Existence and stability of modulating pulse solutions in Maxwell's equations describing nonlinear optics

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Abstract

Modulating pulse solutions play a big rôle in modern long distance high speed communication. Such solutions consist of a traveling pulse–like envelope modulating an underlying electromagnetic wave. In this paper we show that under certain assumptions such solutions exist and are dynamically stable for the associated nonlinear partial differential equations, namely Maxwell's integro–differential equations describing nonlinear optics. The analysis is worked out in detail for bulk media, and we discuss how the results extend to optical fibers and to parametrically forced systems.

1 Introduction

The transport of information by light pulses in optical fibers has become a key technology in long distance high speed communication. From a physical point of view such a bit consists of a pulse-like envelope modulating an underlying spatially and temporarily oscillating monochromatic electromagnetic wave train, a so called modulating pulse. For the transport of information over large distances it is essential that these modulating pulses retain their shape over long times and that they are dynamically stable with respect to perturbations.

In order to analyze mathematically such a situation a number of models as the Nonlinear Schrödinger equation or generalized Ginzburg–Landau equations have been derived by multiple scaling to describe the envelope of the wave packets. For these model equations there exist a number of results, as the existence of exponentially stable pulse families. Hence the analysis of these model problems predicts the possibility of exponentially stable modulating pulse families in the system of nonlinear partial differential equations describing nonlinear optics. It is the purpose of this paper to prove such a result rigorously.

Maxwell's equations for light in nonlinear optical material are given by

$$\triangle \vec{E} - \nabla(\nabla \cdot \vec{E}) - \partial_t^2 \vec{E} = \partial_t^2 \vec{P}, \tag{1.1}$$

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where $\vec{E} = \vec{E}(t,\vec{x}) \in \mathbb{R}^3$ is the electric field, t is the time, $\vec{x} = (x,y,z) \in \mathbb{R}^3$, $\vec{P} = \vec{P}(t,\vec{x}) \in \mathbb{R}^3$ is the material polarization, and where the speed of light in vacuum and the dielectric constant are scaled to 1. The polarization $\vec{P} = \vec{P}_{\rm l} + \vec{P}_{\rm nl}$ splits into a linear and into a nonlinear part, which in general both depend on the history of the electric field. In centrosymmetric isotropic bulk material, the constitutive law for the linear response $\vec{P}_{\rm l}$ is given by

$$\vec{P}_{\rm l}(t,\vec{x}) = \vec{P}_{\rm l}(\vec{E})(t,x) = (\chi_1 *_t \vec{E})(t,\vec{x}) = \int_{-\infty}^{\infty} \chi_1(t-\tau)\vec{E}(\tau,\vec{x})\,\mathrm{d}\tau, \tag{1.2}$$

where χ_1 in (1.2) is a scalar function, independent of $\vec{x}$, with $\chi_1(t) = 0$ for $t < 0$ due to causality, and similar for the nonlinear polarization. The case of optical fibers where χ_1 also depends on the transverse directions y, z will be discussed later. We also postpone the specification of $\vec{P}_{\rm nl}$. We refer to the textbooks [NM92, HK95] for a comprehensive presentation of the physical background and various mathematical aspects concerning nonlinear optics. See also [Gow93, Dut98] for concrete technological aspects of fiber communication.

Through most of our analysis we assume the simplest case that $\vec{E}$ is linearly polarized and only depends on x, i.e.

$$\vec{E}(t,\vec{x}) = u(t,x)\hat{k} \quad \text{with} \quad \|\hat{k}\|_{\mathbb{R}^3} = 1, \quad (1,0,0)\cdot\hat{k} = 0. \tag{1.3}$$

Then (1.1) simplifies to

$$\partial_t^2 u(t,x) = \partial_x^2 u(t,x) - \partial_t^2 p_{\rm l}(t,x) - \partial_t^2 p_{\rm nl}(t,x), \tag{1.4}$$

with $p_{\rm l}(t,x), p_{\rm nl}(t,x) \in \mathbb{R}$ such that $\vec{P}_{\rm l}(t,\vec{x}) = p_{\rm l}(t,x)\hat{k}$, $\vec{P}_{\rm nl}(t,\vec{x}) = p_{\rm nl}(t,x)\hat{k}$. Note that due to (1.2) Maxwell's equations become integro–differential equations. Then equation (1.4) is an evolutionary problem with respect to time t with the initial conditions

$$u(t,x)|_{t=0} = u_0(x), \quad \partial_t u(t,x)|_{t=0} = u_1(x) \tag{1.5}$$

and the "history condition"

$$u(t,x) = u_{\rm h}(t,x) \text{ for } (t,x) \in (-\infty,0)\times\mathbb{R}. \tag{1.6}$$

Note that as usual we do not assume u_h to be a solution of equation (1.4).

As already said, in nonlinear optics one is in particular interested in modulating pulse solutions of (1.1) since these are used to transport information as bits and bytes through optical fibers. Typically, they are approximated by a slowly varying envelope formalism, also called Ginzburg–Landau ansatz. In a first step the system is analyzed by inserting $\vec{E}(t,\vec{x}) = \mathrm{e}^{\mathrm{i}(kx-\omega t)}\hat{k}$ with $(1,0,0)\cdot\hat{k}=0$ into the linearization of (1.1) at $\vec{E}=0$. This gives the dispersion relation

$$k^2 = \omega^2(1+\hat{\chi}_1(\omega)), \tag{1.7}$$

where the linear susceptibility $\hat{\chi}$ is given by

$$\hat{\chi}_1(\omega) = (\mathcal{F}_t\chi_1)(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\chi_1(t)\,\mathrm{d}t = \hat{\chi}_{1r}(\omega) + \mathrm{i}\hat{\chi}_{1i}(\omega),$$

and where $\hat{n}(\omega) = \sqrt{1+\hat{\chi}_1(\omega)}$ is called the (linear) refractive index.

As the second step we let

$$\vec{E}(t,\vec{x}) = [\varepsilon^{1/2}q_+(\varepsilon^2 t, \varepsilon(x-\omega_r' t))\mathrm{e}^{\mathrm{i}(k_c x-\omega_r t)} + \text{c.c.}]\hat{k} + \text{h.o.t}, \tag{1.8}$$

with a small parameter $0<\varepsilon\ll 1$, where $(1,0,0)\cdot\hat{k}=0$, $q_+ = q_+(T,X)\in\mathbb{C}$ is the envelope, $e_+(x,t) = \mathrm{e}^{\mathrm{i}(k_c x-\omega_r t)}$ is the carrier wave, and where the (critical) spatial wavenumber k_c, the temporal wavenumber $\omega_r>0$, and the group speed $\omega_r' = \frac{d\omega}{dk}|_{k=k_c} > 0$ are related by the dispersion relation (1.7). See Section 2 for details. In (1.8) c.c. means complex conjugate and h.o.t denotes terms of higher order in ε. Inserting (1.8) into (1.1) one finds that, under certain assumptions, q_+ has to satisfy a generalized Ginzburg–Landau equation (gGLe) of the form

$$\partial_T q_+ = c_2\partial_X^2 q_+ + c_0 q_+ + c_3|q_+|^2 q_+ + c_5|q_+|^4 q_+, \tag{1.9}$$

with constants $c_j = c_{jr}+\mathrm{i}c_{ji}\in\mathbb{C}$. Because of the quintic nonlinearity ($c_5\neq 0$) and due to dissipation ($c_{2r}>0$) and damping ($c_{0r}<0$), (1.9) is also called perturbed cubic-quintic Nonlinear Schrödinger equation.

