Almost global existence and transient self similar decay for Poiseuille flow at criticality for exponentially long times

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Abstract

We consider nonlinear diffusion equations with critical exponent as $\partial_t u = \partial_x^2 u + u^3$ with $x \in \mathbb{R}$ for small initial data in $L^1 \cap L^\infty$. It is well known that almost all solutions of this system explode in finite time. However, we make the observation that in terms of the norm of the initial conditions it takes an exponentially long time. Moreover, before explosion the L^∞-norm of such solutions becomes exponentially small which makes it almost impossible to observe the instability in experiments. As an application we consider the long time transient self similar decay to unstable Poiseuille flow at criticality for exponentially long times. This, together with a subcritical bifurcation and short time transient amplification, is a principal obstruction in all attempts to measure the critical Reynolds number for this experiment more and more precisely.

1 Introduction

There exists a number of situations in the theory of ordinary and partial differential equations in which an instability manifests after an exponentially long time in terms of the norm of the perturbation. Famous examples are Arnold diffusion, also called the theorem of Nehorosev [Ne77], which shows that in nearly integrable Hamiltonian systems it takes an exponentially long time for a solution to diffuse through the remaining tori in terms of the perturbation parameter, the long time existence of solutions on a time interval of length of order $\mathcal{O}(\exp(1/\varepsilon))$ for nonlinear wave equations [JK84] with initial data of order $\mathcal{O}(\varepsilon)$, and the exponentially slow evolution of interfaces in bistable nonlinear diffusion equations with ε being the order of the diffusion coefficient [CP90].

It is the purpose of this paper to outline a similar phenomenon in nonlinear diffusion equations which to our knowledge has not been documented explicitely before. For nonlinear diffusion equations as

$$\partial_t u = \partial_x^2 u + u^p, \quad u|_{t=0} = u_0, \quad t \geq 0, \; x \in \mathbb{R}, \quad u = u(x,t) \in \mathbb{R}, \tag{1}$$

for small positive initial data in $L^1 \cap L^\infty$ it has been known for a long time that all solutions explode in finite time for $p \leq 3$ [We81]. For $p > 3$ solutions to small initial data in $L^1 \cap L^\infty$ exist globally in time, where the L^1-norm stays bounded and the L^∞-norm decays with a polynomial rate $\mathcal{O}(t^{-1/2})$; see for instance [BKL94, Wa97, STW01].

Our interest is exactly in the threshold $p = 3$, where the trivial solution $u \equiv 0$ has not yet changed from unstable to stable in $L^1 \cap L^\infty$. Our result is as follows. Take an initial condition in $L^1 \cap L^\infty$ of norm less than ε with $\varepsilon > 0$ sufficiently small. Then the associated solution will be less than 4ε for all $t \in [0, \exp(q/\varepsilon^2)]$ with a constant $q > 0$ independent of ε, i.e. in terms of the norm of the initial conditions it takes an exponentially long time for the solution to leave a neighborhood of the origin. This instability is almost not observable since on this very long time interval the solution decays with a rate $\mathcal{O}(t^{-1/2})$ in L^∞, i.e. for $t = \exp(q/\varepsilon^2)$ in L^∞ the solution has a norm of order $\mathcal{O}(\exp(-q/\varepsilon^2))$. Before the explosion takes place the solution becomes flatter and flatter until the mass, i.e. the L^1-norm, is sufficiently big to start the explosion.

The proof which is given in Section 2 is remarkably easy and goes along the lines of the global existence proof for $p > 3$.

This observation is not restricted to the above equation (1). Another example would be

$$\partial_t u = \Delta u + u^2$$

for $x \in \mathbb{R}^2$ or more general

$$\partial_t u = (-1)^{s+1} (\Delta)^s u + u^p$$

for $x \in \mathbb{R}^d$, $s \in \mathbb{N}$ with $p = 1 + 2s/d$. The phenomenon is robust under adding higher order terms, i.e. it also holds for

$$\partial_t u = \partial_x^2 u + u^3 + H(u, \partial_x u, \partial_x^2 u)$$

where $|H(u, \partial_x u, \partial_x^2 u)| \leq C(|u|^4 + |\partial_x u|^2 + |u||\partial_x^2 u|)$, with a constant C, is called an irrelevant nonlinearity. For the exact definition of an irrelevant nonlinearity see [BKL94]. Adding a term $u\partial_x u$ affects the asymptotic behavior [BKL94], but not the transient decay rates and the above phenomenon. However, the proof in this case will be less trivial.

Diffusive behavior is not restricted to obvious diffusion operators [Schn96, Schn98, Uec99], and whenever the lowest order nonlinear terms are critical in the above sense we can expect the above long time stability, where after a time $\mathcal{O}(\exp(1/\varepsilon^{\gamma}))$, for a $\gamma > 0$, an explosion may occur.

There are a number of real world applications of this phenomenon. The example which we consider in this paper in Section 3 is unstable Poiseuille flow at criticality. For this classical hydrodynamical stability problem the basic laminar flow becomes unstable at a critical Reynolds number $\mathcal{R}_c$ in a subcritical bifurcation which makes the measurement of the critical Reynolds number $\mathcal{R}_c$ a delicate experiment. Moreover, perturbations are transported by the flow and may be amplifi ed on a short transient time scale due to the non-normality of the linearization around the laminar flow, see e.g.[SH94]. Our result is another principal obstruction in all attempts to measure the critical Reynolds number $\mathcal{R}_c$ more and more precisely. The better the experiment is performed the more and more the above observation plays a role and the longer it takes to observe the growth of localized perturbations. After a short time transient growth due to the non–normality of the linearization, the solutions seem to decay for a very long time, i.e. at criticality for an exponentially long time.

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2 Formulation and proof of the result for the nonlinear diffusion equation with critical exponent

In section 2.1 we formulate and prove the result for the nonlinear diffusion equation (1) with critical exponent, using simple $L^1 \cap L^\infty$ estimates in order to make clear where the exponentially long times come from. In section 2.2 we reconsider (1) using discrete renormalization to introduce a robust method which is applicable to Poiseuille flow, too.

2.1 Direct L^1-L^∞-estimates

The following presentation is based on [MSU01] where the case $p > 3$ has been explained.

Theorem 2.1 *Let $p = 3$. There exist positive constants q and ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. For initial condition u_0 with $\|u_0\|_{L^1} + \|u_0\|_{L^\infty} \leq \varepsilon$ the associated solution $u = u(t)$ of (1) with $u(0) = u_0$ exists for all $t \in [0, \exp(q/\varepsilon^2)]$ and satisfies*

$$\sup_{t\in[0,\exp(q/\varepsilon^2)]} \|u(t)\|_{L^1\cap L^\infty} \leq 4\varepsilon.$$

Proof. The solution of the linear diffusion equation

$$\partial_t u = \partial_x^2 u, \quad u|_{t=0} = u_0, \;\; x \in \mathbb{R}, \;\; t \geq 0, \;\; u(x,t) \in \mathbb{R}, \tag{2}$$

can be written explicitly as

$$u(x,t) = \tfrac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2/(4t)} u_0(y)\mathrm{d}y = \int_{\mathbb{R}} G(x-y,t) u_0(y)\mathrm{d}y \tag{3}$$

which is abbreviated as $u(t) = e^{\partial_x^2 t} u_0$. By Young's inequality for convolutions with $p > q$ we obtain

$$\|u(t)\|_{L^p} \leq C t^{-1/(2r)} \|u_0\|_{L^q}, \quad \text{where} \quad 1/p = 1/q - 1/r, \; 1 \leq p, r \leq \infty, \tag{4}$$

for some constant C independent of time. Next we consider the variation of constant formula

$$u(t) = \mathrm{e}^{\partial_x^2 t} u_0 + \int_0^t \mathrm{e}^{\partial_x^2 (t-s)} u^3(s)\mathrm{d}s$$

for (1). With $\|u^3\|_{L^\infty} \leq \|u\|_{L^\infty}^3$ and $\|u\|_{L^1}^3 \leq \|u\|_{L^\infty}^2 \|u\|_{L^1}$, the abbreviations

$$a(t) = \sup_{0 \leq s \leq t} \|u(s)\|_{L^1} \quad \text{and} \quad b(t) = \sup_{0 \leq s \leq t} \|(1+s)^{1/2} u(s)\|_{L^\infty}$$

and the estimate (4) we obtain

$$\begin{aligned} (1+t)^{1/2} \left\| \int_0^t \mathrm{e}^{\partial_x^2 (t-s)} u^3(s)\mathrm{d}s \right\|_{L^\infty} &\leq (1+t)^{1/2} \int_0^t \|\mathrm{e}^{\partial_x^2 (t-s)}\|_{L^1 \to L^\infty} \|u^3(s)\|_{L^1}\mathrm{d}s \\ \leq (1+t)^{1/2} \int_0^t (t-s)^{-1/2} (1+s)^{-1}\mathrm{d}s \cdot b(t)^2 a(t) &\leq C_1 \ln(1+t) b(t)^2 a(t) \end{aligned}$$

with a constant C_1 independent of t. Furthermore, we have

$$\begin{aligned} \left\| \int_0^t \mathrm{e}^{\partial_x^2 (t-s)} u^3(s)\mathrm{d}s \right\|_{L^1} &\leq \int_0^t \|e^{\partial_x^2 (t-s)}\|_{L^1 \to L^1} \|u^3(s)\|_{L^1}\mathrm{d}s \\ \leq C_2 \int_0^t (1+s)^{-1}\mathrm{d}s \cdot b(t)^2 a(t) &\leq C_2 \ln(1+t) b(t)^2 a(t) \end{aligned}$$

with a constant C_2 independent of t. Together we obtain

$$a(t) \leq a(0) + C_1 \ln(1+t) b(t)^2 a(t), \quad b(t) \leq b(0) + a(0) + C_2 \ln(1+t) b(t)^2 a(t).$$

We introduce $\tilde{a}(t)$ and $\tilde{b}(t)$ by $a(t) = \varepsilon \tilde{a}(t)$ and $b(t) = \varepsilon \tilde{b}(t)$ which satisfy

$$\begin{aligned} \tilde{a}(t) &\leq \tilde{a}(0) + C_1 \varepsilon^2 \ln(1+t) \tilde{b}(t)^2 \tilde{a}(t), \\ \tilde{b}(t) &\leq \tilde{b}(0) + \tilde{a}(0) + C_2 \varepsilon^2 \ln(1+t) \tilde{b}(t)^2 \tilde{a}(t). \end{aligned}$$

If $\tilde{a}(0) + \tilde{b}(0) \leq 1$ we have $\tilde{a}(t) + \tilde{b}(t) \leq 4$ as long as $\max\{C_1, C_2\} \varepsilon^2 \ln(1+t) \leq 1/64$, i.e. for all $t \in (0, \exp(q/\varepsilon^2))$ for a positive constant q. □

We have additionally proved

Proposition 2.2 *Under the assumptions of Theorem 2.1 we have a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that*

$$\|u(\cdot, t)\|_{L^\infty} \leq C\varepsilon(1+t)^{-1/2}$$

for all $t \in [0, \exp(q/\varepsilon^2)]$, i.e.

$$\|u(\cdot, \exp(q/\varepsilon^2))\|_{L^\infty} \leq \tilde{C}\varepsilon \exp(-q/(2\varepsilon^2))$$

with another constant $\tilde{C} > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$.

