argument we prove also for the nonlinear system (7) that the renormalized solution $\sqrt{t}\,u(t, \sqrt{t}\,x)$ converges towards a multiple of the Gaussian $\mathrm{e}^{-x^2/4}$. There are two approaches, a discrete and a continuous one. We sketch the first one very briefly and the second one in a little more detail.

The discrete approach In the discrete approach (cf. [BK92,BKL94,Gal94,Sch96]) a sequence of problems is considered which converges towards the linear diffusion equation. Define $u_n(\tau, x) = L^n u(L^{2n}\tau, L^n x)$ with $L > 1$ and $n \in \mathbb{N}$. Then u_n satisfies

$$\partial_\tau u_n = \partial_x^2 u_n + L^{n(3-p)}u_n^p \quad \text{for } \tau \in [L^{-2}, 1]; \qquad u_n(L^{-2}, x) = Lu_{n-1}(1, Lx). \tag{8}$$

Obviously, for $p > 3$ the influence of the nonlinear terms tends exponentially to 0 as $n \to \infty$ and in the limit we obtain the linear diffusion equation. Solving the sequence of problems (8) is equivalent to solving (7). By a fixed point argument it then follows (see for instance [BK92,Sch96]) that for spatially localized initial conditions u_0 the functions $u_n|_{\tau=1} = L^n u(L^{2n}, L^n\cdot)$ converge towards a multiple of the Gaussian $\mathrm{e}^{-x^2/4}$.

The continuous approach Here the system satisfied by the renormalized solution is considered directly, where additionally a logarithmic time scale is taken to transfer the polynomial decay rates into exponential ones. We follow the lines of [Way97] and introduce the new variable w and the new coordinates ξ and τ by

$$u(t, x) = t^{-1/2}w(\log t, x/\sqrt{t}) = \mathrm{e}^{-\tau/2}w(\tau, \xi). \tag{9}$$

The transformed equation is given by

$$\partial_\tau w = w/2 + (\xi/2)\partial_\xi w + \partial_\xi^2 w + \mathrm{e}^{(3-p)\tau/2}w^p \tag{10}$$

with $w|_{\tau=0} \in H^2(2)$. The linearization around $w = 0$ leads to the spectral problem which reads in Fourier space

$$-k^2\hat{w} - (k/2)\partial_k\hat{w} - \lambda\hat{w} = \hat{f}, \tag{11}$$

with $\hat{f} \in H^2(2)$. The eigenfunctions $\hat{\psi}_s(k) = k^s e^{-k^2}$ to the real eigenvalues $\lambda = -s/2$ are parameterized with $s \in \mathbb{R}$. See also (4). Since $\partial_k^j\hat{\psi}_s \in L^2$ is required for $j = 0, 1, 2$ (for which the possible singularity of $\partial_k^j\hat{\psi}_s$ at $k = 0$ plays the crucial role) this leads in (11) to two discrete eigenvalues 0 and $-1/2$ and to essential spectrum $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda < -3/4\}$ due to Sobolev's embedding theorem. Since the solutions of the linearized problem are uniformly bounded and since the nonlinear terms vanish with an exponential rate there exist $\varepsilon, C > 0$ such that $\sup_{\tau\geq 0}\|w(\tau)\|_{H^2(2)} < C$ for the solutions w of (10) if $\|w|_{\tau=0}\|_{H^2(2)} < \varepsilon$ and $p > 3$. If we denote with w_0 the part of w belonging to the eigenvalue 0 and with w_1 the rest of w we can conclude by integration of the variation of constants formula with respect to time that

$$w_0(\tau) = w_{\lim} e^{-(\cdot)^2/4} + O(e^{\frac{3-p}{2}\tau}) \quad \text{and} \quad w_1(\tau) = O(e^{\max\{-\frac{1}{2},\frac{3-p}{2}\}\tau}) \text{ for } \tau \to \infty.$$

Herein, $\psi_0 = e^{-(\cdot)^2/4}$ is the eigenvector to 0 in physical space, and $w_{\lim} \in \mathbb{R}$ is a constant only depending on the initial conditions. This leads to the following convergence result, cf. [Way97].

Theorem 5. *Let $p > 3$. Then there exist $\varepsilon, C > 0$ such that the following holds. Let u be a solution of (7) with $\|u|_{t=0}\|_{H^2(2)} \leq \varepsilon$. Then there exists a $w_{\lim} \in \mathbb{R}$ such that*

$$\|\sqrt{t}\,u(t, \cdot\sqrt{t}) - w_{\lim} e^{-(\cdot)^2/4}\|_{H^2(2)} \leq C(1+t)^{-\max\{1/2,(3-p)/2\}} \quad \text{for all } t \geq 0.$$

3.4 Some remarks

Clearly, the above theory can be adjusted to more general nonlinearities as well as to higher space dimensions. If we call the exponent p of the nonlinear term u^p degree of irrelevance then $u^{p_1}(\partial_x u)^{p_2}(\partial_x^2 u)^{p_3}$ has the degree of irrelevance $p = p_1 + 2p_2 + 3p_3$. Thus, nonlinear terms with derivatives give some additional irrelevance. For instance, Lemma 3 holds for all nonlinearities with $p > 3$. In higher space dimensions the solutions of the linear diffusion equation $\partial_t u = \Delta u$, $x \in \mathbb{R}^d$ satisfy the estimate

$$\|u(t)\|_{L^\infty} \leq Ct^{-d/2}\|u\|_{L^1}.$$

From the above analysis it is easy to see, that all nonlinear terms with degree of irrelevance $p{>}1{+}2/d$ are irrelevant. As a consequence, in dimensions $d \geq 3$ all sufficiently smooth nonlinear terms are irrelevant. Physically this means that there are enough directions in which the energy can diffuse away before the quadratic nonlinear terms have time to act. See also [BL99a] for a generalization to larger classes of initial data.

4 Diffusive behavior in the Ginzburg–Landau equation

The (real) Ginzburg–Landau equation

$$\partial_t u = \partial_x^2 u + u - |u|^2 u, \quad u = u(t,x) \in \mathbb{C}, \quad u|_{t=0} = u_0 \tag{12}$$

occurs as an amplitude equation for bifurcation problems on infinitely long cylindrical domains, cf. for instance [CE90b,Sch94b,MS96,Mie99,Sch99b]. It possesses so–called stationary roll solutions, i.e. spatially periodic steady states of the form $u_{q,\beta}(x) = \sqrt{1-q^2}\,\mathrm{e}^{\mathrm{i}(qx+\beta)}$. Letting $u(t,x) = u_{q,\beta}(x) + \mathrm{e}^{-\mathrm{i}(qx+\beta)}v$ and linearizing in v one obtains that a roll $u_{q,\beta}$ is linearly stable if and only if $q^2 \leq 1/3$, cf. [Eck65]. For $q^2 > 1/3$ a roll $u_{q,\beta}$ is sideband– or Eckhaus–unstable. This means that $u_{q,\beta}$ is unstable with respect to perturbations $\mathrm{e}^{\mathrm{i}\tilde{q}x}$ with a slightly different wavenumber $\tilde{q} \approx q$, $\tilde{q} \neq q$.