For small $c_{0r}, c_{2r}, c_{3r}, c_{5r}$ the gGLe has a two-parameter family of rotating pulse solutions $q_{+,\text{pu}}(X;X_0,\theta_0)\mathrm{e}^{\mathrm{i}\omega_+ T}$ with $\omega_+ = \omega_+(c_{0r},c_2,c_3,c_5)$, parametrized by space translation $X_0\in\mathbb{R}$ and phase $\theta_0\in[0,2\pi)$. Moreover, in [KS98] it is shown that for certain parameters

$$(c_{0r}, c_1, c_3, c_5)\in\mathcal{P},$$

where $\mathcal{P}\subset\mathbb{R}^7$ is an open set, the family of pulses $q_{+,\text{pu}}$ is exponentially stable in the gGLe.

Inserting $q_{+,\text{pu}}$ into (1.8) we thus obtain an approximate family of exponentially stable modulating pulse solutions for (1.1), consisting of a traveling pulse-like envelope which modulates the spatially and temporarily oscillating monochromatic wave train $\mathrm{e}^{\mathrm{i}(k_c x-\omega_r t)}$; see Figure 1 for a sketch.

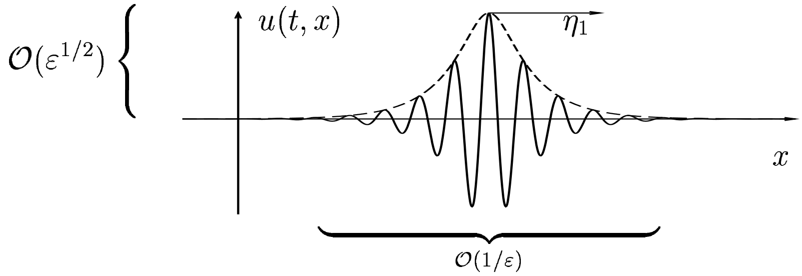

Figure 1: Sketch of a modulating pulse traveling at some speed $\eta_1 \approx \omega_r'(k_c)$.

As already said it is the purpose of the present paper to go beyond these formal arguments and to prove rigorously the existence and exponential stability of such modulating pulse solutions directly for the equations of nonlinear optics (1.1). It will be explained below that such a result is not obvious at all from the formal analysis. Moreover, it is the purpose of this paper to go beyond the consideration of phenomenological model problems as in [Sch00, Uec01].

Existence (E) For linearly polarized solutions, the result reads as follows, see Section 2 for the precise assumptions and Theorem 2.8 for the exact statement.

Under suitable assumptions on $\vec{P}_\text{l}$ and $\vec{P}_\text{nl}$, such that in particular in (1.9) we have $(c_{0r}, c_1, c_3, c_5) \in \mathcal{P}$, (1.1) has a two parameter family of modulating pulse solutions in the form

$$\begin{aligned} \vec{E}(t,\vec{x}) &= u_\text{pu}(x-\eta_1 t - x_0, k_c x - \eta_0 t - \theta_0)\hat{k} \\ u_\text{pu}(\xi,p) &= \varepsilon^{1/2} q_{+,\text{pu}}(\varepsilon\xi)\mathrm{e}^{\mathrm{i}p} + \text{c.c.} + \mathcal{O}(\varepsilon^{3/2}) \in \mathbb{R}, \\ \lim_{\xi\to\pm\infty} u_\text{pu}(\xi,p) &= 0, \quad u_\text{pu}(\xi,p+2\pi) = u_\text{pu}(\xi,p), \end{aligned}$$

where $0 < \varepsilon \ll 1$, $(1,0,0)\cdot\hat{k} = 0$, $x_0 \in \mathbb{R}$, $\theta_0 \in [0,2\pi)$, and where $\eta_1 = \omega_r' + \mathcal{O}(\varepsilon^2)$ and $\eta_0 = \omega_r + \mathcal{O}(\varepsilon^2)$ are small corrections of the linear group speed and the temporal wavenumber, respectively.

This result is proven in Section 4 using a spatial dynamics formulation for (1.1) and center manifold theory. The reduced equations on the center manifold can be interpreted as a small perturbation of the stationary gGLe (1.9). Due to the spectral properties of the pulses for the gGLe we can show that these pulses also exist in the reduced equation with the two small corrections of the temporal wavenumber and the group speed.

Stability (S) In a frame moving with speed η_1 the modulating pulses are time-periodic with period $2\pi/(\eta_0 - \eta_1)$. Therefore in Section 5 we will use Floquet theory to prove their stability. The Floquet spectrum is computed by combining a validity result for the gGLe as a modulation equation for (1.1) with the stability properties of the pulses in the gGLe. This gives the following result, see Theorem 2.9.

Fix $m \geq 1$. Under the assumptions of (E), for all $C_2 > 0$ there exists a $C_1 > 0$ such that for $\varepsilon > 0$ sufficiently small the following holds. If for some $x_0 \in \mathbb{R}$, $\theta_0 \in [0, 2\pi)$ we have

$$\begin{aligned} \|u_0(\cdot) - u_{\rm pu}(\cdot{-}x_0, k_c \cdot{-}\theta_0)\|_{H^{m+1}(\mathbb{R})} &\leq C_1\varepsilon^{1/2}, \\ \|u_1(\cdot) - \frac{\rm d}{{\rm d}t}u_{\rm pu}(\cdot{-}x_0, k_c \cdot{-}\theta_0)\|_{H^{m}(\mathbb{R})} &\leq C_1\varepsilon^{1/2}, \\ \sup_{t\leq 0}\|u_h(t,\cdot) - u_{\rm pu}(\cdot - \eta_1 t - x_0, \cdot - \eta_0 t - \theta_0)\|_{H^{m+1}(\mathbb{R})} &\leq C_1\varepsilon^{1/2}, \end{aligned}$$

then for all $t > 0$ there exist a unique solution of (1.4)–(1.6) and constants $x_1 \in \mathbb{R}$, $\theta_1 \in [0, 2\pi)$ such that

$$\begin{aligned} &\|u(t,\cdot) - u_{\rm pu}(\cdot{-}\eta_1 t{-}x_1, k_c \cdot{-}\eta_0 t{-}\theta_1)\|_{H^{m+1}(\mathbb{R})} \\ &\quad + \|\partial_t u(t,\cdot) - \frac{\rm d}{{\rm d}t}u_{\rm pu}(\cdot{-}\eta_1 t{-}x_1, k_c \cdot{-}\eta_0 t{-}\theta_1)\|_{H^{m}(\mathbb{R})} \leq C_2\varepsilon^{1/2}{\rm e}^{-b\varepsilon^2 t}. \end{aligned}$$

as $t \to \infty$.

We close this introduction with a number of remarks and a plan of the paper.

Remark 1.1 All our calculations can be done for circular polarized waves and also for the full vector Maxwell equation (1.1). However, the ansatz (1.3) is not appropriate for susceptibilities dependent on $\vec{x}$, that is, for optical fibers. We comment on this case in Section 6.

Remark 1.2 In contrast to most of the existing literature we treat Maxwell's equation (1.1) as an evolution equation in t which is somewhat inconvenient due to the memory of the polarization. If the unbounded space variable x is taken as evolutionary variable, then the memory term can of course be converted to a simple multiplication operator by considering the Fourier transform of (1.1) with respect to t. We obtain

$$\partial_x^2 \tilde{u}(\omega, x) = -\omega^2(1 + \hat{\chi}_1(\omega))\tilde{u}(\omega, x) - \omega^2 \tilde{P}_{\rm NL}(\omega, x), \tag{1.10}$$

where $\tilde{u}(\omega, x) = (\mathcal{F}_t u)(\omega, x) = \int_{-\infty}^{\infty} {\rm e}^{{\rm i}\omega t} u(t, x)\,{\rm d}t$ and $\tilde{P}_{\rm NL}(\omega, x)$ contains convolutions of $\tilde{u}(\cdot, x)$ with respect to $\omega \in \mathbb{R}$. For (1.10), by a slowly varying envelope approximation, again the gGLe (1.9) can be derived with T and X interchanged and in a different scaling.