2.2 Renormalization

To explain the method used later for the Poiseuille problem and to thus make the paper somewhat self-contained, we reconsider the long time behavior of (1) using discrete renormalization [BK92]. For notational convenience we assume the initial conditions for (1) to be given at $t = 1$. Moreover we assume that

$$u(\cdot, 1) \in H_2^2 \quad \text{with} \quad \|u(\cdot, 1)\|_{H_2^2} = \mathcal{O}(\varepsilon), \tag{5}$$

where, for $j, m \in \mathbb{N}$, the Hilbert spaces H_j^m are defined as

$$H_j^m := \{u \in H^m(\mathbb{R}) : \|u\|_{H_j^m}^2 = \sum_{i=0}^{m} \|(\partial_x^i u)\rho^j\|_{L^2}^2 < \infty\}, \quad \rho(x) = (1+x^2)^{1/2}.$$

Fourier transform $\hat{u}(k, t) = \mathcal{F}u(t)(k) = \frac{1}{\sqrt{2\pi}} \int e^{ikx} u(x, t) dx$ is an isomorphism between H_j^m and H_m^j.

In Fourier space (1) becomes $\partial_t \hat{u}(k, t) = -k^2 \hat{u}(k, t) + \hat{u}^{*3}(k, t)$, where $\hat{u}^{*3} = \hat{u} * \hat{u} * \hat{u}$ and $(\hat{u} * \hat{v})(k) = \int_{\mathbb{R}} \hat{u}(k-\ell)\hat{u}(\ell) d\ell$. With $\sigma \in (0, 1)$, we let $\hat{u}_n(\varkappa, \tau) = \hat{u}(\sigma^n \varkappa, \sigma^{-2n}\tau)$ and obtain

$$\partial_T \hat{u}_n = -\varkappa^2 \hat{u}_n + \hat{u}_n^{*3} \tag{6}$$

due to the scaling invariance of (1). Hence solving (1) on the time interval $t \in [1, \sigma^{-2n}]$ is equivalent to iterating n times the renormalization process:

$$\text{solve (6) on the time interval } \tau \in [\sigma^2, 1] \text{ with initial datum } u_n(\sigma^2, \varkappa) = u_{n-1}(1, \sigma\varkappa). \tag{7}$$

The variation of constant for (7) reads

$$\hat{u}_n(\varkappa, 1) = e^{-\varkappa^2(1-\sigma^2)} \hat{u}_{n-1}(\sigma\varkappa, 1) + \int_{\sigma^2}^{1} e^{-\varkappa^2(1-s)} \hat{u}_n^{*3}(s, \varkappa) ds. \tag{8}$$

Obviously we have $\|\hat{u}^{*3}\|_{H^2(2)} \leq C\|\hat{u}\|_{H^2(2)}^3$ and

$$\|\hat{u}(\sigma\cdot)\|_{H^2(2)} \leq C\sigma^{-5/2}\|\hat{u}\|_{H^2(2)} \tag{9}$$

with C independent of σ. Introducing

$$\rho_n = \|\hat{u}_n(1)\|_{H^2(2)} \quad \text{and} \quad R_n = \sup_{\tau\in[\sigma^2,1]} \|\hat{u}_n(\tau)\|_{H^2(2)}, \tag{10}$$

a standard contraction mapping argument yields

Lemma 2.3 *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $n \in \mathbb{N}$ the following holds. If $\rho_{n-1} \leq \varepsilon_0$ then there exists a unique solution $\hat{u}_n \in C([\sigma^2, 1], H^2_2)$ of (8) with $R_n \leq C\sigma^{-5/2}\rho_{n-1}$.*

We now decompose $\hat{u}_n(\cdot, 1)$ as

$$\hat{u}_n(\varkappa, 1) = A_n \mathrm{e}^{-\varkappa^2} + \hat{r}_n(\varkappa) \quad \text{with} \quad A_n = \Pi\hat{u}_n(\cdot, 1) = \hat{u}_n(0, 1),$$

where $\Pi : H^2_2 \to \mathbb{R}$ is well defined due to Sobolev embedding. Moreover we have

$$\|\mathrm{e}^{-(1-\sigma^2)\varkappa^2}\hat{r}(\sigma\cdot)\|_{H^2_2} \leq C\sigma\|\hat{r}(\cdot)\|_{H^2_2} \quad \text{if} \quad \hat{r}(0) = 0, \tag{11}$$

i.e. $\mathrm{e}^{-(1-\sigma^2)\varkappa^2}\hat{r}(\sigma\cdot)$ is a contraction when acting on functions that vanish at $k = 0$. Here the smoothness in Fourier space is crucial. Applying Π to (8) gives $|A_{n+1}-A_n| \leq CR_n^3$ and $\rho_{n+1} \leq C\sigma\rho_n + CR_n^3$ and with Lemma 2.3

$$|A_{n+1}-A_n| \leq C(\sigma^{-5/2}\rho_{n-1})^3 \quad \text{and} \quad \rho_{n+1} \leq C\sigma\rho_n + C(\sigma^{-5/2}\rho_{n-1})^3.$$

By scaling $A_n = \varepsilon\tilde{A}_n$ and $\rho_n = \varepsilon\tilde{\rho}_n$ we obtain

$$|\tilde{A}_{n+1} - \tilde{A}_n| \leq C\varepsilon^2(\sigma^{-5/2}\tilde{\rho}_{n-1})^3 \quad \text{and} \quad \tilde{\rho}_{n+1} \leq C\sigma\tilde{\rho}_n + C\varepsilon^2(\sigma^{-5/2}\tilde{\rho}_{n-1})^3. \tag{12}$$

For given $b > 0$ we choose $\sigma > 0$ such that $C \leq \sigma^{-2b}$. Hence

$$|\tilde{A}_{n+1} - \tilde{A}_n| \leq C\varepsilon^2 \quad \text{and} \quad \tilde{\rho}_{n+1} \leq \sigma^{1-2b}\tilde{\rho}_n + C\varepsilon^2, \tag{13}$$

as long as $\tilde{A}_n$ and $\tilde{\rho}_n$ stay $\mathcal{O}(1)$ bounded. By the last estimates this is guaranteed for $n=\mathcal{O}(\varepsilon^{-2})$. Doing back the scalings we obtain

$$\begin{aligned} \|u(\sigma^{-n}x, \sigma^{-2n}) - \sigma^n \frac{A_n}{\sqrt{2\pi}}\mathrm{e}^{-x^2/4}\|_{H^2_2} &\leq C\,\sigma^n\|\hat{u}(\sigma^n\varkappa, \sigma^{-2n}) - A_n\mathrm{e}^{-\varkappa^2}\|_{H^2_2} \\ = C\sigma^n\|\hat{u}_n(\varkappa, 1) - A_n\mathrm{e}^{-\varkappa^2}\|_{H^2_2} &= C\sigma^n\|\hat{r}_n(\varkappa)\|_{H^2_2} \leq C\sigma^n\varepsilon^3 + C\varepsilon\sigma^{2(1-b)n}. \end{aligned} \tag{14}$$

Since $\|u(x)\|_{L^\infty} = \|u(\sigma^{-n}x)\|_{L^\infty} \leq C\|u(\sigma^{-n}x)\|_{H^2(2)}$ and

$$\begin{aligned} \|u(x)\|_{L^1} &= \sigma^{-n}\|u(\sigma^{-n}x)\|_{L^1} \\ &\leq \sigma^{-n}\|u(\sigma^{-n}x)(1+x^2)\|_{L^2}\|(1+x^2)^{-1}\|_{L^2} \leq C\sigma^{-n}\|u(\sigma^{-n}x)\|_{H^2_2} \end{aligned}$$

we obtain $\|u(\cdot,t)\|_{L^1} \leq C$, and in particular

$$\|u(x,t) - \frac{\varepsilon \tilde{A}_n}{\sqrt{2\pi\, t}} \mathrm{e}^{-x^2/4}\|_{L^\infty} \leq Ct^{-1/2}\varepsilon^3 + C\varepsilon t^{-1+b}, \tag{15}$$

first for all $t = \sigma^{-2n} \leq \mathcal{O}(\exp(\varepsilon^{-2}))$. Since (1) generates a local semi-flow we have that (15) holds for all $t \in [0, \exp(q/\varepsilon^2)]$ with some constant $q > 0$ independent of ε, where $\varepsilon \tilde{A}_n$ is replaced by $\varepsilon \tilde{A}(t)$ with a suitable smooth $\mathcal{O}(1)$ bounded function $\tilde{A}$.

The method of discrete renormalization is very robust, since it essentially only uses the parabolic form $-k^2$ of the spectral curve locally at $k = 0$, and it gives the more detailed asymptotics (15) under slightly stronger assumptions.