In this section we review results from [CEE92,BK92] and [GM98] concerning the nonlinear diffusive stability and the diffusive mixing of rolls, respectively.

4.1 Diffusive stability of equilibria

To understand the Ginzburg–Landau equation (12) in the vicinity of a roll we introduce coordinates $u(t,x) = r(t,x)\,\mathrm{e}^{\mathrm{i}\phi(t,x)}$. Then

$$\partial_t \phi = \partial_x^2 \phi - 2\frac{\partial_x r}{r}\partial_x \phi, \quad \partial_t r = \partial_x^2 r + r[1-r^2-(\partial_x\phi)^2]. \tag{13}$$

The roll solution $u_{q,\beta}$ now takes the form $(\phi_{q,\beta}, r_q) = (qx+\beta, \sqrt{1-q^2})$. On the linear level we see that $\phi-\phi_{q,\beta}$ behaves diffusively while $r-r_q$ is linearly exponentially damped with rate $-2(1-q^2)$. Moreover, ϕ itself does not appear on the right–hand side.

This means roughly that the amplitude is slaved to the local wave length $\eta = \partial_x\phi$. Heuristically, we obtain asymptotically $r = \sqrt{1-(\partial_x\phi)^2} + \text{h.o.t.}$. Neglecting the higher order terms and inserting the relation in the equation for ϕ we arrive at the *phase diffusion equation*

$$\partial_t \phi = \frac{1-3(\partial_x\phi)^2}{1-(\partial_x\phi)^2}\partial_x^2\phi. \tag{14}$$

Writing $\phi = \phi_{q,\beta} + \psi$ allows us to study the question of diffusive stability

$$\partial_t \psi = a(q)\partial_x^2\psi + [a(q+\partial_x\psi) - a(q)]\partial_x^2\psi \quad \text{where } a(q) = (1-3q^2)/(1-q^2).$$

Clearly, we need $a(q) > 0$ which characterizes the Eckhaus–stable domain. The lowest order nonlinear terms are a multiple of $(\partial_x\psi)^{p_2}\partial_x^2\psi$ for a $p_2 \geq 1$. Following the remarks given in Section 3.4 we have $p_3 = 1$ and hence $p = 2p_2 + 3 > 3$. Thus, for the phase diffusion equation the nonlinearity is irrelevant.

Of course, the proof of diffusive stability (in the sense of $(H^2(2), L^\infty)$ stability) of Eckhaus–stable rolls $u_{q,\beta}$ for the full Ginzburg–Landau equation (12) is much more involved as the coupling between $\partial_x \phi$ and r has to be studied precisely. See [BK92] for a proof using renormalization theory as described in subsection 3.3 and see [Kap94] for a proof which is based on Lyapunov functions and L^2–L^∞ estimates.

4.2 Diffusive mixing

The next question is the evolution of u for an initial condition $u_0(x) = r_0(x)\mathrm{e}^{\mathrm{i}\phi_0(x)}$ that converges to two different stable rolls $u_{q_\pm,\beta_\pm}$ for $x \to \pm\infty$, where $(\beta_+ - \beta_-)^2 + (q_+ - q_-)^2 \neq 0$. Clearly, the associated solution will, for all $t > 0$, satisfy the same boundary conditions at infinity:

$$u(t,x) - u_{q_\pm,\beta_\pm}(x) \to 0 \quad \text{for} \quad x \to \pm\infty.$$

The question is how the solutions behave in the intermediate regime. Diffusive mixing is seen, if the initial condition u_0 is chosen properly. Then, for large t the solution develops an intermediate wave length q_* which solely depends on q_- and q_+. However, in general the phase β does not converge but grows as $\sqrt{t}$.

For $q_- = q_+ = q \in (-1/\sqrt{3}, 1/\sqrt{3})$ and small $\delta = \beta_+ - \beta_- \neq 0$ the problem has been treated in [CEE92]. It is shown that, if $r_0 - \sqrt{1-q^2}$ and $\phi_0(x) - qx$ are small and spatially localized, then the solution (r,ϕ) of (13) satisfies

$$\|r(t) - \sqrt{1-q^2}\|_{L^\infty} + \|\partial_x \phi(t) - q\|_{L^\infty} \to 0 \quad \text{as } t \to \infty. \tag{15}$$

The extension to the case $q_- \neq q_+$ with small $|q_\pm|, |\beta_\pm|$ and more detailed asymptotics have been obtained in [BK92] using the discrete renormalization approach. In [GM98] these results are generalized to arbitrary $q_-, q_+ \in (-1/\sqrt{3}, 1/\sqrt{3})$. In the case $q_+ \neq q_-$ we may assume $\beta_- = \beta_+ = 0$ without loss of generality (use the translation $x \mapsto x+y$ and the phase invariance $\phi \mapsto \phi+\alpha$).

The first step in the analysis is the construction of a limiting profile $\tilde\eta$ for the local wave vector. It is obtained as the unique similarity solution $\phi(t,x) = \tilde\eta(x/\sqrt{t})$ of the phase diffusion equation (14) which satisfies

$$[a(\tilde\eta)\tilde\eta']' + \frac{\xi}{2}\tilde\eta' = 0 \text{ for } \xi \in \mathbb{R} \quad \text{and} \quad \tilde\eta(\xi) \to q_\pm \text{ for } \xi \to \pm\infty.$$

Then we introduce

$$\tilde\phi(x) = q_- x + \textstyle\int_{-\infty}^{x} [\tilde\eta(\xi) - q_-]\,\mathrm{d}\xi$$

and define the limiting profile

$$\widetilde{U}(t,x) = \sqrt{1 - \tilde\eta(x/\sqrt{t})^2}\, \mathrm{e}^{\mathrm{i}\sqrt{t}\,\tilde\phi(x/\sqrt{t})}.$$

The following result was proved by using a nonlinear change of variables for the amplitude r and the phase ϕ, the continuous renormalization approach from Subsection 3.3 combined with energy estimates in the sense of Section 3.2 and the theory of nonlinear monotone operators.