However, the validity of the gGLe (1.9) as a modulation equation does not hold in the "nonconservative" case considered here (expressed by dissipation $c_{2r} > 0$ and damping $c_{0r} < 0$ in (1.9)). To see this, consider an error in the approximation at "time" $x = 0$, located at, say, $t = 0$, that corresponds to a small wave packet traveling to the left (in negative x-direction) for positive t. As it evolves for $t > 0$ it is $\mathcal{O}(1)$–damped. This means that it grows in negative time t and positive space x direction. For the x-evolution this means that the small initial error grows exponentially, thus causing the approximation to break down already on an $\mathcal{O}(1)$–time scale which is much smaller than the natural time scale $\mathcal{O}(1/\varepsilon^2)$.

Remark 1.3 Since due to the previous remark we consider (1.1) as an evolution equation in t we have to cope with the memory of the polarization. Therefore we will restrict the linear and nonlinear susceptibilities to a special (physically motivated) class of functions, which will allow us to convert (1.1) into a system of autonomous differential equations without memory. This approach is widely used for numerical simulations of Maxwell's equation in the time domain, see [HK96] and the references therein. This ansatz is in particular used for the spatial dynamics formulation appearing in the construction of the modulating pulses, see Section 4.

Remark 1.4 As already said the method used in this paper has been applied to two model problems in [Sch00, Uec01]. The idea to construct special traveling wave solutions to systems on unbounded domains via spatial dynamics and center manifold reduction goes back on [Kir82] (see [Mi88a] for a general result). It is nowadays a well-established theory extensively used in a number of applications, as water wave problems ([Mi86, Ki88, IK92]) and elasticity problems ([Mi88b, Mi90]).

Remark 1.5 Apart from the problems associated with the memory and the fact that (1.1) is a system of hyperbolic (and not parabolic) type, the main new difficulty comes from the fact that due to reflection symmetry of the problem the gGLe is only one possible modulation equation for (1.1). The determining system of amplitude equations is given by a nonlinearly coupled system of gGLe (2.11) which contains (1.9) as invariant subsystem. Since (1.1) is reflection symmetric the associated amplitude equation (2.11) cover bi-directional motion, where (1.9) only covers unidirectional motion.

The plan of the paper is as follows. In Section 2 we first describe in detail model (1.4) to (1.6), and formulate the constitutive laws for the linear and nonlinear polarization. Next we explain the reduction to the system of nonlinearly coupled gGLe and the results from [KS98] concerning the existence and stability of pulse–solutions in the single gGLe. Then we state our main results Theorems 2.8 and 2.9. In the short but important Section 3 we convert equation (1.4) with the memory term into an extended system of autonomous partial differential equations without memory. This extended system will be the basis of the proof of Theorems 2.8 and 2.9. In Section 4 we show the existence of the modulating pulses, using a spatial dynamics formulation for the extended system and a center manifold reduction. The stability of the modulating pulses is shown in Section 5 by proving the existence and stability of the pulses in the nonlinear coupled system of gGLes and combining it with the validity of this system as the modulation equation for (1.4). In Section 6 we show how our results extend to optical fibers and comment on the periodically forced Nonlinear Schrödinger equation as modulation equation and on modulating multi-pulse solutions. In the Appendices we collect some results about the derivation, approximation properties and pulse solutions of the associated modulation equations.

Notations. Throughout this paper we assume $0 \leq \varepsilon \ll 1$. Many constants are uniformly denoted by C.

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2 Setup and statement of results

2.1 The linear response

For ω in the optical range, the linear susceptibility $\hat{\chi}_1(\omega) = \hat{\chi}_{1r}(\omega) + \mathrm{i}\hat{\chi}_{1i}(\omega)$ is usually modeled in the form

$$\hat{\chi}_1(\omega) = \sum_{j=1}^{n_1} \frac{b_{j,1}d_{j,1}}{\delta_{j,1}^2 + d_{j,1}^2 - \omega^2 - 2\mathrm{i}\delta_{j,1}\omega}, \tag{2.1}$$

where the constants $b_{j,1}, d_{j,1}, \delta_{j,1} > 0$ are fitted from experimental data for the refractive index $\hat{n}(\omega) = \sqrt{1 + \hat{\chi}_1(\omega)}$, see, e.g., [Mal65].

Here we extend (2.1) to all frequencies. Figure 2a) shows the typical graphs of $\hat{\chi}_{1r} + \chi_0$ and $\hat{\chi}_{1i}$ (with $n_1 = 2$). The constant $\chi_0 > 0$ appearing in Figure 2 corresponds to an instantaneous part of the linear response and will ensure that the group-velocity is always below 1, the (nondimensionalized) vacuum speed of light.

Note that the ansatz (2.1) gives almost no absorption ($\hat{\chi}_i(\omega) > 0$) except near the material resonances ω_j.

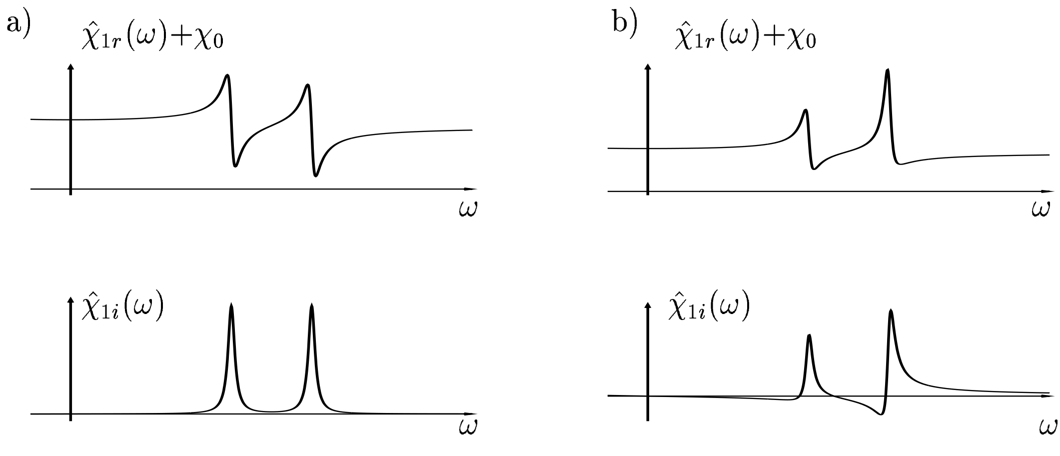

Figure 2: Susceptibility and absorption for typical materials; a) with ansatz (2.1), b) with ansatz (2.2) modeling the transmission window.

A more involved ansatz which we will use in the following is given by

$$\begin{aligned}\hat{\chi}_1(\omega) = \sum_{j=1}^{n_1}\Bigg(&\frac{b_{j,1}d_{j,1}}{\delta_{j,1}^2 + d_{j,1}^2 - \omega^2 - 2\mathrm{i}\delta_{j,1}\omega}\\ &+ \frac{a_{j,1}\gamma_{j,1}(\gamma_{j,1}^2 + c_{j,1}^2 + \omega^2) + \mathrm{i}a_{j,1}\omega(\gamma_{j,1}^2 - c_{j,1}^2 + \omega^2)}{(\gamma_{j,1}^2 + c_{j,1}^2 - \omega^2)^2 + 4\gamma_{j,1}^2\omega^2}\Bigg),\end{aligned} \tag{2.2}$$

which is written in such a way that

$$\chi_1(t) = H(t)\sum_{j=1}^{n_1}(a_{j,1}\cos(c_{j,1}t)\mathrm{e}^{-\gamma_{j,1}t} + b_{j,1}\sin(d_{j,1}t)\mathrm{e}^{-\delta_{j,1}t}) \tag{2.3}$$

has a simple form. Here H is the Heaviside function, $H(t)$=1 for $t\geq 0$ and $H(t)$=0 for $t\leq 0$.