Remark 2.4 Another possibility to prove a result similar to Theorem 2.1 or (15) would be to work with self similar coordinates, i.e.

$$u(x,t) = t^{-1/2} w(t^{-1/2}x, \log t) = e^{-\tau/2} w(\xi, \tau)$$

and to apply the invariant manifold theorem of [Wa97]. On the one-dimensional center manifold, with coordinate z, we obtain

$$\dot{z} = z^3 + \mathrm{h.o.t},$$

i.e., for initial conditions less than ε it takes a time τ of order $\mathcal{O}(1/\varepsilon^2)$ to reach 2ε. Doing back the above transformation for time, i.e. $t = e^\tau$ would show our result, too. Moreover, we obtain that indeed

$$u(x,t) = \varepsilon t^{-1/2} A^*(\varepsilon^2 \log t) \mathrm{e}^{-x^2/4} + \mathcal{O}(t^{-1/2}\varepsilon^2)$$

with a function $A^*(\tau) = \varepsilon z(\varepsilon^2 \tau)$ that satisfies $\partial_\tau A^* = (A^*)^3 + \mathrm{h.o.t.}$. The assumptions on the initial conditions to apply the invariant manifold theorem of [Wa97] are similar as for the discrete renormalization approach, but we consider the latter to be somewhat more robust.

3 Transient self similar decay to unstable Poiseuille flow at criticality for exponentially long times

3.1 The equations

Poiseuille flow [HS72, DR81] is a classical hydrodynamical stability problem for the theoretical and experimental study of the transition scenario from laminar to turbulent flow. The evolution of the viscous incompressible fluid in a cylindrical domain $(x, y) \in \Omega = \mathbb{R} \times \Sigma$, with Σ a bounded cross section, subject to a pressure gradient, is described by the Navier-Stokes equations.

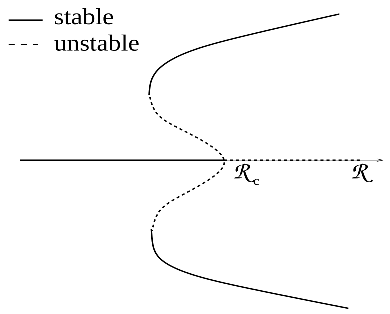

Figure 1: Simplified schematic bifurcation diagram for Poiseuille flow.

The equations describing Poiseuille flow are

$$\begin{aligned} \partial_t U &= -\nabla p + \frac{1}{\mathcal{R}}\Delta U - (U\cdot\nabla)U, \\ \nabla\cdot U &= 0, \quad U|_{\mathbb{R}\times\partial\Sigma} = 0, \end{aligned} \tag{16}$$

where $\mathcal{R}$ is called the Reynolds number. For simplicity we restrict ourselves to two-dimensional flows, i.e., w.l.o.g. $\Sigma = (0,\pi)$. However, our result also holds for three-dimensional flows for which a transition to instability occurs, e.g., for elliptical pipes with sufficient ellipticity, see [Ho77].

We shall assume that there is a basic flow $U_0 = (y - y^2, 0)$, $p_0 = -\frac{2}{\mathcal{R}}x$. The deviation from the basic flow, defined by $U = U_0 + U'$, $p = p_0 + p'$, satisfies

$$\begin{aligned} \partial_t U' &= -\nabla p' + \frac{1}{\mathcal{R}}\Delta U' - (U'\cdot\nabla)U_0 - (U_0\cdot\nabla)U' - (U'\cdot\nabla)U', \\ \nabla\cdot U' &= 0, \qquad U'|_{y=0,\pi} = 0, \end{aligned} \tag{17}$$

In the following we will drop the primes $'$. In order to solve the problem uniquely for the velocity field $U = U(x,y,t) = (U_1,U_2)(x,y,t) \in \mathbb{R}^2$ we additionally assume the vanishing mean flow condition

$$\int_\Sigma U_1(x,y,t)\mathrm{d}y = 0,$$

cf. [CI94]. There exists a critical Reynolds number $\mathcal{R}_c$ such that for $\mathcal{R} < \mathcal{R}_c$ the trivial flow is asymptotically stable, i.e., after some possible transient growth due to the non-normality of the linearization of (17) perturbations decay with some exponential rate towards this trivial solution. If $\mathcal{R}$ becomes larger than $\mathcal{R}_c$ the trivial solution becomes unstable via a subcritical bifurcation. See Figure 1.

As usual we eliminate the pressure gradient by introducing a projection $\mathcal{P}$ on the divergence free vector fields by defining $u = \mathcal{P}f$, where u solves

$$u + \nabla p = f, \qquad \nabla\cdot u = 0, \qquad u\cdot\vec{n}|_{\mathbb{R}\times\{0,\pi\}} = 0,$$

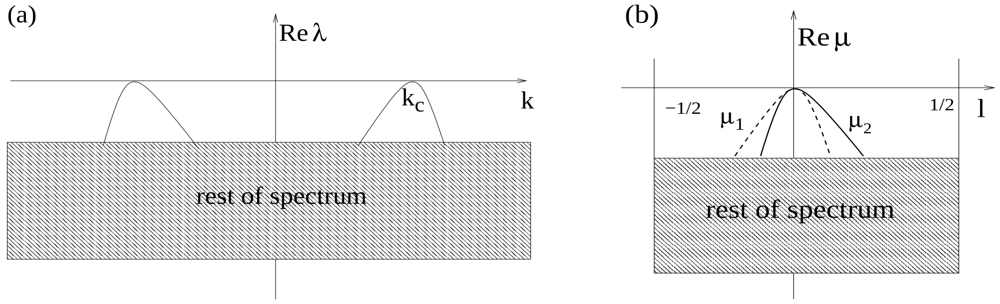

Figure 2: Sketch of the spectrum for Poiseuille flow at the threshold of instability; critical curve of eigenvalues over the Fourier wave number k (a), and over the Bloch wave number ℓ (b).

with $\vec{n}$ the outer normal on $\mathbb{R} \times \partial\Sigma$. Then we consider

$$\partial_t u = \mathcal{M}u + \mathcal{N}(u), \tag{18}$$

where

$$\mathcal{M}u = \frac{1}{\mathcal{R}}\mathcal{P}\Delta u - \mathcal{P}[(u \cdot \nabla)U_0 - (U_0 \cdot \nabla)u], \qquad \mathcal{N}(u) = -\mathcal{P}[(u \cdot \nabla)u],$$

in a space of functions satisfying $\nabla \cdot u = 0$, the above Dirichlet boundary conditions and the vanishing mean flow condition. For the functional analytic properties of the projection $\mathcal{P}$ and the other terms in this equation see below.

3.2 Linear analysis

Since (18) is translation invariant along the x-axis the solutions of the linearized system for $u = u(x, y, t)$ are of the form

$$\varphi_{k,n}(y)\mathrm{e}^{\lambda_n(k)t}\mathrm{e}^{\mathrm{i}kx}$$

with $k \in \mathbb{R}, n \in \mathbb{N}$, and $\varphi_{k,n}(y) \in \mathbb{C}^2$. Then for $\mathcal{R} = \mathcal{R}_c$ all curves, except λ_1, have strictly negative real parts. More precisely, there exists a $\sigma > 0$ such that

$$\sup_{n=2,3,\ldots} \sup_{k\in\mathbb{R}} \mathrm{Re}\lambda_n(k) < -\sigma \ .$$

The curve $k \mapsto \mathrm{Re}\lambda_1(k)$ touches 0 at a wave number $k_c \neq 0$; see Figure 2. From the form of $k \mapsto \mathrm{Re}\lambda_1(k)$ close to wave number k_c, namely

$$\mathrm{Re}\lambda_1(k) = -C_1(k - k_c)^2 + \mathcal{O}((k - k_c)^3)$$

for a constant $C_1 > 0$, it is clear that the linearized system exhibits some diffusive behavior. For $t \to \infty$ the linearized system behaves asymptotically as

$$u(x, y, t) = A_* t^{-1/2} e^{\frac{(x-\nu t)^2}{4(C_1+\mathrm{i}C_2)t}} e^{\mathrm{i}(k_c x - \omega_0 t)} \varphi_{k_c,1}(y) + \text{c.c.} + \mathcal{O}(t^{-1}) \ .$$

for a constant $A_* \in \mathbb{C}$, where

$$\mathrm{Im}\lambda_1(k) = \omega_0 + \nu(k-k_c) - C_2(k-k_c)^2 + \mathcal{O}((k-k_c)^3)$$

with real valued constants ω_0, ν, and C_2.

3.3 The result

In order to understand heuristically the nonlinear system at criticality we make a so called Ginzburg-Landau ansatz [Schn99, Mi01], namely

$$u(x,y,t) = \delta A(\delta(x-\nu t), \delta^2 t)\mathrm{e}^{\mathrm{i}(k_c x - \omega_0 t)}\varphi_{k_c,1}(y) + \text{c.c.} + \mathcal{O}(\delta^2) \tag{19}$$

with $0 < \delta \ll 1$ a small perturbation parameter. Inserting this into (18) shows that at criticality, i.e. $\mathcal{R} = \mathcal{R}_c$, the complex-valued amplitude $A(X,T)$ with $X = \delta x, T = \delta^2 t$ satisfies in lowest order a Ginzburg-Landau equation

$$\partial_T A = (C_1 + \mathrm{i}C_2)\partial_X^2 A + \gamma A|A|^2 \ ,$$

with $\mathrm{Re}\gamma > 0$ a constant which can be computed by classical perturbation analysis [HS72, BSvH95, Bol96]. Hence, for $\delta \to 0$ the critical modes concentrate at the Fourier wave number $\pm k_c$ and corresponding to the curve of eigenvalues λ_1 they satisfy an equation similar to the nonlinear diffusion equation of the Introduction. This motivates the following result for Poiseuille flow at criticality, where H_2^2 is defined as

$$H_2^2 = \{u \in H^2(\mathbb{R}\times\Sigma, \mathbb{R}^2) : \|u\|_{H_2^2}^2 = \sum_{|\alpha|=0}^{2} \|(\partial^\alpha u)\rho^2\|_{L^2}^2 < \infty\}, \quad \rho(x) = (1+x^2)^{1/2},$$

together with the divergence and boundary conditions. For notational convenience the initial conditions are taken at $t = 1$.