Theorem 6. *[GM98, Theorem 4.2] For all* $q_-, q_+ \in (-1/\sqrt{3}, 1/\sqrt{3})$ *there exist* $t_0 > 0$ *and* $\varepsilon > 0$ *such that for all* $\nu \in (0,1)$ *and for all* $u_0 \in H^2_{\rm ul}(\mathbb{R})$ *satisfying* $\|u_0 - \widetilde{U}(t_0,\cdot)\|_{H^2} \le \varepsilon$ *the unique solution of (12) in* $H^2_{\rm ul}(\mathbb{R})$ *with* $u(0,\cdot) = u_0$ *satisfies*

$$\|u(t,\cdot)-\widetilde{U}(t_0+t,\cdot)\|_{H^1_{\rm ul}} = O(t^{-\nu/4}), \quad \|\,|u(t,\cdot)|-|\widetilde{U}(t_0+t,\cdot)|\,\|_{H^1_{\rm ul}} = O(t^{-3\nu/4}) \tag{16}$$

as $t \to \infty$.

Here the spaces $H^m_{\rm ul}$ are the uniformly local Sobolev spaces introduced in [MS95]. Using (16) and simple properties of $\tilde{\eta}$ one obtains, for all fixed ℓ,

$$\sup_{|x|\le\ell}\left|u(t,x) - \sqrt{1-q^{*2}}\,\mathrm{e}^{\mathrm{i}(q^*x+\sqrt{t}\,\phi^*)}\right| = O(t^{-\nu/4}),$$

where $q^* = \tilde{\eta}(0)$ and $\phi^* = \tilde{\phi}(0)$ with $\mathrm{sign}(\phi^*) = \mathrm{sign}(q_+-q_-)$. The weaker decay rate compared to [BK92] is due to less restrictive conditions on u_0.

5 General pattern forming systems

So far we have studied the diffusive stability and the diffusive mixing of special periodic states of the Ginzburg–Landau equation which has the phase invariance as an SO(2) symmetry. The rotating waves $u_{q,\beta}$ are in fact relative equilibria (in the stationary problem) with respect to this symmetry. As a consequence it is possible to factor out the phase and treat the problem as a spatially homogeneous one. In the following we want to indicate, that it is also possible to apply the above techniques to general spatially periodic solutions in parabolic systems. A basic tool for this situation is Bloch analysis which generalizes Fourier analysis.

5.1 The Bloch analysis

For simplicity we only treat the one–dimensional case. For the general case we refer to the standard reference [RS78] as well as to the recent work [Mie97b,Sca99], see also [OZ00]. We consider a differential operator

$$\mathcal{L}v = M\partial_x^2 v + N(x)\partial_x v + P(x)v$$

where $u \in \mathbb{R}^m$, $M \in \mathbb{R}^{m\times m}$ is invertible and N, P are periodic matrices with period p. The basic observation is that $\mathcal{L}$ maps functions of the form $\mathrm{e}^{\mathrm{i}lx}V(x)$ with $V(x+p) = V(x)$ into itself. Functions of this form are called Bloch waves and $l \in \mathcal{T}_p = \mathbb{R}/p\mathbb{Z}$ is called their Bloch wave number.

Defining the Hilbert spaces

$$\mathbb{H}^k_l(\mathbb{R}) = \{v \in H^k_{\rm loc}(\mathbb{R}) \mid v(x+p) = \mathrm{e}^{\mathrm{i}lp}v(x)\},$$

the operator $\mathcal{L}$ restricts to a closed unbounded operator $\mathcal{L}_l$ with compact resolvent from $\mathbb{H}_l^0(\mathbb{R})$ into itself. Its domain is $D(\mathcal{L}_l) = \mathbb{H}_l^2(\mathbb{R})$. In fact, the transformation $V = T_l v$ with $V(x) = \mathrm{e}^{-ilx}v(x)$ maps $\mathbb{H}_l^k(\mathbb{R})$ into $H_{\mathrm{per}}^k((0,p)) := \mathbb{H}_0^k(\mathbb{R})$. This transformation defines the *Bloch operator*

$$B_l = T_l \mathcal{L}_l T_l^{-1} \quad \text{with} \quad B_l V = MV'' + [N(x) + 2\mathrm{i}lM]V' + [P(x) + \mathrm{i}lN(x) - l^2 M]V.$$

All these operators map $H_{\mathrm{per}}^2((0,p))$ into $L_{\mathrm{per}}^2((0,p))$ and they depend polynomially on l. Thus classical perturbation arguments apply.

The main tool of the Bloch analysis is the decomposition of $L^2(\mathbb{R})$ into the orthogonal, direct sum of the spaces $\mathbb{H}_l^0(\mathbb{R})$:

$$L^2(\mathbb{R}) = \bigoplus_{l\in\mathcal{T}_p} \mathbb{H}_l^0, \quad v(x) = \int_{l\in[0,p)} v_l(x)\,\mathrm{d}l,$$

where the integral is called the *direct integral* which has to be understood in the $L^2(\mathbb{R})$ sense. We have $\|v\|_{L^2(\mathbb{R})}^2 = \int_0^p \|v_l\|_{L^2(\mathbb{R})}^2 \,\mathrm{d}l$ and v_l can be expressed through the Fourier transform $\hat{u}$ as $v_l(x) = (2\pi)^{-1/2}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(l+mp)}\hat{v}(l+mp)$. Thus, the Bloch decomposition is a partial Fourier transform which is exactly adjusted to the periodicity of the underlying problem.

Moreover, the operator $\mathcal{L}$ takes the form $\oplus_{l\in\mathbb{S}^1}\mathcal{L}_l$ with respect to this decomposition. As a result the exponential and the resolvent take the form

$$\mathrm{e}^{t\mathcal{L}} = \bigoplus_{l\in\mathcal{T}_p} \mathrm{e}^{t\mathcal{L}_l}, \quad (\lambda - \mathcal{L})^{-1} = \bigoplus_{l\in\mathcal{T}_p} (\lambda - \mathcal{L}_l)^{-1}.$$

The orthogonality of the decomposition implies the norm identities

$$\|\mathrm{e}^{t\mathcal{L}}\|_{H^k(\mathbb{R})\to H^k(\mathbb{R})} = \sup_{l\in\mathcal{T}_p} \|\mathrm{e}^{tB_l}\|_{\mathbb{H}_l^k\to\mathbb{H}_l^k},$$
$$\|(\lambda - \mathcal{L})^{-1}\|_{H^k(\mathbb{R})\to H^k(\mathbb{R})} = \sup_{l\in\mathcal{T}_p} \|(\lambda - B_l)^{-1}\|_{\mathbb{H}_l^k\to\mathbb{H}_l^k}.$$