In optical material one usually has a small amount of absorption for all frequencies, with a minimum of absorption at certain frequencies, called the transmission windows and modeled by (2.2). In order to model the small frequency–independent amount of absorption not yet present in (2.2) we modify (1.4)–(1.6) by introducing a small constant $0 < \gamma_0$ to

$$(1+\chi_0)\partial_t^2 u = \partial_x^2 u - \partial_t^2(\chi_1 *_t u) - 2\gamma_0\partial_t u - \gamma_0^2 u - \partial_t^2 p_{\mathrm{nl}}(t,x), \tag{2.4}$$

$$u(t,x)|_{t=0} = u_0(x), \quad \partial_t u(t,x)|_{t=0} = u_1(x), \quad u(t,x) = u_h(t,x) \text{ for } t<0. \tag{2.5}$$

Remark 2.1 As already said there are a number of parameters which allow us to fit this ansatz to the experimental data in the optical range. We will assume in the following that these coefficients are chosen in such a way that the system has the subsequent properties. These coefficients have to be considered as effective (or averaged) coefficients taking also into account some spatially distributed amplifiers.

Remark 2.2 The crucial reason for modeling $\hat{\chi}_1$ by a rational function as in (2.2) and in the existing literature is that by this choice the memory term $\partial_t^2(\chi_1 *_t u)$ can be converted later on into a system of differential equations, see Section 3.

We start to analyze (2.4) by considering the linearized system. The modified dispersion relation reads

$$k^2 = \omega^2(1 + \chi_0 + \hat{\chi}_1(\omega)) - \gamma_0^2 + 2\mathrm{i}\gamma_0\omega \tag{2.6}$$

Figure 3 shows the solution $k = k(\omega) = k_r(\omega) + \mathrm{i}k_i(\omega) \in \mathbb{C}$ for $\omega \in \mathbb{R}$ (with $\hat{\chi}$ from figure 2b)). There exist an optimal frequency ω_c such that for right traveling wave trains $\mathrm{e}^{\mathrm{i}(k(\omega)x-\omega t)} = \mathrm{e}^{-k_i(\omega)x}\mathrm{e}^{\mathrm{i}(k_r(\omega)x-\omega t)}$ the damping $k_i(\omega)$ has a minimum at ω_c. However, $-k(\omega)$ is also a solution of (2.6), such that in this picture left traveling

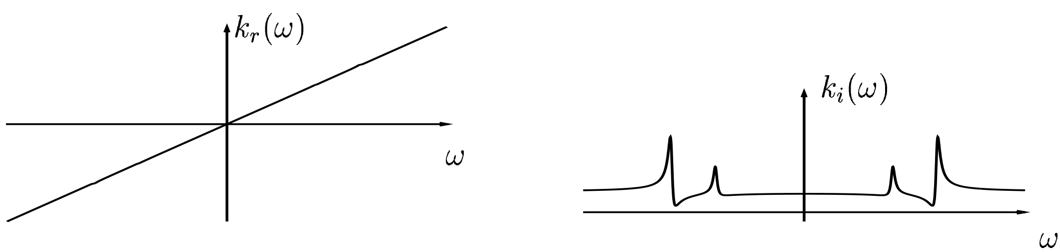

Figure 3: $k(\omega)$ as obtained from solving the dispersion relation (2.6) for k. Note that $k_r(\omega)$ is no straight line. The picture shows the asymptotic behavior for $|\omega| \to \infty$

wave trains are exponentially amplified for negative times and positive x. Therefore, as explained in Remark 1.2, we treat Maxwell's equation as an evolution equation in t. Thus, we need to solve (2.6) for $\omega = \omega_r + \mathrm{i}\omega_i \in \mathbb{C}$ as a function of $k \in \mathbb{R}$. For $\hat{\chi}$ in the form (2.2) the dispersion relation is equivalent to a polynomial in ω of even order $2N$, with k^2 as a parameter. We have $2N = 2\widetilde{N}+2$, where $\widetilde{N}$ is the number of different vectors $(c_{j,1}, \gamma_{j,1}, d_{j,1}, \beta_{j,1})$ of parameters in (2.2), corresponding to different resonances in the material. Thus we obtain $2N$ solutions $\omega_j(k)$ of (2.6). Due to the special form of $\hat{\chi}$ these fulfill $\omega_{2j} = -\overline{\omega}_{2j-1}$, $j = 1, \ldots, N$. Figure 4 shows typical graphs for ω_j, (with $\hat{\chi}(\omega)$ from figure 2b)): There is one critical curve $\omega_1(k)$ (and

$-\overline{\omega}_1(k))$ and a critical wavenumber k_c such that ω_{1i} has a maximum at $k = k_c$, $\omega_{1i}(k_c) \approx 0$, and is strictly below the k–axes for k away from k_c, and all other curves $\omega_{ji}(k)$, $j \geq 3$, are strictly negative for all k.

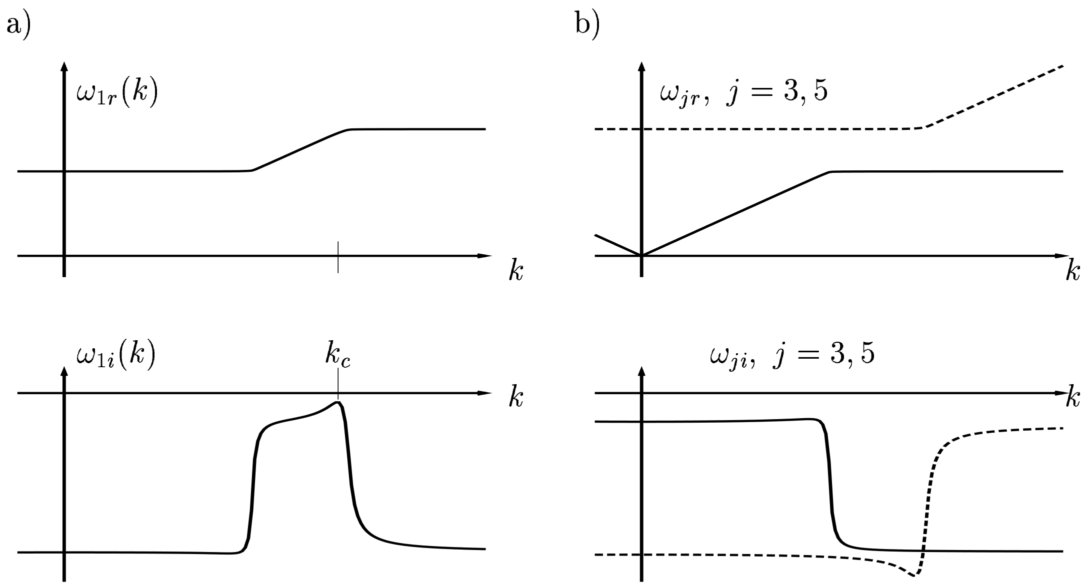

Figure 4: $\omega_j(k) = \omega_{jr}(k) + \mathrm{i}\omega_{ji}(k)$ as obtained from solving the dispersion relation (2.6). a) the critical curve $\omega_1(k)$, b) the damped modes.