Theorem 3.1 *Consider (18) for $\mathcal{R} = \mathcal{R}_c$. For all $b > 0$ there exist positive constants ε_0, q, and C such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. For initial conditions u_0 with $\|u_0\|_{H_2^2} \leq \varepsilon$ the associated solution $u = u(t)$ of (18) with $u|_{t=1} = u_0$ exists for all $t \in [1, \exp(q\varepsilon^{b-2})]$ and satisfies*

i) $\sup_{t\in[1,\exp(q\varepsilon^{b-2})]} \|u(t)\|_{L^1} \leq 16\varepsilon$,

ii) $\|u(t)\|_{L^\infty} \leq C\varepsilon t^{-\frac{1}{2}}$ *for all* $t \in [1, \exp(q\varepsilon^{b-2})]$*; in particular we have*

$$\|u|_{t=\exp(q\varepsilon^{b-2})}\|_{L^\infty} \leq C\varepsilon \exp(-q\varepsilon^{b-2}/2).$$

Remark 3.2 In order to show this result we prove, similar to (15), the detailed transient asymptotic behavior of our solution $u(t)$ for $t \in [1, \exp(q\varepsilon^{-2})]$. It behaves (which we do not prove in this optimal ($b = 0$) formulation) like

$$u(x, y, t) = \varepsilon A^*(\varepsilon^2 \log t) t^{-1/2} \mathrm{e}^{-\frac{(x-\nu t)^2}{4(C_1 \mathrm{i} C_2)t}} \mathrm{e}^{\mathrm{i}(k_c x - \omega_0 t)} \varphi_{k_c,1}(y) + \text{c.c.} + \mathcal{O}(\varepsilon^2/\sqrt{t}) + \mathcal{O}(\varepsilon/t) ,$$

where $A^* = A^*(\tau) = \mathcal{O}(1)$ solves a problem of the form

$$\partial_\tau A^* = \tilde{\gamma} A^* |A^*|^2$$

on a time interval of order $\mathcal{O}(1)$ with initial conditions $A^*|_{\tau=0}{=}\mathcal{O}(1)$ and a constant $\tilde{\gamma}{=}\mathcal{O}(1)$ with $\mathrm{Re}\tilde{\gamma}{>}0$, compare Remark 2.4.

Remark 3.3 The approach of Remark 2.4 immediately shows the instability of $u = 0$ in a number of spaces, for instance, there exists a $\delta_1 > 0$ such that for all initial conditions $u_0 \in H_2^2$ we have a $t_0 > 0$ such that the associated solution satisfies $\|u(t_0)\|_{L^\infty} > \delta_1$ no matter how small $\|u_0\|_{H_2^2} > 0$ has been.

Remark 3.4 Due some technical details in our approach we prove a slightly weaker result than in Theorem 2.1 and Proposition 2.2 with respect to the length of the possible time interval, namely $t \in [1, \exp(q\varepsilon^{b-2})]$ instead of $t \in [1, \exp(q/\varepsilon^2)]$. By including higher order terms into our analysis it is possible to prove the assertions of the theorem also for $b = 0$.

Remark 3.5 There exist attractivity results [Eck93] showing that all small solutions of pattern forming systems, as Poiseuille flow, develop in such a way that after a time of order $\mathcal{O}(1/\delta^2)$ they are of the form (19). Hence an optimal description in terms of δ of the set of initial conditions for which our result holds would be adequate. Here we refrain from such a very technical description, cf. [Schn98].

Remark 3.6 In case of weak linear instability, i.e. $\mathcal{R} - \mathcal{R}_c = \delta^2 > 0$ small, the dynamics stated in the theorem holds for all $0 \le t \le \min(\delta^{-2}, \exp(q\varepsilon^{-2}))$, i.e. especially $\|u|_{t=\delta^{-2}}\|_{L^\infty} \le C\varepsilon\delta$ which is much smaller than ε if $0 \le \delta \ll \varepsilon$.

Remark 3.7 A physical relevant cross section Σ where such a behavior could be observed is for instance an elliptic domain with strong ellipticity [Ho77]. In case of a circular domain no instability occurs. The case $\Sigma = (0, \pi)$ is an idealization of span-wise independent flows in $\Sigma = (0, \pi) \times \mathbb{R}$. For the last unbounded cross section the trivial solution $u = 0$ at criticality is stable with respect to small spatially localized perturbations since the nonlinear terms $A|A|^2$ are asymptotically irrelevant with respect to diffusion in $\mathbb{R}^2$.

The proof of Theorem 3.1 is based on Bloch wave analysis and renormalization theory. It stands in a line of papers where results about the dynamics of simple nonlinear diffusion equations have been transfered to the Ginzburg-Landau equation, cf. [BK92, CEE92, BK94], or to hydrodynamical stability problems [Schn98, ES00].

For notational convenience it is advantageous to go into a frame moving with velocity ν, i.e. we introduce the coordinate $x' = x - \nu t$. Again for notational convenience we drop the prime $'$, i.e. w.l.o.g. we can assume $\nu = 0$ in the following.

3.4 Bloch wave analysis

This section is based on [ES02, Section 5] and contains a short summary about the information needed for the use of Bloch waves.

By rescaling the space variable x, w.l.o.g. we can assume that $k_c = 1$. We suppress the notation of the variable y throughout this section. The starting point of Bloch wave analysis in case of a 2π-periodic underlying pattern is the (formal) relation

$$\begin{aligned} u(x) &= \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{\mathrm{i}kx} \tilde{u}(k) \mathrm{d}k = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}(n+\ell)x} \tilde{u}(n+\ell) \mathrm{d}\ell \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(n+\ell)x} \tilde{u}(n+\ell) \mathrm{d}\ell = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}\ell x} \hat{u}(\ell, x)\, \mathrm{d}\ell, \end{aligned}$$

where now $\tilde{u} = \mathcal{F}u$ is the Fourier transform of u and where we define

$$\big(\mathcal{T}u\big)(\ell, x) \equiv \hat{u}(\ell, x) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nx} \tilde{u}(n+\ell)\ . \tag{20}$$

Thus Bloch wave transform $\mathcal{T}$ can be seen as a generalization of Fourier transform $\mathcal{F}$. For the rest of the paper we use the following notation: if f is a function, then $\hat{f}$ is defined by $\hat{f}=\mathcal{T}f$, and if $\mathcal{A}$ is an operator, then $\hat{\mathcal{A}}$ is defined by $\hat{\mathcal{A}}=\mathcal{T}\mathcal{A}\mathcal{T}^{-1}$.

Note that by Parseval's identity we have

$$\begin{aligned} \int_{\mathbb{R}} |u(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |\tilde{u}(k)|^2 \mathrm{d}k = \sum_{n\in\mathbb{Z}} \int_{-1/2}^{1/2} |\tilde{u}(n+\ell)|^2 \mathrm{d}\ell \\ = \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} |\tilde{u}(n+\ell)|^2 \mathrm{d}\ell = \int_{-1/2}^{1/2} \int_0^{2\pi} |\hat{u}(\ell, x)|^2\, \mathrm{d}x \mathrm{d}\ell \end{aligned} \tag{21}$$

The sum and the integral can be interchanged due to Fubini's theorem when u is in the Schwartz space $\mathcal{S}$. By density (21) extends to $u \in L^2$. We shall use the following fundamental properties

$$\begin{aligned} \hat{u}(\ell, x) &= \mathrm{e}^{\mathrm{i}x} \hat{u}(\ell+1, x), \quad \hat{u}(\ell, x) = \hat{u}(\ell, x+2\pi), \quad \text{and} \\ \hat{u}(\ell, x) &= \overline{\hat{u}}(-\ell, x) \text{ for real-valued } u\ . \end{aligned}$$

Multiplication in position space corresponds to a modified convolution for the Bloch-functions,

$$\big(\widehat{u\cdot v}\big)(\ell,x) \;=\; \int_{-1/2}^{1/2}\hat{u}(\ell-\ell',x)\hat{v}(\ell',x)\mathrm{d}\ell \;\equiv\; \big(\hat{u}\star\hat{v}\big)(\ell,x)\;,$$

which follows from

$$\begin{aligned}\big(\widehat{u\cdot v}\big)(\ell,x) &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\tilde{u}(\ell+m-k)\tilde{v}(k)\mathrm{e}^{\mathrm{i}mx}\mathrm{d}k\\ &= \int_{-1/2}^{1/2}\sum_{m,n\in\mathbb{Z}}\tilde{u}(\ell+m-\ell'-n)\tilde{v}(\ell'+n)\mathrm{e}^{\mathrm{i}(m-n)x}\mathrm{e}^{\mathrm{i}nx}\mathrm{d}\ell.\end{aligned}$$

We introduce the norms

$$\begin{aligned}\|f\|^2_{\mathrm{H}^n_s} &= \sum_{j=0}^{s}\sum_{m_1+m_2=0,m_1,m_2\geq 0}^{n}\int\int|\partial_x^{m_1}\partial_y^{m_2}f(x,y)|^2x^{2j}\mathrm{d}y\mathrm{d}x,\\ \|\hat{f}\|^2_{\hat{\mathrm{H}}^s_n} &= \sum_{j_1+j_2=0,j_1,j_2\geq 0}^{n}\sum_{m=0}^{s}\int_{-1/2}^{1/2}\int_0^{2\pi}\int_{\Sigma}|\partial_x^{j_1}\partial_y^{j_2}\partial_\ell^m\hat{f}(\ell,x)|^2\mathrm{d}y\mathrm{d}x\mathrm{d}\ell,\end{aligned}$$

Similar to (21), from Parseval's identity we get

$$C^{-1}\|u\|_{\mathrm{H}^n_s} \;\leq\; \|\hat{u}\|_{\hat{\mathrm{H}}^s_n} \;\leq\; C\|u\|_{\mathrm{H}^n_s}\;, \tag{22}$$

for some $C>0$. In the following we mainly work with the spaces to $s=n=2$ and $s=0$, $n=2$. We have

$$\|\widehat{u\,v}\|_{\hat{\mathrm{H}}^2_2} \;=\; \|\hat{u}\star\hat{v}\|_{\hat{\mathrm{H}}^2_2}\leq C\|\hat{u}\|_{\hat{\mathrm{H}}^2_2}\;\|\hat{v}\|_{\hat{\mathrm{H}}^2_2}\,. \tag{23}$$

3.5 The linearized problem in Bloch space

The linearized problem is given by

$$\partial_t u \;=\; \mathcal{M}u. \tag{24}$$

Since the spectrum of $\mathcal{M}$ is well-documented, we just summarized the results in Section 3.2. Although the linearized problem has constant coefficients it is advantageous to work in Bloch space. This is due to the fact that the critical eigenvalue curve $\lambda_1(k)$ touches the real axis at a wave-number $k_c\neq 0$. Renormalization in Fourier space as in Section 2.2 would be much more complicated due to the fact that the Fourier modes do not concentrate at the wave numbers $\pm k_c$ alone, but also at all integer multiples of k_c. Working in Bloch space turns out to be more convenient.