As a consequence we have a useful result for the spectrum of $\mathcal{L}$:

$$\mathrm{spec}_{L^2(\mathbb{R})}\mathcal{L} = \bigcup_{l\in\mathcal{T}_p} \mathrm{spec}_{\mathbb{H}_l^0(\mathbb{R})}\mathcal{L}_l = \bigcup_{l\in\mathcal{T}_p} \mathrm{spec}_{L_{\mathrm{per}}^2((0,p))} B_l .$$

We refer to [RS78] for the self–adjoint case and to [Mie97b] for the case of general elliptic operators. In [Sca99] the theory is developed for the Navier–Stokes equation where slight deviations arise, cf. also [Mie97c] for a concrete treatment of the operator arising in the Rayleigh–Bénard problem.

5.2 The Swift–Hohenberg equation in one and two dimensions

A widely studied model problem for the pattern formation over unbounded domains is the Swift–Hohenberg equation [SH77,CE90a],

$$\partial_t u = -(1+\Delta)^2 u + \varepsilon^2 u - u^3, \;\; t \geq 0, \;\; x \in \mathbb{R}^d, \;\; u = u(t,x) \in \mathbb{R}. \tag{17}$$

First we consider the case $d = 1$ and below the case $d = 2$. For small $\varepsilon > 0$ there exist stationary roll solutions $u_{\varepsilon,\kappa}$ of (17) with $\kappa \in (-\varepsilon, \varepsilon)$, which bifurcate from $(\varepsilon, u) \equiv (0, 0)$. These rolls have an amplitude $r \approx \widetilde{a} = \widetilde{a}(\varepsilon, \kappa) = \sqrt{4(\varepsilon^2 - \kappa^2)/3}$, are even in x and periodic with period $2\pi/k$, where $k = \sqrt{\kappa + 1}$. They may be expanded as

$$u_{\varepsilon,\kappa}(x) = a_1 \cos(kx) + a_3 \cos(3kx) + O(\widetilde{a}^5), \tag{18}$$

where $a_1 = \widetilde{a} + \widetilde{a}^3/512 + O(\widetilde{a}^5)$ and $a_3 = O(\widetilde{a}^3)$, see e.g. [CE90a,Mie95].

The nonlinear diffusive stability of rolls $u_{\varepsilon,\kappa}$ with respect to spatially localized perturbations has been shown in [Sch96]. Here we outline the ideas. Letting $u(t, x) = u_{\varepsilon,\kappa}(x) + v(t, x)$, the perturbations v have to satisfy

$$\partial_t v = \mathcal{L}v + F(v), \tag{19}$$

where $\mathcal{L}v = -(1+\partial_x^2)^2 v + \varepsilon^2 v - 3u_{\varepsilon,\kappa}^2 v$ and $F(v) = -3u_{\varepsilon,\kappa} v^2 - v^3$. The operator $\mathcal{L}$ can be treated with the Bloch analysis of Section 5.1. The associated Bloch operators are

$$B(\varepsilon, \kappa, l)V \stackrel{\text{def}}{=} -(1+(\partial_x + \mathrm{i}l)^2)^2 V + (\varepsilon^2 - 3u_{\varepsilon,\kappa}^2)V, \tag{20}$$

which are unbounded operators on $L^2_{\text{per}}(\mathcal{T}_{2\pi/k})$ with domain $H^4_{\text{per}}(\mathcal{T}_{2\pi/k})$. For every fixed $l \in \mathcal{T}_k$ the eigenvalue problem (20) is self–adjoint with a discrete set of real eigenvalues $\{\lambda_j^{(\varepsilon,\kappa)}(l) \in \mathbb{R} : j \in \mathbb{N}\}$, $\lambda_j^{(\varepsilon,\kappa)}(l) \geq \lambda_{j+1}^{(\varepsilon,\kappa)}(l) \to -\infty$ for $j \to \infty$.

For $(\varepsilon, \kappa) = (0, 0)$ the eigenvalues are $\lambda_j^{(0,0)}(l) = -(1-(kj+l)^2)^2$, and by perturbation arguments $\lambda_j^{(\varepsilon,\kappa)}(l)$ can be calculated for all small (ε, κ). The stability of $u_{\varepsilon,\kappa}$ is then determined from the behavior of the smooth function $l \mapsto \lambda_1^{(\varepsilon,\kappa)}(l)$ for l close to 0. In fact, we always have $\lambda_1^{(\varepsilon,\kappa)}(0) = 0$. This eigenvalue 0 comes from the fact that we have a family (w.r.t. β) of stationary solutions and that translations along this family lead to the associated eigenvector $\partial_x u_{\varepsilon,\kappa}$. For small l we have the expansion

$$\lambda_1^{(\varepsilon,\kappa)}(l) = -c_1(\varepsilon, \kappa)l^2 + O(l^4), \tag{21}$$

where expansions for the coefficient $c_1(\varepsilon, \kappa)$ are given in [CE90a,Mie95,Mie97b]. If $c_1 < 0$ then $u_{\varepsilon,\kappa}$ is linearly unstable (Eckhaus' sideband instability) which occurs for $\varepsilon^2 < E_{\text{Eckh}}(\kappa) = 3\kappa^2 + O(|\kappa|^3)$. If ε^2 lies above the Eckhaus stability boundary $E_{\text{Eckh}}(\kappa)$, then the rolls are linearly stable.

Moreover, the parabolic expansion (21) of the critical eigenvalue suggests that solutions to the linear problem $v_t = \mathcal{L}v$ decay like solutions to the linear diffusion equation (1).

Regarding the nonlinear equation (19) the nonlinearity doesn't seem to be irrelevant by naive power counting in the sense of Section 3. However, in the spatially periodic case a more elaborate way of power counting involving the Bloch analysis is necessary. This can be understood by relating the problem on the unbounded domain to the center–manifold theory for (17) with periodic boundary conditions. Then via normal form transformations and appropriate convolution identities it turns out that the nonlinearity in Bloch space vanishes up to a sufficiently high order. For details see Section 3 of [Sch98b]. Using the renormalization group approach from Subsection 3.3 one obtains the following theorem.