2.2 The nonlinear response

The nonlinear polarization $\vec{P}_{\mathrm{nl}}(t,x) = p_{\mathrm{nl}}(t,x)\hat{k}$ also splits into an instantaneous part and a part with memory see, e.g. [Men99]. For our analysis we don't allow for an instantaneous part, but see Remark 3.1, and we assume a focusing saturable nonlinearity, where the third order term is assumed to be small, proportional to the small bifurcation parameter $\varepsilon > 0$, see, e.g. [GH91]. Then in general $p_{\mathrm{nl}}^{\mathrm{m}}$ is given by

$$p_{\mathrm{nl}}^{\mathrm{m}}(t,x) = \varepsilon p_{\mathrm{nl},3}(t,x) + p_{\mathrm{nl},5}(t,x) \text{ with}$$
$$p_{\mathrm{nl},3}(t,x) = \iiint \tilde{\chi}_3(t-\tau_1, t-\tau_2, t-\tau_3) u(\tau_1,x)u(\tau_2,x)u(\tau_3,x)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3,$$
$$p_{\mathrm{nl},5}(t,x) = \int \cdots \int \tilde{\chi}_5(t-\tau_1,\dots,t-\tau_5) u(\tau_1,x)\cdots u(\tau_5,x)\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_5.$$

However, for most materials we have $\tilde{\chi}_3(t_1,t_2,t_3) = \delta(t_1-t_2)\delta(t_1-t_3)\chi_3(t_1)$ and $\tilde{\chi}_5(t_1,\dots,t_5) = \delta(t_1-t_2)\cdots\delta(t_1-t_5)\chi_5(t_1)$ where δ is the Dirac delta distribution.

Therefore, we restrict our analysis to the case

$$p_{\rm nl}(t,x) = \varepsilon \int \chi_3(t-\tau)u^3(\tau,x)\,{\rm d}\tau + \int \chi_5(t-\tau)u^5(\tau,x)\,{\rm d}\tau. \tag{2.7}$$

We assume the nonlinear susceptibilities to be given also in the form (2.2), i.e.,

$$\begin{aligned}\hat\chi_3(\omega) = \sum_{j=1}^{n_3}\Bigg(&\frac{b_{j,3}d_{j,3}}{\gamma_{j,3}^2+d_{j,3}^2-\omega^2-2{\rm i}\gamma_{j,3}\omega}\\ &+\frac{a_{j,3}\gamma_{j,3}(\gamma_{j,3}^2+c_{j,3}^2+\omega^2)+{\rm i}a_{j,3}\omega(\gamma_{j,3}^2-c_{j,3}^2+\omega^2)}{(\gamma_{j,3}^2+c_{j,3}^2-\omega^2)^2+4\gamma_{j,3}^2\omega^2}\Bigg),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\hat\chi_5(\omega) = \sum_{j=1}^{n_5}\Bigg(&\frac{b_{j,5}d_{j,5}}{\gamma_{j,5}^2+d_{j,5}^2-\omega^2-2{\rm i}\gamma_{j,5}\omega}\\ &+\frac{a_{j,5}\gamma_{j,5}(\gamma_{j,5}^2+c_{j,5}^2+\omega^2)+{\rm i}a_{j,5}\omega(\gamma_{j,5}^2-c_{j,5}^2+\omega^2)}{(\gamma_{j,5}^2+c_{j,5}^2-\omega^2)^2+4\gamma_{j,5}^2\omega^2}\Bigg).\end{aligned} \tag{2.9}$$

Remark 2.3 Remarks 2.1 and 2.2 also hold for the nonlinear responses.

2.3 The results

Let us first collect our assumptions; all constants are $\mathcal{O}(1)$ with respect to the small bifurcation parameter ε which is introduced in (A2).

(A1) $\hat\chi_1$ is given by (2.2).

This implies that the dispersion relation (2.6) has $2N$ roots $\omega_j(k)\in\mathbb{C}$, $1\le j\le 2N$ with $\omega_{2l}(k)=-\overline{\omega}_{2l-1}(k)$, $l=1,\dots,N$, with $N=\tilde N+1$, where $\tilde N$ is the number of different vectors $(c_{j,1},\gamma_{j,1},d_{j,1},\beta_{j,1})$ in (2.2). For these roots $\omega_j(k)$ we assume:

(A2) There exists a $k_c>0$ such that $\omega_{1i}(k_c)=\alpha_0\varepsilon^2$ with $\alpha_0<0$, $0<\varepsilon\ll 1$, $\omega_{1i}'(k_c)=0$, $\omega_{1i}''(k_c)<0$, and $\omega_{1i}(k)<-\sigma$ for all $k\in\mathbb{R}\setminus(B_\delta(k_c)\cup B_\delta(-k_c))$ for some $\sigma,\delta>0$. Moreover, $\omega_{1r}(k_c)=\nu_0>0$, $\omega_{1r}'(k_c)=\nu_1>0$, with $\nu_0/k_c-\nu_1>0$.

(A3) for $3\le j\le 2N$ we have $\omega_{ji}(k)<-\sigma$ for all $k\in\mathbb{R}$ (damped modes).

For the nonlinearity we assume that

(A4) $p_{\rm nl}$ is given by (2.7) with $\hat\chi_3$ and $\hat\chi_5$ given by (2.8) and (2.9).

Remark 2.4 The first assumption of (A2) implies that ω_1 is of the form sketched in Figure 4 fundamental for our analysis. The last assumption in (A2), that the group velocity ν_1 is $\mathcal{O}(1)$–different (smaller than) from the phase velocity ν_0/k_c of the critical mode, is for technical reasons. It is used in the spatial dynamics formulation to construct a center manifold of size independent of ε. If $\nu_0/k_c - \nu_1 = \mathcal{O}(\varepsilon)$ then roughly speaking the results stated below also hold, but the analysis becomes more complicated, cf. [HCS99]. Moreover we remark that for the following derivation of the gGLe it is not needed that $\hat{\chi}_1$, $\hat{\chi}_3$ and $\hat{\chi}_5$ are of the rational form (2.2), (2.8) and (2.9). The rational form is only used in the spatial dynamics formulation in Section 4.

Due to the reflection symmetry $x \to -x$ of Maxwell's equations there are two critical complex conjugate curves of eigenvalues $k \mapsto \omega_1(k)$ and $k \mapsto \omega_2(k)$. Therefore, the amplitude equations of the system are given by two gGLe, one for the right moving wave packets modulating $e_+ = e_+(t,x) = \mathrm{e}^{\mathrm{i}(k_c x - \nu_0 t)}$ and one for the left moving wave packets modulating $e_- = e_-(t,x) = \mathrm{e}^{\mathrm{i}(k_c x + \nu_0 t)}$. Under the assumptions (A1)–(A4) it is straightforward to derive the equation for the modulations of these critical modes. Let

$$\begin{aligned} u(t,x) &= \varepsilon^{1/2}\psi_q(t,x) \\ &= \varepsilon^{1/2} q_+(\varepsilon^2 t, \varepsilon(x-\nu_1 t)) e_+(t,x) + \varepsilon^{1/2} q_-(\varepsilon^2 t, \varepsilon(x+\nu_1 t)) e_-(t,x) + \text{c.c.} \end{aligned} \tag{2.10}$$

with $q_\pm = q_\pm(T,X) \in \mathbb{C}$. Inserting (2.10) into (2.4) and equating the coefficients of $\varepsilon^{5/2} e_+$ and $\varepsilon^{5/2} e_-$ to zero gives the system of nonlinear coupled gGLe

$$\begin{aligned} \partial_T q_+ &= c_2 \partial_X^2 q_+ + c_0 q_+ + c_3 |q_+|^2 q_+ + c_5 |q_+|^4 q_+ \\ &\quad + c_4 |\tau_{2\nu_1 T/\varepsilon} q_-|^2 q_+ + c_6 |\tau_{2\nu_1 T/\varepsilon} q_-|^4 q_+, \\ \partial_T q_- &= \overline{c_2} \partial_X^2 q_- + \overline{c_0} q_- + \overline{c_3} |q_-|^2 q_- + \overline{c_6} |q_-|^4 q_- \\ &\quad + \overline{c_4} |\tau_{-2\nu_1 T/\varepsilon} q_+|^2 q_- + \overline{c_5} |\tau_{2\nu_1 T/\varepsilon} q_+|^4 q_-, \end{aligned} \tag{2.11}$$