Thus, we consider

$$\partial_t\hat{u} = \widehat{\mathcal{M}}\hat{u}, \tag{25}$$

where the operator $\widehat{\mathcal{M}} = \mathcal{T}\mathcal{M}\mathcal{T}^{-1}$ equals a direct integral $\int \mathcal{M}_\ell \mathrm{d}\ell$. Each $\mathcal{M}_\ell$ acts on the subspace with fixed Bloch wave number ℓ. The eigenfunctions of $\mathcal{M}_\ell$ are given by Bloch waves of the form $\mathrm{e}^{\mathrm{i}\ell x} w_{\ell,n}$ with 2π-periodic $w_{\ell,n}$, where $n \in \mathbb{N}$ indexes various eigenvalues for fixed ℓ. For each $\ell \in \mathbb{R}$ these are solutions of

$$\big(\mathcal{M}_\ell w_{\ell,n}\big)(x) = \mathrm{e}^{-i\ell x}\mathcal{M}(w_{\ell,n}(x)\mathrm{e}^{-i\ell x}) = \mu_n(\ell) w_{\ell,n}(x) .$$

The spectrum takes the familiar form of two curves $\mu_1(\ell) = \lambda_1(k_c+\ell)$ and $\mu_2(\ell) = \lambda_1(-k_c+\ell)$ with an expansion

$$\mu_j(\ell) = i\omega_0 - (C_1 + \mathrm{i}C_2)\ell^2 + \mathcal{O}(\ell^3) ,$$

for $j = 1, 2$ with $C_1 > 0$ and the remainder of the spectrum has negative real part bounded away from 0. The eigenfunctions associated with $\mu_1(0)$ and $\mu_2(0)$ are $\mathrm{e}^{\pm \mathrm{i}k_c x}\varphi_{\pm k_c,1}(y)$. There is an $\ell_0 > 0$ such that for fixed $\ell \in (-\ell_0, \ell_0)$ the eigenfunctions $w_{\ell,j}(x)$ of the main branches $\mu_j(\ell)$ for $j = 1, 2$ are well defined as ℓ is varied away from 0. Corresponding to this we define the projections $\hat{P}_j(\ell)$ by

$$\hat{P}_j(\ell)\hat{f}(l) = \langle w^*_{\ell,j}, \hat{f}(l)\rangle w_{\ell,j} ,$$

for $j = 1, 2$ and $\hat{P}_\mathrm{c}(\ell)$ by

$$\hat{P}_\mathrm{c}(\ell)\hat{f}(l) = \hat{P}_1(\ell)\hat{f}(l) + \hat{P}_2(\ell)\hat{f},$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2([0, 2\pi] \times \Sigma)$ and $w^*_{\ell,j}$ the associated eigenfunction of the adjoint problem. We will need a version of $\hat{P}_c$ that depends smoothly on ℓ. Therefore we fix once and for all a non-negative smooth cutoff function χ with support in $[-\ell_0/2, \ell_0/2]$ which equals 1 on $[-\ell_0/4, \ell_0/4]$. Then we define the operators $\hat{E}_\mathrm{c}$ and $\hat{E}_\mathrm{s}$ by

$$\hat{E}_\mathrm{c}(\ell) = \chi(\ell)\hat{P}_\mathrm{c}(\ell) , \quad \hat{E}_\mathrm{s}(\ell) = \mathbf{1}(\ell) - \hat{E}_\mathrm{c}(\ell) .$$

It will be useful to define auxiliary mode filters $\hat{E}^\mathrm{h}_\mathrm{c}$ and $\hat{E}^\mathrm{h}_\mathrm{s}$ by $\hat{E}^\mathrm{h}_\mathrm{c}(\ell) = \chi(\ell/2)\hat{P}_\mathrm{c}(\ell)$ and $\hat{E}^\mathrm{h}_\mathrm{s}(\ell) = \mathbf{1}(\ell) - \chi(2\ell)\hat{P}_\mathrm{c}(\ell)$. These definitions are made in such a way that

$$\hat{E}^\mathrm{h}_\mathrm{c}\hat{E}_\mathrm{c} = \hat{E}_\mathrm{c} , \quad \hat{E}^\mathrm{h}_\mathrm{s}\hat{E}_\mathrm{s} = \hat{E}_\mathrm{s} ,$$

which will be used to replace the (missing) projection property of $\hat{E}_\mathrm{c}$ and $\hat{E}_\mathrm{s}$.

As already said, our proof of Theorem 3.1 is based on discrete renormalization theory similar to section 2.2. Thus, with $\sigma \in (0, 1)$, we let

$$\big(\widehat{\mathcal{L}}\hat{u}\big)(\varkappa, x, y) = \hat{u}(\sigma\varkappa, x, y) .$$

Note that here, and elsewhere, the scaling does not act on the (x, y) variable, only on the Bloch wave number $\varkappa$. The integration region over the ℓ variable is finite and it will change with the scaling. Therefore, we introduce

$$\mathcal{K}_\sigma = \{\hat{u} \,:\, \|\hat{u}\|_{\mathcal{K}_\sigma} < \infty,\ \mathrm{div} u = 0,\ u|_{\mathbb{R}\times\partial\Sigma} = 0,\ \int_\Sigma u_1 dy = 0\} ,$$

where

$$\|\hat{u}\|^2_{\mathcal{K}_\sigma} \equiv \sum_{n=0}^{2} \sum_{n'+n''=0,n',n''\geq 0}^{2} \int_{-1/(2\sigma)}^{1/(2\sigma)} \int_0^{2\pi} \int_\Sigma |\partial_\ell^n \partial_x^{n'} \partial_y^{n''} \hat{u}(\ell,x,y)|^2 \mathrm{d}y \mathrm{d}x \mathrm{d}\ell.$$

For the nonlinear terms we need a space

$$\mathcal{K}^*_\sigma = \{\hat{u} \ : \ \|\hat{u}\|_{\mathcal{K}^*_\sigma} < \infty, \ \mathrm{div} u = 0, \ u \cdot \vec{n}|_{\mathbb{R}\times\partial\Sigma} = 0, \ \int_\Sigma u_1 dy = 0\} \ ,$$

where $\vec{n}$ is the outer normal on $\mathbb{R} \times \partial\Sigma$ and where

$$\|\hat{u}\|^2_{\mathcal{K}^*_\sigma} \equiv \sum_{n=0}^{2} \sum_{n'+n''=0,n',n''\geq 0}^{1} \int_{-1/(2\sigma)}^{1/(2\sigma)} \int_0^{2\pi} \int_\Sigma |\partial_\ell^n \partial_x^{n'} \partial_y^{n''} \hat{u}(\ell,x,y)|^2 \mathrm{d}y \mathrm{d}x \mathrm{d}\ell.$$

From (22) we have that $\mathcal{T}$, as defined in (20), is an isomorphism between H^2_2 and $\mathcal{K}_1$. As in (11) we have

$$\|\widehat{\mathcal{L}}\hat{f}\|_{\mathcal{K}_{\sigma^n}} \leq C\sigma^{-5/2}\|\hat{f}\|_{\mathcal{K}_{\sigma^{n-1}}} \ , \tag{26}$$

for $0 < \sigma \leq 1$ and similarly for $\mathcal{K}^*_\sigma$, cf. [ES02].

Because of the nature of the spectrum $\mu_1(\ell)$ and $\mu_2(\ell)$, the critical part of the solution satisfies

$$\hat{E}_\mathrm{c}\hat{u}(t^{-1/2}\ell,x,y,t) = \mathrm{e}^{-(C_1+\mathrm{i}C_2)\ell^2}(a_1\mathrm{e}^{\mathrm{i}(k_c x-\omega_0 t)}\varphi_{k_c,1}(y) + \overline{a_1}\mathrm{e}^{-\mathrm{i}(k_c x-\omega_0 t)}\varphi_{-k_c,1}(y)) + \mathcal{O}(t^{-1/2})$$

with $a_1 \in \mathbb{C}$ a number depending on the initial conditions. Using this observation and the fact that the $\hat{E}_\mathrm{s}$-part is exponentially damped, we obtain the following result for the linearized system.

Proposition 3.8 *The solution $\hat{u}$ of the problem (25) with initial data $\hat{u}|_{t=1} = \hat{u}_0$ satisfies*

$$\|\big(\hat{u}(t^{-1/2}\ell,x,y,t) - \mathrm{e}^{-(C_1+\mathrm{i}C_2)\ell^2}(\hat{P}_1(0)e^{\mathrm{i}\omega_0 t} + \hat{P}_{-1}(0)e^{-\mathrm{i}\omega_0 t})\hat{u}_0(0)(x,y)\big)\|_{\mathcal{K}_{1/\sqrt{t}}} \leq Ct^{-1/2}\|\hat{u}_0\|_{\hat{\mathrm{H}}^2_2} \ ,$$

for a constant $C > 0$ and all $t \geq 1$. Moreover, there is a constant $\gamma_- > 0$ such that

$$\|(\ell,x,y) \mapsto \big(\hat{E}_\mathrm{s}\hat{u}\big)(t^{-1/2}\ell,x,y,t)\|_{\mathcal{K}_{1/\sqrt{t}}} \leq C\mathrm{e}^{-\gamma_- t}\|\hat{u}_0\|_{\hat{\mathrm{H}}^2_2} \ ,$$

for all $t \geq 1$.

3.6 The renormalization process for the full problem

Here we start the proof of our main result. In addition to the statement in Theorem 3.1 we give the detailed asymptotic behavior for large t.