Theorem 7. *[Sch96] There exists an $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, all Eckhaus–stable rolls $u_{\varepsilon,\kappa}$ and all $p \in (0, 1/2)$ the following holds. There exist $\delta, C > 0$ such that for all initial conditions v_0 satisfying*

$$v_0(x) = \sum_n \varepsilon A_n(\varepsilon x)\mathrm{e}^{\mathrm{i}nx} + \text{c.c.} \quad \textit{with} \quad \sum_n n^2 \|A_n\|_{H^2(2)} < \delta \tag{22}$$

we have a constant $\alpha = O(1)$ such that the solution v of (19) with $v|_{t=0} = v_0$ satisfies

$$\|v(t, x) - \tfrac{\alpha}{\sqrt{t}}\, \mathrm{e}^{-x^2/(4c_1 t)} \partial_x u_{\varepsilon,\kappa}(x)\|_{L^\infty(\mathbb{R})} \le C\varepsilon^{-1+2p} t^{-1+p} \quad \textit{for } t \to \infty,$$

with $c_1 = c_1(\varepsilon, \kappa)$ from (21).

Remark 8. From (22) one obtains that the attracted neighborhood $\mathcal{U}$ of $u_{\varepsilon,\kappa}$ is of order ε in $L^\infty(\mathbb{R})$. Defining

$$\mathcal{B}_1 = \{u : u(x) = \sum_n \varepsilon A_n(\varepsilon x)\mathrm{e}^{\mathrm{i}nx} \text{ with } \sum_n n^2 \|A_n\|_{H^2(2)} < \infty\}$$

we have asymptotic $(\mathcal{B}_1, L^\infty)$ stability of $u_{\varepsilon,\kappa}$. A more refined characterization of $\mathcal{U}$ is given in [Sch96].

The above results have been transfered to the two–dimensional Swift–Hohenberg equation in [Uec99]. Here the rolls considered are $u_{\varepsilon,\kappa}(x_1, x_2) = u^{1\mathrm{d}}_{\varepsilon,\kappa}(x_1)$, i.e., they are independent of x_2. Inserting $u(t, x) = u_{\varepsilon,\kappa}(x_1) + v(t, x)$ into (17) one obtains

$$\partial_t v = \mathcal{L}v + F(v) \tag{23}$$

with $F(v) = -3u_{\varepsilon,\kappa}v^2 - v^3$ as before, but now $\mathcal{L}v = -(1+\Delta)^2 v + \varepsilon^2 v - 3u^2_{\varepsilon,\kappa} v$. The eigenvalue problem $\mathcal{L}v = \lambda v$ can still be treated by the Bloch analysis with $v(x) = \mathrm{e}^{\mathrm{i}l\cdot x} V(x_1)$ where $l = (l_1, l_2) \in \mathcal{T}_k \times \mathbb{R}$ is now a two–dimensional Bloch wave vector. The Bloch operators read

$$B(\varepsilon, \kappa, l)V \stackrel{\text{def}}{=} -(1+(\partial_x+\mathrm{i}l_1)^2 - l_2^2)^2 V + (\varepsilon^2 - 3u^2_{\varepsilon,\kappa})V = \lambda V. \tag{24}$$

Again, the stability properties of $u_{\varepsilon,\kappa}$ are determined by the behavior of the spectral surface $(l_1, l_2) \mapsto \lambda_1^{(\varepsilon,\kappa)}(l_1, l_2)$. Rigorous stability results and a complete characterization of the set of unstable Bloch wave vectors $\{l \in \mathcal{T}_k \times \mathbb{R} \mid \lambda_1^{(\varepsilon,\kappa)}(l) > 0\}$ in dependence of (ε, κ) have been obtained in [Mie97b]. In addition to the Eckhaus instability for $\varepsilon^2 < E_{\mathrm{Eckh}}(\kappa)$ there occurs a second instability mechanism called zigzag–instability. This means instability with respect to Bloch waves $\mathrm{e}^{\mathrm{i}l\cdot x}V(x_1)$ with wave vectors $l = (0, l_2)$ with small $l_2 \neq 0$. The linearized stability results may be summarized as follows.

Theorem 9. *[Mie97b] There exist an* $\varepsilon_1 > 0$ *and curves* $E_{\rm Eckh}, K_{\rm zigzag}$ *with expansions* $E_{\rm Eckh}(\kappa) = 3\kappa^2 - \kappa^3 + O(\kappa^4)$ *and* $K_{\rm zigzag}(\varepsilon) = -\varepsilon^4/512 + O(\varepsilon^6)$ *such that a roll* $u_{\varepsilon,\kappa}$ *with* $\varepsilon \in (0, \varepsilon_1]$ *is linearly stable if and only if*

$$\varepsilon^2 > E_{\rm Eckh}(\kappa) \quad \text{and} \quad \kappa > K_{\rm zigzag}(\varepsilon).$$

In this case the surface $\lambda_1^{(\varepsilon,\kappa)}$ *of the largest eigenvalue of (24) has the expansion*

$$\lambda_1^{(\varepsilon,\kappa)}(l) = -c_1(\varepsilon,\kappa)l_1^2 - c_2(\varepsilon,\kappa)l_2^2 + O(|l|^4) \quad \text{for } l \to 0 \tag{25}$$

with $c_j(\varepsilon,\kappa) > 0$ *for* $j = 1, 2$.

In [Uec99] these results are combined with the techniques developed in [Sch96] to show the nonlinear diffusive stability of marginally stable rolls. In order to obtain estimates on the size of the domain of attraction we need to take care of the dependence of (c_1, c_2) on (ε, κ). Here it is useful to consider the special parameterization $\kappa = \beta\varepsilon$ with $\beta \in (0, 1/\sqrt{3})$ and the transformed parameter set $\mathcal{P} = (0, 1/\sqrt{3}) \times (0, \varepsilon_1)$. With the definition (5) adapted to $H^m(k) = \{u : \mathbb{R}^2 \to \mathbb{C} : \|u\|_{H^m(k)} = \|u\rho^k\|_{H^m(\mathbb{R}^2)} < \infty\}$, $\rho(x) = (1 + |x|^2)^{1/2}$, we have the following result.