where

$$c_0 = \alpha_0, \quad c_2 = \frac{\mathrm{i}}{2}\omega_1''(k_c) = \frac{1}{2}\big(-\omega_{1i}''(k_c) + \mathrm{i}\omega_{1r}''(k_c)\big) \tag{2.12}$$

$$c_3 = c_4 = \frac{\nu_1^2 \nu_0}{2k_c}\big(-\hat{\chi}_{3i}(\nu_0) + \mathrm{i}\hat{\chi}_{3r}(\nu_0)\big), \tag{2.13}$$

$$c_5 = c_6 = 10\frac{\nu_1^2 \nu_0}{2k_c}\big(-\hat{\chi}_{5i}(\nu_0) + \mathrm{i}\hat{\chi}_{5r}(\nu_0)\big). \tag{2.14}$$

Some details of the derivation of the gGLes are given in Appendix A.1. We used the abbreviation $(\tau_{cT} u)(X) = u(X - cT)$. Hence, these two gGLe are coupled by some terms singularly depending on the small bifurcation parameter ε. For the case of classical partial differential equations (no memory) the mathematical justification of the GLes as modulation equations can be found in [Sch99]. A sketch of the theory can be found in the appendix. For spatially localized solutions, we find that the interaction time of counter propagating wave packets is very small and so the coupling terms are then of higher order and can be neglected [Sch97], i.e., then effectively $c_4 = c_6 = 0$. Therefore, the pulse solutions will still be described by the single gGLe (1.9) from the Introduction.

As already explained we will use heavily the properties of this single gGLe. We will make this more precise in the following. In (2.11) we always have $c_{0i}=0$ due to the derivation. We let $\tilde{q}_+(T,X) = q_+(T,X)\mathrm{e}^{\mathrm{i}\omega_0 T}$ and search for stationary pulse solutions $q_{+,\mathrm{pu},\omega_0}$ of

$$\partial_T \tilde{q}_+ = c_2 \partial_X^2 \tilde{q}_+ + (\alpha_0 - \mathrm{i}\omega_0)\tilde{q}_+ + c_3|\tilde{q}_+|^2\tilde{q}_+ + c_5|\tilde{q}_+|^4\tilde{q}_+. \tag{2.15}$$

For $\alpha_0 = c_{2r} = c_{3r} = c_{5r} = 0$ a pulse $q^{(0)}_{+,\mathrm{pu},\omega_0}$ is explicitly given by

$$q^{(0)}_{+,\mathrm{pu},\omega_0}(X) = \left(\frac{\beta_1\omega_0}{1+\beta_2\cosh(\beta_3\sqrt{\omega_0}X)}\right)^{1/2},$$

$$\beta_1 = 4/c_{3i},\ \beta_2 = \sqrt{1+\frac{16c_{5i}\omega_0}{3c_{3i}^2}},\ \beta_3 = 2/\sqrt{c_{2i}}.$$

For small $\alpha_0, c_{2r}, c_{3r}, c_{5r} \neq 0$ and suitable ω_0 these pulses persist. Even analytic expressions can be found, see, e.g. [vSH92]. Moreover, in [KS98] it is shown that for a certain choice of parameters $\alpha_0, c_2, c_3, \omega_0$ the pulses are exponentially stable.

Theorem 2.5 *[KS98, Theorem 1.3] There exist an open set $\mathcal{P} \subset \mathbb{R}^7$ such that for $(\alpha_0, c_2, c_3, c_5) \in \mathcal{P}$ the following holds. There exists an $\omega_+ = \omega_+(\alpha_0, c_2, c_3, c_5) > 0$ such that (2.15) has a two-parameter family*

$$\mathcal{M}_q = \{q_{+,\mathrm{pu},\omega_+}(X - X_0)\mathrm{e}^{\mathrm{i}\theta_0} : X_0 \in \mathbb{R}, \theta_0 \in [0, 2\pi)\} \tag{2.16}$$

of pulse solutions. Moreover, $q_{+,\mathrm{pu},\omega_+}$ is exponentially orbital stable, i.e., there exists a constant $b > 0$ such that the following holds. For all $C_2 > 0$ there exists a $C_1 > 0$ such that from

$$\|q_+(0,\cdot) - q_{+,\mathrm{pu},\omega_+}(\cdot - X_0)\mathrm{e}^{\mathrm{i}\theta_0}\|_{H^1} \leq C_1$$

for some $X_0 \in \mathbb{R}$, $\theta_0 \in [0, 2\pi)$, it follows

$$\|q_+(T,\cdot) - q_{+,\mathrm{pu},\omega_+}(\cdot - X_1)\mathrm{e}^{\mathrm{i}\theta_1}\|_{H^1} \leq C_2\mathrm{e}^{-bT}$$

for some $X_1 \in \mathbb{R}$, $\theta_1 \in [0, 2\pi)$.

Remark 2.6 A characterization for $(\alpha_0, c_2, c_3, c_5) \in \mathcal{P}$ is $0 < -\alpha_0, c_{2r}, c_{3r}, -c_{5r}$ and small, $0 < c_{2i}, c_{3i}, -c_{5i}$, and

$$\frac{4c_{3r}}{c_{3i}} > \frac{c_{2r}}{c_{2i}} > 0 \quad \text{and} \quad \left(\frac{4c_{3r}}{c_{3i}} - \frac{c_{2r}}{c_{2i}}\right)^2 > \frac{24}{5}\alpha_0 \frac{8c_{5r}}{c_{3i}^2}.$$

From (2.12)–(2.14) it is clear that we can tune $\hat{\chi}_1, \hat{\chi}_3$ and $\hat{\chi}_5$ in such a way that $(\alpha_0, c_2, c_3, c_5) \in \mathcal{P}$. However, note that $c_{3r} > 0$ implies that in some (nonlinear) way energy has to be supplied to the system to compensate for the (linear) loss $\alpha_0 < 0$. See the Remarks 2.1 and 2.2. As already said for ε small the system (2.4) is completely determined by the system of the two gGLe (2.11), and for spatially localized solutions, as the pulses we are interested in, these equations will decouple. Since the pulse q_{+,pu,ω_+} is then stable in the first equation and the origin $q_- = 0$ is still stable in the second equation, the pulse will also be stable in the complete system (2.11) of amplitude equations, see Appendix A.3.

Remark 2.7 There exists a second family of pulse solutions $q_{+,\text{pu},\omega_-} \approx q^{(0)}_{+,\text{pu},\omega_-}$ of (2.15) with $\omega_- = \omega_-(\alpha_0, c_2, c_3, c_5) < \omega_+$. However, these are unstable [KS98]. The constants $\omega_\pm$ are given by

$$\omega_\pm = \frac{5c_{3i}^2}{8c_{5r}} \left(\frac{4c_{3r}}{c_{3i}} - \frac{c_{2r}}{c_{2i}} \pm \sqrt{\left(\frac{4c_{3r}}{c_{3i}} - \frac{c_{2r}}{c_{2i}}\right)^2 - \frac{24}{5}\alpha_0 \frac{8c_{5r}}{c_{3i}^2}} \right).$$

We may now state our main results.