Theorem 3.9 *For all $b>0$ there are positive constants ε_0, q and C such that for all $\varepsilon\in(0,\varepsilon_0)$ the following holds. Assume $\|u_0\|_{\mathrm{H}^2_2}\leq\varepsilon$ and let u be the solution of (18) with initial condition $u|_{t=1}=u_0$. Let $\tilde{\psi}(\ell)=\exp(-C_1\ell^2)$. Then the rescaled solution $\hat{u}^{\mathrm{r}}(\ell,x,y,t)=\hat{u}(t^{-1/2}\ell,x,y,t)$ satisfies*

$$\begin{aligned}&\|(\ell,x,y)\mapsto\hat{u}^{\mathrm{r}}(\ell,x,y,t)-(\varepsilon A_*(\varepsilon^{2-b}\log t)\mathrm{e}^{\mathrm{i}(k_c x-\omega_0 t)}\varphi_{1,k_c}(y)\tilde{\psi}(\ell)+\mathrm{c.c.})\|_{\mathcal{K}_{1/\sqrt{t}}}\\&\qquad\leq C\varepsilon t^{b-1/2}+C\varepsilon^{2-b}\end{aligned}\tag{27}$$

for all $t\in[1,\exp(q\varepsilon^{b-2})]$, where $A_*=A_*(\tau)$ is a function with $A_*|_{\tau=0}=\mathcal{O}(1)$ and

$$\sup_{\tau\in[0,q]}|\partial_\tau^j A_*(\tau)|<C=\mathcal{O}(1),\quad j=0,1.$$

Remark 3.10 Similar to (15), this means that a spatially localized initial perturbation $u_0(x)$ of order $\mathcal{O}(\varepsilon)$ behaves like

$$u(x,t)\sim\varepsilon\left[A_*(\varepsilon^{2-b}\log t)t^{-1/2}\exp\left(-x^2/(4(C_1+\mathrm{i}C_2)t)\right)\mathrm{e}^{\mathrm{i}(k_c x-\omega_0 t)}\varphi_{1,k_c}+\mathrm{c.c.}\right],$$

for $\varepsilon\to 0$, uniformly for $x\in\mathbb{R}$, for all $t\in[1,\exp(\varepsilon^{b-2})]$, cf. [Schn98]. The function A_* is denoted different from the function A^* defined in Remark 3.2 due to the different time scales. On the interval $[1,\exp(q\varepsilon^{b-2})]$ the function A^* does not make any $\mathcal{O}(1)$ change.

Proof. The idea of the proof is similar to [Schn96], i.e. the solution $\hat{u}$ is split into a diffusive part $\hat{u}_{\mathrm{c}}$ and into an exponentially damped part $\hat{u}_{\mathrm{s}}$. In Bloch space the initial conditions satisfy $\|\hat{u}_0\|_{\hat{\mathrm{H}}^2_2}\leq\varepsilon$. The system for the variables $\hat{u}_{\mathrm{c}}$ and $\hat{u}_{\mathrm{s}}$ with initial conditions $\hat{u}_{\mathrm{c}}|_{t=0}=\hat{E}_{\mathrm{c}}\hat{u}|_{t=0}$, $\hat{u}_{\mathrm{s}}|_{t=0}=\hat{E}_{\mathrm{s}}\hat{u}|_{t=0}$ is given in Bloch space by

$$\begin{aligned}\partial_t\hat{u}_{\mathrm{c}}&=\widehat{\mathcal{M}}\hat{u}_{\mathrm{c}}+\hat{E}_{\mathrm{c}}\widehat{\mathcal{N}}(\hat{u}_{\mathrm{c}},\hat{u}_{\mathrm{s}}),\\\partial_t\hat{u}_{\mathrm{s}}&=\widehat{\mathcal{M}}\hat{u}_{\mathrm{s}}+\hat{E}_{\mathrm{s}}\widehat{\mathcal{N}}(\hat{u}_{\mathrm{c}},\hat{u}_{\mathrm{s}}),\end{aligned}\tag{28}$$

where with $\hat{u}=\hat{u}_{\mathrm{c}}+\hat{u}_{\mathrm{s}}$,

$$\widehat{\mathcal{M}}=\mathcal{T}\mathcal{M}\mathcal{T}^{-1}\quad\text{and}\quad\widehat{\mathcal{N}}(\hat{u}_{\mathrm{c}},\hat{u}_{\mathrm{s}})=\mathcal{T}\mathcal{N}(\mathcal{T}^{-1}\hat{u}).$$

We start with the renormalization process by introducing the scalings

$$\begin{aligned}\hat{u}_{\mathrm{c},n}(\varkappa,x,y,\tau)&=\hat{u}_{\mathrm{c}}(\sigma^n\varkappa,x,y,\sigma^{-2n}\tau),\\\hat{u}_{\mathrm{s},n}(\varkappa,x,y,\tau)&=\sigma^{-n}\hat{u}_{\mathrm{s}}(\sigma^n\varkappa,x,y,\sigma^{-2n}\tau).\end{aligned}$$

The variation of constant formula yields now

$$\begin{aligned}\hat{u}_{\mathrm{c},n}(\varkappa,x,y,\tau)&=\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\sigma^2)}\hat{u}_{\mathrm{c},n-1}(\sigma\varkappa,x,y,1)\\&\quad+\sigma^{-2n}\int_{\sigma^2}^{\tau}\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')}\left(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{u}_{\mathrm{c},n},\hat{u}_{\mathrm{s},n})\right)(\varkappa,x,y,\tau')\mathrm{d}\tau',\end{aligned}\tag{29}$$

$$\begin{aligned}\hat{u}_{\mathrm{s},n}(\varkappa,x,y,\tau)&=\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\sigma^2)}\sigma^{-1}\hat{u}_{\mathrm{s},n-1}(\sigma\varkappa,x,y,1)\\&\quad+\sigma^{-3n}\int_{\sigma^2}^{\tau}\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')}\left(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{u}_{\mathrm{c},n},\hat{u}_{\mathrm{s},n})\right)(\varkappa,x,y,\tau')\mathrm{d}\tau',\end{aligned}\tag{30}$$

with

$$\widehat{\mathcal{M}}_{\mathrm{c},n} = \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{c}}^{\mathrm{h}} \widehat{\mathcal{M}} \widehat{\mathcal{L}}^{-n} , \quad \widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n}) = \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{c}} \widehat{\mathcal{N}}(\widehat{\mathcal{L}}^{-n}\hat{u}_{\mathrm{c},n}, \sigma^n \widehat{\mathcal{L}}^{-n}\hat{u}_{\mathrm{s},n}) ,$$
$$\widehat{\mathcal{M}}_{\mathrm{s},n} = \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{s}}^{\mathrm{h}} \widehat{\mathcal{M}} \widehat{\mathcal{L}}^{-n} , \quad \widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n}) = \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{s}} \widehat{\mathcal{N}}(\widehat{\mathcal{L}}^{-n}\hat{u}_{\mathrm{c},n}, \sigma^n \widehat{\mathcal{L}}^{-n}\hat{u}_{\mathrm{s},n}) .$$

3.7 Bounds on the linear and nonlinear terms

Lemma 3.11 *For all $\rho_1 \geq \rho_2 \geq 0$ there exist $C_{\rho_1,\rho_2} > 0$ and $\gamma_- > 0$ such that for all $n \in \mathbb{N}$, for all $1 \geq \tau > \tau' \geq \sigma^2$ and all $\sigma \in (0,1)$ one has*

$$\begin{aligned}
\|\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')} \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{c}}^{\mathrm{h}} \widehat{\mathcal{L}}^{-n}\hat{g}\|_{\mathcal{K}_{\sigma^n}} &\leq C\|\hat{g}\|_{\mathcal{K}_{\sigma^n}} , \\
\|\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')} \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{s}}^{\mathrm{h}} \widehat{\mathcal{L}}^{-n}\hat{g}\|_{\mathcal{K}_{\sigma^n}} &\leq C\mathrm{e}^{-\gamma_-\sigma^{-2n}(\tau-\tau')}\|\hat{g}\|_{\mathcal{K}_{\sigma^n}} , \\
\|\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')} \widehat{\mathcal{L}}^n \hat{E}_{\mathrm{s}}^{\mathrm{h}} \widehat{\mathcal{L}}^{-n}\hat{g}\|_{\mathcal{K}_{\sigma^n}} &\leq C\mathrm{e}^{-\gamma_-\sigma^{-2n}(\tau-\tau')} \max(1, \sigma^{3n/2}(\tau-\tau')^{-3/4})\|\hat{g}\|_{\mathcal{K}^*_{\sigma^n}} .
\end{aligned}$$

Proof. The first estimate follows directly from the fact that

$$\widehat{\mathcal{M}}_{\mathrm{c},n}(\ell) f = \mu_1(\ell)\hat{P}_1(\ell) f + \mu_2(\ell)\hat{P}_2(\ell) f = -C_1\ell^2\hat{P}_{\mathrm{c}}(\ell) f + \mathcal{O}(\ell^3) .$$

The last two estimates follow from the fact that the real part of the spectrum of $\widehat{\mathcal{M}}_{\mathrm{s},n}(\ell)$ as a function of ℓ can be bounded from above by a strictly negative parabola. It is easy to see that the estimates can be chosen independent of n. For more details see [Schn98, Lemma 7.1] which is based on estimates of [Io71]. □

Lemma 3.12 *Suppose $\max\{\|\hat{u}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n}}, \|\hat{u}_{\mathrm{s},n}\|_{\mathcal{K}_{\sigma^n}}\} \leq 1$. Then there exists a $C_1 > 0$ such that for all $\sigma \in (0,1]$ one has*

$$\begin{aligned}
\|\widehat{\mathcal{N}}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n}} &\leq C_1\sigma^{2n}(\|\hat{u}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n}}\|\hat{u}_{\mathrm{s},n}\|_{\mathcal{K}_{\sigma^n}} + \sigma^n\|\hat{u}_{\mathrm{s},n}\|^2_{\mathcal{K}_{\sigma^n}}) \\
\|\widehat{\mathcal{N}}_{\mathrm{s},n}\|_{\mathcal{K}^*_{\sigma^n}} &\leq C_1\sigma^{n}(\|\hat{u}_{\mathrm{c},n}\|^2_{\mathcal{K}_{\sigma^n}} + \sigma^n\|\hat{u}_{\mathrm{c},n}\|_{\mathcal{K}_{\sigma^n}}\|\hat{u}_{\mathrm{s},n}\|_{\mathcal{K}_{\sigma^n}} + \sigma^{2n}\|\hat{u}_{\mathrm{s},n}\|^2_{\mathcal{K}_{\sigma^n}})
\end{aligned}$$

Proof. The projection $\mathcal{P}$ on the divergence free vector fields is estimated in [Schn98, Lemma 8.1 and Lemma A.8] and is a bounded operator in all spaces considered with bounds independent of $n \in \mathbb{N}$ and $\sigma \in (0,1)$. Using the equation $\mathrm{div} u = 0$ the nonlinear terms can be written as

$$\mathcal{N}(u) = -\mathcal{P}[\nabla(u^T u)].$$

Since $H^2(\mathbb{R}\times(0,\pi))$ is a Banach algebra, i.e. $\|uv\|_{H^2} \leq C\|u\|_{H^2}\|v\|_{H^2}$, the nonlinearity maps $\mathcal{K}_{\sigma^n}$ into $\mathcal{K}^*_{\sigma^n}$. Since $\hat{P}_c(\ell)$ has a finite dimensional image and a compact support w.r.t. ℓ the part $\widehat{\mathcal{N}}_{\mathrm{c},n}$ maps $\mathcal{K}_{\sigma^n}$ still in $\mathcal{K}_{\sigma^n}$.