Theorem 10. *[Uec99] There exist continuous functions* $\delta, C : (0, 1/\sqrt{3}) \to \mathbb{R}_+$ *and a continuous function* $A : \mathcal{P} \times H^2(3) \to \mathbb{R}$ *such that for all* $(\beta, \varepsilon) \in \mathcal{P}$ *the following holds. Let* $\kappa = \beta\varepsilon$ *and let* $v = v(t, x)$ *be the solution to (23) with the initial condition* v_0 *satisfying*

$$v_0(x) = \varepsilon^{3/2} \sum_{n \geq 0} A_n(\varepsilon x_1, \sqrt{\varepsilon} x_2) e^{inx_1} + \text{c.c.} \tag{26}$$

$$\text{with} \quad \sum_{n \geq 0} (1+n^2) \|A_n\|_{H^2(3)} \leq \delta(\beta). \tag{27}$$

Then we have

$$\|v(t, x) - \tfrac{\alpha}{\sqrt{c_1 c_2}\, t}\, e^{-[d_1 x_1^2 + d_2 x_2^2]/(4t)} \partial_{x_1} u_{\varepsilon,\kappa}^{\rm 1d}(x_1)\|_{L^\infty(\mathbb{R}^2)} \leq C(\beta)\varepsilon^{-3/2} t^{-3/2},$$

with $\alpha = A(\beta, \varepsilon, v_0)$ *and* $d_j = 1/c_j(\varepsilon, \beta\varepsilon)$ *from (25),* $j = 1, 2$.

Remark 11. Similar to Remark 8 one obtains from (26) that in two dimension the attracted neighborhood $\mathcal{U}$ of $u_{\varepsilon,\kappa}$ is of order $\varepsilon^{3/2}$ in $L^\infty(\mathbb{R}^2)$, see [Uec99].

5.3 Application to hydrodynamical stability problems

With the above methods it was possible to solve a class of hydrodynamical stability problems on unbounded domains which have been open for almost 30 years, namely the nonlinear stability of linearly Eckhaus–stable spatially periodic equilibria in infinite cylinders with respect to spatially localized perturbations. The linear stability analysis leads to a similar situation as for the Swift–Hohenberg equation, i.e. continuous spectrum up to the imaginary axis and no obvious sign for the nonlinear terms.

The physical problems which we have in mind are the Taylor–Couette problem and Rayleigh–Bénard's problem. The Taylor–Couette problem consists in finding the flow of a viscous incompressible fluid filling the domain between two rotating infinitely extended cylinders. Bénard's problem consists in finding the flow of a viscous incompressible fluid filling an infinitely extended strip subjected to some heating from below.

In both problems the velocity field is governed by the Navier–Stokes equations, and in both problems there exist a spatially homogeneous flow, the Couette flow and pure heat conduction, respectively, which gets unstable and bifurcates into a family of spatially periodic equilibria, the Taylor vortices and roll solutions, respectively.

The result for the Taylor–Couette problem is formulated in [Sch98b]. For completeness we will give here an explicit formulation of the nonlinear diffusive stability result for roll solutions in the Rayleigh–Bénard problem. The linear Bloch theory with one unbounded direction was first studied in [KvW97] and that with two unbounded directions in [Mie97c]. Here we only treat the two–dimensional problem in the strip $\Omega = \mathbb{R} \times (0, \pi)$. The velocity field $u = (u_1, u_2)$, the temperature T and the pressure p satisfy

$$\begin{aligned} \partial_t u &= \nu \Delta u - \nabla p - T \boldsymbol{e}_2 - (u \cdot \nabla) u \\ \partial_t T &= \kappa \Delta T + u_2 - (u \cdot \nabla) T \\ 0 &= \nabla \cdot u \end{aligned}$$

for all $(x, y) \in \Omega$, with $\Delta = \partial_x^2 + \partial_y^2$, the mean flux condition $\int_0^\pi u_1 \, \mathrm{d}y = 0$ and the boundary conditions

$$\partial_y u_1 = u_2 = 0 \ \text{ for } \ y = 0, \pi, \quad T = T_0 \ \text{ for } \ y = 0, \ \text{ and } \ T = T_1 \ \text{ for } \ y = \pi$$

where $T_0 < T_1$.

There exists a trivial spatially homogeneous solution $u = 0, T = T_0 + y(T_1 - T_0)/\pi$, which becomes unstable when the parameter $\mu = T_1 - T_0$ is sufficiently large. Then, a one–dimensional family of spatially periodic equilibria $(u_{q,\mu}, T_{q,\mu})$ with

$$(u_{q,\mu}, T_{q,\mu})(x, y) = (u_{q,\mu}, T_{q,\mu})(x + 2\pi/q, y),$$

bifurcates, where the horizontal wave number q lies in the interval $(q_{\text{ex}}^-(\mu), q_{\text{ex}}^+(\mu))$. The linear stability analysis leads to a similar situation as for the Swift–Hohenberg equation. We have the linear Eckhaus–stability for all $(u_{q,\mu}, T_{q,\mu})$ with

$$q \in (q_{\text{stab}}^-(\mu), q_{\text{stab}}^+(\mu)) \subset (q_{\text{ex}}^-(\mu), q_{\text{ex}}^+(\mu)).$$

In [Sch98b] the following result has been proved.

Theorem 12. *There exist* $\varepsilon, C > 0$ *such that the following holds. Let the initial condition* $(u,T)|_{t=0} = (u_{q,\mu}, T_{q,\mu}) + (v,\theta)|_{t=0}$ *satisfy* $q \in (q^-_{\rm stab}(\mu), q^+_{\rm stab}(\mu))$ *and* $\|(v,\theta)\|_{H^2(2)} \leq \varepsilon$. *Then the associated solution* $(u,T) = (u_{q,\mu}, T_{q,\mu}) + (v,\theta)$ *satisfies*

$$\left\| \begin{pmatrix} v \\ \theta \end{pmatrix} - \frac{A^*}{\sqrt{t}} \mathrm{e}^{-dx^2/t} \partial_x \begin{pmatrix} u_{q,\mu} \\ T_{q,\mu} \end{pmatrix} \right\|_{L^\infty} \leq \frac{C}{t} \quad \textit{for all } t > 0,$$

where $d = d(q,\mu) > 0$ *and* $A^* \in \mathbb{R}$ *is a constant depending on the initial condition.*

In a different situation the diffusive stability method has been used to prove the nonlinear stability of Kolmogorov flow (cf. [Sch99a]).

5.4 An open problem: Diffusive mixing in pattern forming systems

In the last section we explained how diffusive behavior occurs near spatially periodic equilibria in pattern forming systems. If such steady state solutions are perturbed in a spatially localized manner, the perturbations are repaired diffusively and decay to 0 algebraically in t as $t \to \infty$.

For the Ginzburg–Landau equation we additionally know that diffusive mixing takes place if two different stable steady states are prescribed for $x \to -\infty$ and $x \to \infty$, see Section 4.2. Although a proof of diffusive mixing of steady states in general pattern forming systems is still missing, it is widely expected that such a behavior occurs. For the one dimensional Swift–Hohenberg equation this conjecture is as follows.