Theorem 2.8 *[Existence] Assume (A1)–(A4). There exists an $\varepsilon_0 > 0$ such that the following holds. Suppose that $\hat{\chi}_1, \hat{\chi}_3$, and $\hat{\chi}_5$, are chosen in such a way that $0 < \varepsilon \le \varepsilon_0$ and $(\alpha_0, c_2, c_3, c_5) \in \mathcal{P}$. Then there exists a two-parameter family of modulating pulse solutions for Maxwell's equation (2.4) in the form*

$$\mathcal{M}_u = \{u(t,x) = u_{\text{pu}}(x - \eta_1 t - x_0, k_c x - \eta_0 t - \theta_0) : x_0 \in \mathbb{R}, \theta_0 \in [0, 2\pi)\},$$
$$u_{\text{pu}}(\xi, p) = \varepsilon^{1/2} q_{+,\text{pu},\omega_+}(\varepsilon\xi) e^{ip} + \text{c.c.} + \mathcal{O}(\varepsilon^{3/2}) \in \mathbb{R},$$
$$\lim_{\xi \to \pm\infty} u_{\text{pu}}(\xi, p) = 0, \quad u_{\text{pu}}(\xi, p + 2\pi) = u_{\text{pu}}(\xi, p),$$

where $\eta_1 = \omega_r' + \mathcal{O}(\varepsilon^2)$ and $\eta_0 = \omega_r + \mathcal{O}(\varepsilon^2)$ are small corrections of the linear group velocity and the linear temporal wavenumber.

Theorem 2.9 *[Stability] Fix $m \geq 1$. Under the assumptions of Theorem 2.8, there exists a constant $b > 0$ such that for all $C_2 > 0$ we have a $C_1 > 0$ such that the following holds. For some $x_0 \in \mathbb{R}$, $\theta_0 \in [0, 2\pi)$, let*

$$\begin{aligned} \|u_0(\cdot) - u_{\text{pu}}(\cdot - x_0, k_c \cdot - \theta_0)\|_{H^{m+1}(\mathbb{R})} &\leq C_1 \varepsilon^{1/2}, \\ \|u_1(\cdot) - \frac{\mathrm{d}}{\mathrm{d}t} u_{\text{pu}}(\cdot - x_0, k_c \cdot - \theta_0)\|_{H^m(\mathbb{R})} &\leq C_1 \varepsilon^{1/2}, \\ \sup_{t \leq 0} \|u_h(t, \cdot) - u_{\text{pu}}(\cdot - \eta_1 t - x_0, \cdot - \eta_0 t - \theta_0)\|_{H^{m+1}(\mathbb{R})} &\leq C_1 \varepsilon^{1/2}, \end{aligned} \tag{2.17}$$

and let u be the solution of (2.4)-(2.5). Then u exists for all $t > 0$, and there exist constants $x_1 \in \mathbb{R}, \theta_1 \in [0, 2\pi)$ such that

$$\begin{aligned} &\|u(t, \cdot) - u_{\text{pu}}(\cdot - \eta_1 t - x_1, k_c \cdot - \eta_0 t - \theta_1)\|_{H^{m+1}(\mathbb{R})} \\ &\quad + \|\partial_t u(t, \cdot) - \frac{\mathrm{d}}{\mathrm{d}t} u_{\text{pu}}(\cdot - \eta_1 t - x_1, k_c \cdot - \eta_0 t - \theta_1)\|_{H^m(\mathbb{R})} \leq C_2 \varepsilon^{1/2} \mathrm{e}^{-b\varepsilon^2 t} \end{aligned}$$

as $t \to \infty$.

Remark 2.10 Since we are dealing with an integro–differential equation, (2.17) is a natural condition, as mentioned before. However, it can be relaxed in the sense that it is sufficient that the linear and nonlinear initial polarizations are close to the ones produced by a modulating pulse. This will be made precise in Section 5, where we prove Theorem 2.9 using the extended system (3.4).

In Appendix A.2, in order to go beyond the formal derivation of the system of nonlinear coupled gGLes (2.11), we also prove its validity as modulation equation for Maxwell's equations; we give exact estimates between the approximations obtained via the gGLe and solutions of (2.4), (2.5).

3 The extended system

We convert Maxwell's equations (2.4) into an extended system of autonomous differential equations without memory term. This can always be done for any $\hat{\chi}_1, \hat{\chi}_3, \hat{\chi}_5$ given by (2.2), (2.8) and (2.9). However, to simplify notation we assume, w.l.o.g. for our purposes,

$$\begin{aligned} n_1 = n_3 = n_5 = 1, \qquad & \delta_1 = \gamma_1, \ \delta_{1,3} = \gamma_{1,3} =: \gamma_3, \ \delta_{1,5} = \gamma_{1,5} =: \gamma_5, \\ & c_1 = d_1, \ c_{1,3} = d_{1,3} =: d_3, \ c_{1,5} = d_{1,5} =: d_5, \end{aligned} \tag{3.1}$$

such that now $N=2$ in (A2). Letting

$$\begin{aligned} \phi_j(t,x) &= \int_{-\infty}^t \chi_j(t-\tau)u^j(\tau,x)\,\mathrm{d}\tau,\ j{=}1,5, \\ \phi_3(t,x) &= \varepsilon\int_{-\infty}^t \chi_3(t-\tau)u^3(\tau,x)\,\mathrm{d}\tau, \end{aligned} \tag{3.2}$$

we obtain

$$\left.\begin{aligned} \partial_t\phi_j(t,x) &= \chi_j(0)u^j(t,x) + \int_0^t \chi_j'(t-\tau)u^j(\tau,x)\,\mathrm{d}\tau, \\ \partial_t^2\phi_j(t,x) &= j\chi_j(0)u^{j-1}(t,x)\partial_t u(t,x) + \chi_j'(0)u^j(t,x) \\ &\quad + \int_0^t \chi_j''(t-\tau)u^j(\tau,x)\,\mathrm{d}\tau, \end{aligned}\right\} j=1,5$$

$$\partial_t\phi_3(t,x) = \varepsilon\left[\chi_3(0)u^3(t,x) + \int_0^t \chi_3'(t-\tau)u^3(\tau,x)\,\mathrm{d}\tau\right],$$

$$\partial_t^2\phi_3(t,x) = \varepsilon\left[3\chi_3(0)u^2(t,x)u_t(t,x){+}\chi_3'(0)u^3(t,x){+}\int_0^t \chi_3''(t{-}\tau)u^3(\tau,x)\,\mathrm{d}\tau\right].$$

Since χ_j, $j=1,3,5$, fulfills

$$\chi_j'' = \beta_{j1}\chi_j + \beta_{j2}\chi_j' \quad \text{with} \quad \beta_{j1} = -(\gamma_j^2+d_j^2),\ \beta_{j2} = -2\gamma_j, \tag{3.3}$$

we can rewrite (2.4) as

$$\begin{aligned} (1{+}\chi_0)\partial_t^2 u &= \partial_x^2 u{-}2\gamma_0\partial_t u{-}\gamma_0^2 u \\ &\quad - \big[\beta_{11}\phi_1{+}\beta_{12}\partial_t\phi_1{+}\big(\chi_1'(0){-}\beta_{12}\chi_1(0)\big)u{+}\chi_1(0)\partial_t u\big] \\ &\quad - \big[\beta_{31}\phi_3{+}\beta_{32}\partial_t\phi_3{+}\varepsilon g_3(u,\partial_t u)\big]{-}\big[\beta_{51}\phi_5{+}\beta_{52}\partial_t\phi_5{+}g_5(u,\partial_t u)\big], \\ \partial_t^2\phi_1 &= \big[\chi_1'(0) - \beta_{12}\chi_1(0)\big]u + \chi_1(0)\partial_t u + \beta_{11}\phi_1 + \beta_{12}\partial_t\phi_1, \\ \partial_t^2\phi_3 &= \beta_{31}\phi_3 + \beta_{32}\partial_t\phi_3 + \varepsilon g_3(u,\partial_t u), \\ \partial_t^2\phi_5 &= \beta_{51}\phi_5 + \beta_{52}\partial_t\phi_5 + g_5(u,\partial_t u), \end{aligned} \tag{3.4}$$

where

$$g_j(u,\partial_t u) = (j-1)\chi_j(0)u^{j-1}\partial_t u + (\chi_j'(0) - \beta_{j2}\chi_j(0))u^j, \quad j=3,5. \tag{3.5}$$