The estimates for $\widehat{\mathcal{N}}_{\mathrm{s},n}$ follow directly from

$$\big(\widehat{\mathcal{L}}(\hat{f}\star\hat{g})\big)(\varkappa) = \sigma\big((\widehat{\mathcal{L}}\hat{f})\star(\widehat{\mathcal{L}}\hat{g})\big)(\varkappa) . \tag{31}$$

The estimates on $\widehat{\mathcal{N}}_{\mathrm{c},n}$ follow from the fact that the quadratic interaction of critical modes $\mathrm{e}^{\pm\mathrm{i}(k_c+\ell)x}$ with $|\ell| \leq \ell_0$ gives non critical modes, i.e. $\hat{P}_{\mathrm{c}}\widehat{\mathcal{N}}_{2,\mathrm{c}}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{c},n}) \equiv 0$. This is the crucial property used in the proof. □

Next we estimate the integrals in the variation of constant formulas (29)-(30) in terms of

$$R_{\mathrm{c},n} = \sup_{\tau\in[\sigma^2,1]} \|\hat{u}_{\mathrm{c},n}(\tau)\|_{\mathcal{K}^c_{\sigma^n}} \qquad \text{and} \qquad R_{\mathrm{s},n} = \sup_{\tau\in[\sigma^2,1]} \|\hat{u}_{\mathrm{s},n}(\tau)\|_{\mathcal{K}^s_{\sigma^n}} .$$

Lemma 3.13 *Assume $R_{\mathrm{c},n} + R_{\mathrm{s},n} \leq 1$. Then for all $1 \geq \tau \geq \sigma^2$ and all $\sigma \in (0, 1]$ one has*

$$\|\sigma^{-2n} \int_{\sigma^2}^{\tau} \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')} \big(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\mathrm{d}\tau'\|_{\mathcal{K}_{\sigma^n}} \leq C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R^2_{\mathrm{s},n}),$$
$$\|\sigma^{-3n} \int_{\sigma^2}^{\tau} \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')} \big(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\mathrm{d}\tau'\,\|_{\mathcal{K}_{\sigma^n}}$$
$$\leq C(R^2_{\mathrm{c},n} + \sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^{2n} R^2_{\mathrm{s},n}).$$

Proof. Using Lemma 3.11 and Lemma 3.12, we bound the integral in (29) by

$$\sup_{\tau\in[\sigma^2,1]} \|\sigma^{-2n} \int_{\sigma^2}^{\tau} \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\tau')} \big(\widehat{\mathcal{N}}_{\mathrm{c},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\mathrm{d}\tau'\|_{\mathcal{K}_{\sigma^n}}$$
$$\leq C\sigma^{-2n}C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R^2_{\mathrm{s},n})\sigma^{2n} \int_{\sigma^2}^{1} \mathrm{d}\tau' \leq C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R^2_{\mathrm{s},n}).$$

For the integral in (30) we find similarly

$$\sup_{\tau\in[\sigma^2,1]} \|\sigma^{-3n} \int_{\sigma^2}^{\tau} \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\tau')} \big(\widehat{\mathcal{N}}_{\mathrm{s},n}(\hat{u}_{\mathrm{c},n}, \hat{u}_{\mathrm{s},n})\big)(\cdot,\cdot,\tau')\mathrm{d}\tau'\|_{\mathcal{K}^s_{\sigma^n}}$$
$$\leq\ C\sigma^{-2n}C(R^2_{\mathrm{c},n}+\sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n}+\sigma^{2n}R^2_{\mathrm{s},n}) \int_{\sigma^2}^{1} \mathrm{e}^{-C\sigma^{-2n}(1-\tau')} \max(1, \sigma^{3n/2}(1-\tau')^{-3/4})\mathrm{d}\tau'$$
$$\leq\ C(R^2_{\mathrm{c},n} + \sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^{2n} R^2_{\mathrm{s},n}).$$

□

Lemma 3.14 *For all $1 \geq \tau \geq \sigma^2$ and all $\sigma \in (0, 1]$ we have*

$$\|\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(\tau-\sigma^2)} \widehat{\mathcal{L}}^n \hat{E}^{\mathrm{h}}_{\mathrm{c}} \widehat{\mathcal{L}}^{-n} \widehat{\mathcal{L}}\hat{g}\|_{\mathcal{K}_{\sigma^n}} \leq C\sigma^{-5/2}\|\hat{g}\|_{\mathcal{K}_{\sigma^{n-1}}} ,$$
$$\|\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{s},n}(\tau-\sigma^2)} \widehat{\mathcal{L}}^n \hat{E}^{\mathrm{h}}_{\mathrm{s}} \widehat{\mathcal{L}}^{-n}\sigma^{-1} \widehat{\mathcal{L}}\hat{g}\|_{\mathcal{K}_{\sigma^n}} \leq C\sigma^{-7/2}\mathrm{e}^{-C\sigma^{-2n}(\tau-\sigma^2)}\|\hat{g}\|_{\mathcal{K}_{\sigma^{n-1}}}.$$

Proof. These bounds follow immediately from Lemma 3.11 and (26). □

3.8 A priori bounds on the non-linear problem

To state a priori bounds on the solution of (29)-(30) we use the quantities

$$\rho_{\mathrm{c},n} = \|\hat{u}_{\mathrm{c},n}|_{\tau=1}\|_{\mathcal{K}_{\sigma^n}} \qquad \text{and} \qquad \rho_{\mathrm{s},n} = \|\hat{u}_{\mathrm{s},n}|_{\tau=1}\|_{\mathcal{K}_{\sigma^n}} .$$

Lemma 3.15 *For all $n \in \mathbb{N}$ there is a constant $\eta_n > 0$ such that the following holds: If $\rho_{\mathrm{c},n-1}$, $\rho_{\mathrm{s},n-1}$ and $\sigma > 0$ are smaller than η_n, the solutions of (29)-(30) exist for all $\tau \in [\sigma^2, 1]$. Moreover, we have the estimates*

$$\begin{aligned} R_{\mathrm{c},n} &\leq C\sigma^{-5/2}\rho_{\mathrm{c},n-1} + C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2), \\ R_{\mathrm{s},n} &\leq C\sigma^{-7/2}\rho_{\mathrm{s},n-1} + C(R_{\mathrm{c},n}^2 + \sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^{2n} R_{\mathrm{s},n}^2), \end{aligned}$$

with a constant C independent of σ and n.

Remark 3.16 There is no need for a detailed expression for η_n since the existence of the solutions is guaranteed if we can show $R_{\mathrm{c},n} + R_{\mathrm{s},n} < \infty$. By Lemma 3.15 we have detailed control of these quantities in terms of the norms of the initial conditions and σ.

Proof. For the derivation of the estimates we assume in the sequel, without loss of generality, that $R_{\mathrm{c},n} + R_{\mathrm{s},n} \leq 1$.

Using Lemma 3.14 the first term in (30) is bounded by $C\sigma^{-7/2}\rho_{\mathrm{s},n-1}$ with $C > 0$ independent of $\sigma \in (0, 1]$ and $n \in \mathbb{N}$. For the second term, Lemma 3.13 yields a bound $C(R_{\mathrm{c},n}^2 + \sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^{2n} R_{\mathrm{s},n}^2)$. Again by Lemma 3.14 the first term in (29) is bounded by $C\sigma^{-5/2}\rho_{\mathrm{c},n-1}$, and the second term by $C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2)$ due to Lemma 3.13.

The proof of Lemma 3.15 now follows by applying the contraction mapping principle to the system consisting of (29) and (30). For $\rho_{\mathrm{c},n-1} > 0$ and $\rho_{\mathrm{s},n-1} > 0$ sufficiently small, the Lipschitz constant on the right hand side of (29), (30) in $\mathcal{C}([\sigma^2, 1], \mathcal{K}_{\sigma^n}^c \times \mathcal{K}_{\sigma^n}^s)$ is smaller than 1. An application of a classical fixed point argument completes the proof of Lemma 3.15. □

3.9 The iteration process

We decompose the solution $\hat{u}_{\mathrm{c},n}(\cdot, \cdot, \tau)$ for $\tau = 1$ into a Gaussian part and a remainder. Let $\tilde{\psi}(\varkappa) = \mathrm{e}^{-(C_1+\mathrm{i}C_2)\varkappa^2}$ and write

$$\begin{aligned} \hat{u}_{\mathrm{c},n}(\varkappa, x, y, 1) &= A_n\tilde{\psi}(\varkappa)w_{\sigma^{-n}\varkappa,1}(x, y)\mathrm{e}^{\mathrm{i}\omega_0\sigma^{-2n}} \\ &\quad + \overline{A_n}\tilde{\psi}(\varkappa)w_{\sigma^{-n}\varkappa,-1}(x, y)\mathrm{e}^{-\mathrm{i}\omega_0\sigma^{-2n}} + \hat{r}_n(\varkappa, x, y) , \end{aligned}$$

where $\hat{r}_n(0, x, y) = 0$, and the amplitude A_n is in $\mathbb{C}$. Moreover we define $\widehat{\Pi} : \mathcal{K}_{\sigma^n} \to \mathbb{C}$ by $(\widehat{\Pi}f)w_{0,1} = \hat{P}_1(0)f\big|_{\varkappa=0}$. Since $\hat{f}$ is in H^2 as a function of ℓ we have

$$|\widehat{\Pi}\hat{f}| \leq C\|\hat{f}\|_{\mathcal{K}_{\sigma^n}} \tag{32}$$

and $\hat{\Pi}$ is well defined. Then (29) can be decomposed accordingly and takes the form

$$A_n = A_{n-1} + \mathrm{e}^{-\mathrm{i}\omega_0\sigma^{-2n}}\widehat{\Pi}\bigg(\int_{\sigma^2}^1 \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\tau')}\big(\sigma^{-2n}(\widehat{\mathcal{N}}_{\mathrm{c},n})\big)\bigg)\,\mathrm{d}\tau', \qquad (33)$$