Conjecture: *Fix* $\varepsilon > 0$ *sufficiently small and let* u_{ε,k_-}, u_{ε,κ_+} *be two stable rolls with* $\kappa_- \neq \kappa_+$. *Then there exist limiting profiles* $\tilde{k} \in C^4_b(\mathbb{R})$ *and* $\tilde{U} \in C^4(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ *with* $\tilde{k}(\xi) \to k_\pm = \sqrt{\kappa_\pm + 1}$ *for* $\xi \to \pm\infty$ *and*

$$\|\tilde{k}(\cdot) - (k_- + (k_+ - k_-)\mathrm{Erf}(\cdot))\|_{C^4} \leq C\varepsilon^2$$
$$\|\tilde{U}(t,\cdot) - a_1(\varepsilon, \tilde{k}(\cdot/\sqrt{t}))\cos(\sqrt{t}\,\tilde{\phi}(\cdot/\sqrt{t}))\|_{C^4} \leq C\varepsilon^3,$$

where $\tilde{\phi}(\xi) = k_-\xi + \int_{-\infty}^{\xi} [\tilde{k}(s) - k_-]\,\mathrm{d}s$ *and* $a_1(\varepsilon, k)$ *from (18), such that the following holds. There exist* $t_0 > 0$, $\delta > 0$ *such that for all* $u_0 \in H^2_{\rm lu}(\mathbb{R})$ *with* $\|u_0 - \tilde{U}(t_0,\cdot)\|_{H^2(2)} \leq \delta$ *we have*

$$\|u(t,\cdot) - \tilde{U}(t,\cdot)\|_{L^\infty} \leq Ct^{-1/3}.$$

In particular, this implies that for all fixed $\ell > 0$,

$$\sup_{|x| \leq \ell} |u(t,x) - u_{\varepsilon,k_*^2-1}(x + \sqrt{t}\,\phi_*)| \leq C_\ell t^{-1/3},$$

where $k_* = \tilde{k}(0) = \frac{1}{2}(k_+ + k_-) + O(\varepsilon^2)$ *and* $\phi_* = \tilde{\phi}(0) = (k_+ - k_-)/\sqrt{\pi} + O(\varepsilon^2)$.

For Bénard's problem a similar conjecture can be stated. The proof of the diffusive repair of the steady states heavily relies on the Bloch wave analysis. So far it is not known how to transfer it to the diffusive mixing case, where we have to combine the Bloch wave analysis of the spatially periodic steady states with a local analysis in x–space to prove these conjectures.

References

[BK92] J. Bricmont and A. Kupianen. Renormalization group and the Ginzburg–Landau equation. *Comm. Math. Phys.*, 150:287–318, 1992.

[BKL94] J. Bricmont, A. Kupianen, and G. Lin. Renormalization group and asymptotics of solutions of nonlinear parabolic equations. *Comm. Pure Appl. Math.*, 6:893–922, 1994.

[BL99a] S. Benachour and P. Laurençot. Global solutions to viscous Hamilton-Jacobi equations with irregular initial data. *Comm. Partial Differential Equations*, 24(11-12):1999–2021, 1999.

[BL99b] W.-J. Beyn and J. Lorenz. Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals. *Numer. Funct. Anal. Optimization*, 20(3-4):201–244, 1999.

[BV90] A.V. Babin and M.I. Vishik. Attractors of partial differential equations in an unbounded domain. *Proc. Royal Soc. Edinb.*, 116A:221–243, 1990.

[CE90a] P. Collet and J.-P. Eckmann. *Instabilities and Fronts in Extended Systems*. Princeton University Press, 1990.

[CE90b] P. Collet and J.-P. Eckmann. The time dependent amplitude equation for the Swift–Hohenberg problem. *Comm. Math. Phys.*, 132:139–152, 1990.

[CE99a] P. Collet and J.-P. Eckmann. The definition and measurement of the topological entropy per unit volume in parabolic PDEs. *Nonlinearity*, 12:451–473, 1999.

[CE99b] P. Collet and J.-P. Eckmann. Extensive properties of the complex Ginzburg–Landau equation. *Comm. Math. Physics*, 200:699–722, 1999.

[CEE92] P. Collet, J.-P. Eckmann, and H. Epstein. Diffusive repair for the Ginzburg–Landau equation. *Helv. Phys. Acta*, 65:56–92, 1992.

[DFKM96] G. Dangelmayr, B. Fiedler, K. Kirchgässner, and A. Mielke. *Dynamics of Nonlinear Waves in Dissipative Systems: Reduction, Bifurcation and Stability*, volume 352 of *Pitman Research Notes in Mathematics*. Addison Wesley Longman, 1996.

[Eck65] W. Eckhaus. *Studies in Non–Linear Stability Theory*. Springer Tracts in Nat. Phil. Vol.6, 1965.

[Eck93] W. Eckhaus. The Ginzburg–Landau manifold is an attractor. *J. Nonlinear Science*, 3:329–348, 1993.

[ES98] J. Escher and B. Scarpellini. Stability properties of parabolic equations in unbounded domains. *Arch. Math.*, 71(1):31–45, 1998.

[ES99a] J.-P. Eckmann and G. Schneider. Nonlinear stability of modulating fronts in the Swift–Hohenberg equation. Preprint, Universität Bayreuth, 1999.

[ES99b] M. Efendiev and A. Scheel. Upper and lower bounds for the Hausdorff dimension of the attractor for reaction–diffusion equations in $\mathbb{R}^n$. *RIMS Kokyuroku*, 1105:81–90, 1999.

[EW94a] J.-P. Eckmann and C.E. Wayne. The nonlinear stability of front solutions for parabolic partial differential equations. *Comm. Math. Phys.*, 161(2):323–334, 1994.

[EW94b] J.-P. Eckmann and C.E. Wayne. The nonlinear stability of front solutions for parabolic partial differential equations. *Comm. Math. Phys.*, 161(2):323–334, 1994.

[EZ99] M. Efendiev and S. Zelik. The attractor for a nonlinear reaction–diffusion system in an unbounded domain. Preprint no. 505, WIAS Berlin, 1999.

[Fei96] E. Feireisl. Bounded, locally compact global attractors for semilinear damped wave equations on $\mathbb{R}^n$. *J. Diff. Integral Eqns.*, 9:1147–1156, 1996.

[Gal94] T. Gallay. Local stability of critical fronts in parabolic partial differential equations. *Nonlinearity*, 7:741–764, 1994.

[GM98] T. Gallay and A. Mielke. Diffusive mixing of stable states in the Ginzburg–Landau equation. *Comm. Math. Phys.*, 199:71–97, 1998.