$$\hat{r}_n(\varkappa,x) = \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}\hat{r}_{n-1}(\sigma\varkappa,x) + \widehat{X}_n(\varkappa,x) + \sigma^{-2n}\int_{\sigma^2}^1 \Big(\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\tau')}(\widehat{\mathcal{N}}_{\mathrm{c},n})\Big)(\varkappa,x)\,\mathrm{d}\tau', \qquad (34)$$

where

$$\begin{aligned}\widehat{X}_n(\varkappa,x,y) \equiv\ & \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}A_{n-1}\tilde{\psi}(\sigma\varkappa)w_{\sigma^{-n}\varkappa,1}(x,y)\mathrm{e}^{\mathrm{i}\omega_0\sigma^{-2(n-1)}}\\ & -A_n\tilde{\psi}(\varkappa)w_{\sigma^{-n}\varkappa,1}(x,y)\mathrm{e}^{\mathrm{i}\omega_0\sigma^{-2n}}\\ & +\mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}\overline{A_{n-1}}\tilde{\psi}(\sigma\varkappa)w_{\sigma^{-n}\varkappa,-1}(x,y)\mathrm{e}^{-\mathrm{i}\omega_0\sigma^{-2(n-1)}}\\ & -\overline{A_n}\tilde{\psi}(\varkappa)w_{\sigma^{-n}\varkappa,-1}(x,y)\mathrm{e}^{-\mathrm{i}\omega_0\sigma^{-2n}}.\end{aligned}$$

If we define next $\rho_{r,n} = \|\hat{r}_n\|_{\mathcal{K}^c_{\sigma^n}}$ then the above construction implies $\rho_{\mathrm{c},n} \le C(|A_n| + \rho_{r,n})$.

Our main estimate is now

Proposition 3.17 *There is a constant $C > 0$ such that for sufficiently small $\sigma > 0$ the solution $(v_{\mathrm{c},n}, v_{\mathrm{s},n})$ of (29)-(30) satisfies*

$$|A_n - A_{n-1}| \le C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2), \qquad (35)$$

$$\rho_{r,n} \le C\sigma\rho_{r,n-1} + C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2) + C\sigma^n R_{\mathrm{c},n}\,, \qquad (36)$$

$$\rho_{\mathrm{s},n} \le C\sigma^{-7/2}\mathrm{e}^{-C\sigma^{-2n}}\rho_{\mathrm{s},n-1} + C(R_{\mathrm{c},n}^2 + \sigma^n R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^{2n}R_{\mathrm{s},n}^2). \qquad (37)$$

Proof. The bound on $|A_n - A_{n-1}|$ follows from (32), (33) and Lemma 3.13.

Next we bound $\hat{r}_n$ in terms of $\hat{r}_{n-1}$, using (34). As in (11), the first term is the one where the projection is crucial. For $\sigma > 0$ sufficiently small and $\hat{r}_{n-1} \in \mathcal{K}_{\sigma^{n-1}}$ with $\hat{r}_{n-1}(0) = 0$ we have

$$\|(\varkappa,x) \mapsto \mathrm{e}^{\sigma^{-2n}\widehat{\mathcal{M}}_{\mathrm{c},n}(1-\sigma^2)}\hat{r}_{n-1}(\sigma\varkappa,x)\|_{\mathcal{K}_{\sigma^n}} \le C\sigma\|\hat{r}_{n-1}\|_{\mathcal{K}_{\sigma^{n-1}}} = C\sigma\rho_{r,n-1}$$

The last term in (34) has been bounded in the proof of Lemma 3.13 by $C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2)$.

Finally we have

$$\|\widehat{X}_n\|_{\mathcal{K}_{\sigma^n}} \le C(R_{\mathrm{c},n}R_{\mathrm{s},n} + \sigma^n R_{\mathrm{s},n}^2) + C\sigma^n R_{\mathrm{c},n}\,,$$

where the last term is due to $\mu_1(\ell) = -C_1\ell^2 + \mathcal{O}(\ell^3)$ not being exactly a parabola. For details see [Schn96]. Collecting the bounds, the estimate (36) for $\rho_{r,n}$ follows.

The estimate in (37) for $\rho_{\mathrm{s},n}$ is immediately clear. Hence, the proof of Proposition 3.17 is complete. □

Proof of Theorem 3.9: The proof is an induction argument, using repeatedly the above estimates. We write C for constants which can be chosen independent of σ and n. From Lemma 3.15 we obtain that for $C_1 > 0$ sufficiently small, there exists a $C_2 > 0$ such that

$$\begin{aligned} R_{\mathrm{c},n} &\leq C_2(\sigma^{-5/2}\rho_{\mathrm{c},n-1} + (\sigma^{-7/2}\rho_{\mathrm{s},n-1})^2), \\ R_{\mathrm{s},n} &\leq C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}), \end{aligned}$$

if

$$|\sigma^{-5/2}\rho_{\mathrm{c},n-1}| + |\sigma^{-7/2}\rho_{\mathrm{s},n-1}| \leq C_1. \tag{38}$$

Using Proposition 3.17 we find

$$\begin{aligned} |A_n - A_{n-1}| &\leq C\Big[C_2(\sigma^{-5/2}\rho_{\mathrm{c},n-1} + (\sigma^{-7/2}\rho_{\mathrm{s},n-1})^2)C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}) \\ &\qquad + \sigma^n(C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}))^2\Big], \\ \rho_{r,n} &\leq C\sigma\rho_{r,n-1} \\ &\quad + C\Big[C_2(\sigma^{-5/2}\rho_{\mathrm{c},n-1} + (\sigma^{-7/2}\rho_{\mathrm{s},n-1})^2)C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}) \\ &\qquad + \sigma^n(C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}))^2\Big], \\ \rho_{\mathrm{c},n} &\leq C(|A_n| + \rho_{r,n}), \\ \rho_{\mathrm{s},n} &\leq C\sigma^{-7/2}\mathrm{e}^{-C\sigma^{-2n}}\rho_{\mathrm{s},n-1} \\ &\quad + C\Big[(C_2(\sigma^{-5/2}\rho_{\mathrm{c},n-1} + (\sigma^{-7/2}\rho_{\mathrm{s},n-1})^2))^2 \\ &\qquad + \sigma^n C_2(\sigma^{-5/2}\rho_{\mathrm{c},n-1} + (\sigma^{-7/2}\rho_{\mathrm{s},n-1})^2)C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}) \\ &\qquad + \sigma^{2n}(C_2((\sigma^{-5/2}\rho_{\mathrm{c},n-1})^2 + \sigma^{-7/2}\rho_{\mathrm{s},n-1}))^2\Big]. \end{aligned}$$

In a **first step** we show $\rho_{\mathrm{s},n} = \mathcal{O}(\varepsilon^2)$ for $n \geq 2$. We introduce $A_n = \varepsilon\tilde{A}_n$, $\rho_{r,n} = \varepsilon\tilde{\rho}_{r,n}$, and $\rho_{\mathrm{s},n} = \varepsilon\tilde{\rho}_{\mathrm{s},n}$. We obtain

$$\begin{aligned} |\tilde{A}_2 - \tilde{A}_1| &\leq \mathcal{O}(\varepsilon), \\ \tilde{\rho}_{r,2} &\leq C\sigma\tilde{\rho}_{r,1} + \mathcal{O}(\varepsilon), \\ \tilde{\rho}_{\mathrm{s},2} &\leq C\sigma^{-7/2}\mathrm{e}^{-C\sigma^{-2}}\tilde{\rho}_{\mathrm{s},1} + \mathcal{O}(\varepsilon). \end{aligned}$$

For given ε and b we choose $\sigma > 0$ so small that

$$C\sigma^{-7/2}\mathrm{e}^{-C\sigma^{-2}} \leq \varepsilon \qquad \text{and} \qquad C \leq \sigma^{-2b}.$$

Since σ goes logarithmically in ε the assumption (38) is still satisfied if $\rho_{\mathrm{c},1} + \rho_{\mathrm{s},1} = \mathcal{O}(\varepsilon)$ for $\varepsilon \to 0$. Then we obtain $\tilde{\rho}_{\mathrm{s},2} = \mathcal{O}(\varepsilon^{1-b})$, or equivalently $\rho_{\mathrm{s},2} = \mathcal{O}(\varepsilon^{2-b})$ with an arbitrary small, but fixed $b > 0$, due to the nonlinear terms.

In a **second step** we prove the assertions of Theorem 3.9. We introduce $A_n = \varepsilon \tilde{A}_n$, $\rho_{r,n} = \varepsilon \tilde{\rho}_{c,n}$, and $\rho_{\mathrm{s},n} = \varepsilon^{2-b} \tilde{\rho}_{\mathrm{s},n}$. This time we obtain a system of the form

$$\begin{aligned} |\tilde{A}_n - \tilde{A}_{n-1}| &\leq \mathcal{O}(\varepsilon^{2-b}), \\ \tilde{\rho}_{r,n} &\leq C\sigma \tilde{\rho}_{r,n-1} + \mathcal{O}(\varepsilon^{2-b}) + C\sigma^n \\ \tilde{\rho}_{\mathrm{s},n} &\leq C\sigma^{-7/2} \mathrm{e}^{-C\sigma^{-2}} \tilde{\rho}_{\mathrm{s},n-1} + C\tilde{\rho}^2_{\mathrm{c},n-1}. \end{aligned}$$

Thus the sequence $\tilde{\rho}_{\mathrm{s},n}$ is mainly bounded by $C\tilde{\rho}^2_{\mathrm{c},n-1}$. The sequence $\tilde{\rho}_{r,n}$ shows some decay as $(C\sigma)^n$ from the first term leading to t^{b-1} by the choice $C \leq \sigma^{-2b}$ and some growth of order $\mathcal{O}(\varepsilon^{2-b})$ from the second term. The sequence $|\tilde{A}_n|$ can be bounded by $|\tilde{A}_1| + n\mathcal{O}(\varepsilon^{2-b})$. Hence $|\tilde{A}_n|$ is of order $\mathcal{O}(1)$ for $n \leq q\varepsilon^{-2+b}$ for a $q > 0$ independent of ε. Doing back the scalings as in (14) shows the estimate (27) for $t \in [0, \exp(q\varepsilon^{b-2})]$. □

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