[GR97] T. Gallay and G. Raugel. Stability of travelling waves for a damped hyperbolic equation. *Z. Angew. Math. Phys.*, 48(3):451–479, 1997.

[GR98] T. Gallay and G. Raugel. Scaling variables and asymptotic expansions in damped wave equations. *J. Diff. Eq.*, 150(1):42–97, 1998.

[HCS99] M. Haragus-Courcelle and G. Schneider. Bifurcating fronts for the Taylor–Couette problem in infi nite cylinders. *ZAMP*, 50:120–151, 1999.

[Kap94] T. Kapitula. On the nonlinear stability of plane waves for the Ginzburg–Landau equation. *Comm. Pure Appl. Math.*, 47:831–841, 1994.

[KS98] T. Kapitula and B. Sandstede. Stability of bright and dark solitary–wave solutions to perturbed nonlinear Schrödinger equations. *Physica D*, 124:58–103, 1998.

[KSM92] P. Kirrmann, G. Schneider, and A. Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. *Proc. Royal Soc. Edinburgh*, 122A:85–91, 1992.

[KvW97] Y. Kagei and W. von Wahl. The Eckhaus criterion for convection roll solutions of the Oberbeck–Boussinesq equations. *Int. J. Non–Linear Mechanics*, 32:563–620, 1997.

[Mie95] A. Mielke. A new approach to sideband instabilities using the principle of reduced instability. In A. Doelman and A. van Harten, editors, *Nonlinear dynamics and pattern formation in the natural environment*, pages 206–222. Longman UK, 1995.

[Mie97a] A. Mielke. The complex Ginzburg–Landau equation on large and unbounded domains: sharper bounds and attractors. *Nonlinearity*, 10:199–222, 1997.

[Mie97b] A. Mielke. Instability and stability of rolls in the Swift–Hohenberg equation. *Comm. Math. Phys.*, 189:829–853, 1997.

[Mie97c] A. Mielke. Mathematical analysis of sideband instabilities with applicaton to Rayleigh–Bénard convection. *J. Nonlinear Science*, 7:57–99, 1997.

[Mie99] A. Mielke. The Ginzburg–Landau equation in its role as a modulation equation. In B. Fiedler, G. Iooss, N. Kopell, and F. Takens, editors, *Handbook of Dynamical Systems III. Towards Applications*. Springer–Verlag, 1999. To appear.

[MS95] A. Mielke and G. Schneider. Attractors for modulation equations on unbounded domains — existence and comparison. *Nonlinearity*, 8(5):743–768, 1995.

[MS96] A. Mielke and G. Schneider. Derivation and justifi cation of the complex Ginzburg–Landau equation as a modulation equation. In Percy Deift, C. David Levermore, and C. Eugene Wayne, editors, *Lect. Appl. Math. 31*, pages 191–216. American Mathematical Society, 1996.

[MSZ00] A. Mielke, G. Schneider, and A. Ziegra. Comparison of inertial manifolds and application to modulated systems. *Math. Nachrichten*, 214:53–69, 2000.

[OZ00] J. H. Ortega and E. Zuazua. Large time behavior in $\mathbb{R}^n$ for linear parabolic equations with periodic coeffi cients. *Asymptot. Anal.*, 22(1):51–85, 2000.

[RK98] G. Raugel and K. Kirchgässner. Stability of fronts for a KPP-system. II. The critical case. *J. Diff. Eqns.*, 146(2):399–456, 1998.

[RS78] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV*. Academic Press, New York, 1978.

[Sca99] B. Scarpellini. *Stability, Instability, and Direct Integrals*, volume 402 of *Pitman Research Notes in Mathematics*. Chapman & Hall/CRC, 1999.

[Sch94a] G. Schneider. Error estimates for the Ginzburg–Landau approximation. *ZAMP*, 45:433–457, 1994.

[Sch94b] G. Schneider. Global existence via Ginzburg–Landau formalism and pseudo–orbits of Ginzburg–Landau approximations. *Comm. Math. Phys.*, 164:157–179, 1994.

[Sch94c] G. Schneider. A new estimate for the Ginzburg–Landau approximation on the real axis. *J. Nonlinear Science*, 4:23–34, 1994.

[Sch96] G. Schneider. Diffusive stability of spatial periodic solutions of the Swift–Hohenberg equation. *Comm. Math. Phys.*, 178:679–702, 1996.

[Sch98a] G. Schneider. Hopf–bifurcation in spatially extended reaction–diffusion systems. *J. Nonlinear Science*, 8:17–41, 1998.

[Sch98b] G. Schneider. Nonlinear stability of Taylor–vortices in infinite cylinders. *Arch. Rational Mech. Analysis*, 144(2):121–200, 1998.

[Sch99a] G. Schneider. Cahn–Hilliard description of secondary flows of a viscous incompressible fluid in an unbounded domain. *ZAMM*, 79:615–626, 1999.

[Sch99b] G. Schneider. Global existence results for pattern forming processes in infinite cylindrical domains – applications to 3d Navier–Stokes problems –. *J. Mathématiques Pures Appliquées*, 78:265–312, 1999.

[Sch00] G. Schneider. Existence and stability of modulating pulse solutions in a phenomenological model of nonlinear optics. *Physica D*, 140(3–4):283–293, 2000.

[SH77] J. Swift and P.C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Physical Review A*, 15(1):319–328, 1977.

[SS98] B. Sandstede and A. Scheel. Spectral stability of modulated travelling waves bifurcating near essential instabilities. *Proc. Roy. Soc. Edinb.*, 1998. Submitted (DANSE Preprint 56/98).

[Uec99] H. Uecker. Diffusive stability of rolls in the two–dimensional real and complex Swift–Hohenberg equation. *Comm. PDE*, 24(11–12):2109–2146, 1999.

[Uec00] H. Uecker. Stable modulating multi–pulse solutions for dissipative systems with resonant spatially periodic forcing. *J. Nonlin. Sci.*, 2000. To appear.

[vH91] A. van Harten. On the validity of Ginzburg–Landau's equation. *J. Nonlinear Science*, 1:397–422, 1991.

[Way97] C.E. Wayne. Invariant manifolds for parabolic partial differential equations on unbounded domains. *Arch. Rational Mech. Analysis*, 138(3):279–306, 1997.

[Zel99] S. Zelik. The attractor for nonlinear reaction-diffusion system in the unbounded domain and Kolmogorov's ε–entropy. *Mathem. Nachr.*, 1999. Submitted (DANSE Preprint 34/99).