This reformulation of (2.4) as a system of differential equations without memory is crucial for the spatial dynamics formulation in Section 4. System (3.4) allows us to apply the standard theory for semilinear hyperbolic problems; see Section 5. Since the equations for the variables ϕ_j contain no spatial derivatives we obtain the local existence of solutions $V = (u, \phi_1, \phi_3, \phi_5)$ in

$$C([0,t_0], H^{m+1}(\mathbb{R}) \times [H^m(\mathbb{R})]^3) \cap C^1([0,t_0], [H^m(\mathbb{R})]^4) \cap \; C^2([0,t_0], [H^{m-1}(\mathbb{R})]^4),$$

for a $t_0 > 0$, to initial conditions $(V(0,\cdot), \partial_t V(0,\cdot)) \in \big[H^{m+1}(\mathbb{R}) \times [H^m(\mathbb{R})]^3\big] \times [H^m(\mathbb{R})]^4$ for $m \geq 1$, cf. Lemma 5.2.

Remark 3.1 It is the possibility to express χ_j'' by lower order derivatives in (3.3) which allows us to remove the memory terms from (2.4) and hence to obtain the extended system (3.4). This system is semilinear due to the absence of an instantaneous nonlinear polarization. If $\vec{P}_{\rm nl}(t,x) = p_{\rm nl}(t,x)\hat{k}$ with

$$p_{\rm nl}(t,x) = p^{\rm i}_{\rm nl}(t,x) + p^{\rm m}_{\rm nl}(t,x),$$

with $p^{\rm m}_{\rm nl}$ as above and where for instance $p^{\rm i}_{\rm nl}(t,x) = \varepsilon\alpha_3 u^3 + \alpha_5 u^5$, with $\alpha_3 > 0$ and $\alpha_5 < 0$, the extended system becomes quasilinear; see also Remark 4.3. For the existence and uniqueness in this case the methods of [HKM76] can be applied.

4 Existence of the modulating pulses

To prove Theorem 2.8 we use a spatial dynamics formulation and a center manifold reduction. The method goes back to [Kir82]. Since then it has been used to construct special traveling wave solutions to systems on unbounded domains in a variety of problems similar to ours, see [EW91, IM91, HCS99, Sch00, SS00, Uec01]. The idea is as follows. We set

$$\big(u(t,x), \phi_1(t,x), \phi_3(t,x), \phi_5(t,x)\big) = \big(\widetilde{W}_1(\xi,p), W_3(\xi,p), W_5(\xi,p), W_7(\xi,p)\big),$$

with $\xi = x - \eta_1 t \in \mathbb{R}$, $p = k_c x - \eta_0 t \in \mathcal{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$, where $\eta_0 = \nu_0 + \varepsilon^2\tilde{\nu}_0$, $\eta_1 = \nu_1 + \varepsilon^2\tilde{\nu}_1$ are small corrections of the linear phase velocity and the linear temporal wavenumber with $\tilde{\nu}_0$ and $\tilde{\nu}_1$ a priori unknown. We then formulate (3.4) as a first order dynamical system $\partial_\xi W = F(\partial_p, W)$ in the "spatial" variable ξ on the unbounded cylinder $(\xi,p) \in \mathbb{R} \times \mathcal{T}_{2\pi}$. For the linearization of this system about the trivial solution $W \equiv 0$ we find four eigenvalues $\mathcal{O}(\varepsilon)$ close to zero, with the rest of the spectrum bounded away from the imaginary axis. We then construct a four dimensional center manifold $\mathcal{M}_c$ for $\partial_\xi W = F(\partial_p, W)$. The reduced equation on $\mathcal{M}_c$ can be interpreted as an $\mathcal{O}(\varepsilon)$ perturbation of the stationary gGLe (1.9) in the moving frame. Thus, for $\varepsilon = 0$ we have pulse solutions for the reduced equation due to Theorem 2.5. We then show that these pulses persist for the $\mathcal{O}(\varepsilon)$ perturbations in the reduced equation, for suitable $\tilde{\nu}_0$ and $\tilde{\nu}_1$. This will prove Theorem 2.8. It remains to work out the details of the approach just sketched.

4.1 The spatial dynamics formulation

Setting

$$W = (W_1, \partial_t \widetilde{W}_1, W_3, \partial_t W_3, W_5, \partial_t W_5, W_7, \partial_t W_7)^T,$$

with $W_1 = (1-\partial_p^2)^{1/2}\widetilde{W}_1$ and using $\partial_x = \partial_\xi + k_c\partial_p$, $\partial_t = -\eta_1\partial_\xi - \eta_0\partial_p$ we write (3.4) as a first order system. With

$$\partial_x^2 u = (\partial_\xi + k_c\partial_p)\left(-\frac{1}{\eta_1}\partial_t + \left(k_c - \frac{\eta_0}{\eta_1}\right)\partial_p\right)\widetilde{W}_1$$

we obtain

$$\partial_\xi W = D\mathcal{M}(\partial_p)W - \frac{\eta_0}{\eta_1}\partial_p W + N(W_1, W_2), \tag{4.1}$$

where

$$D = \begin{pmatrix} D_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{pmatrix}, \quad \begin{array}{l} D_1 = \mathrm{diag}(-1/\eta_1, -\eta_1/(1-(1+\chi_0)\eta_1^2), -1/\eta_1, -1/\eta_1), \\ D_2 = \mathrm{diag}(-1/\eta_1, -1/\eta_1, -1/\eta_1, -1/\eta_1), \end{array}$$

$$\mathcal{M}(\partial_p) = \begin{pmatrix} \mathcal{M}^{(0)}(\partial_p) & \mathcal{M}^{(1)}(\partial_p) \\ \mathbf{0} & \mathcal{M}^{(2)}(\partial_p) \end{pmatrix}, \quad \mathcal{M}^{(0)}(\partial_p) = \begin{pmatrix} 0 & (1-\partial_p^2)^{1/2} & 0 & 0 \\ m_{21} & m_{22} & \beta_{11} & \beta_{12} \\ 0 & 0 & 0 & 1 \\ m_{41} & \chi_1(0) & \beta_{11} & \beta_{12} \end{pmatrix},$$

$$\mathcal{M}^{(1)}(\partial_p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_{51} & \beta_{52} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{M}^{(2)}(\partial_p) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta_{51} & \beta_{52} \end{pmatrix},$$

$$m_{21} = \left[\left(k_c^2 - 2k_c\frac{\eta_0}{\eta_1} + \frac{\eta_0^2}{\eta_1^2}\right)\partial_p^2 - \gamma_0^2 - \chi_1'(0) - \beta_{12}\chi_1(0)\right](1-\partial_p^2)^{-1/2},$$

$$m_{22} = \left(\frac{\eta_0}{\eta_1^2} - \frac{2k_c}{\eta_1}\right)\partial_p - 2\gamma_0 - \chi_1(0), \quad m_{41} = (\chi_1'(0) - \chi_1(0)\beta_{12})(1-\partial_p^2)^{-1/2},$$

and

$$N(W_1,W_2)=D\big(0,\varepsilon g_3+g_5,0,0,0,g_3/\eta_1,0,g_5/\eta_1\big)^T$$

with g_j, $j=3,5$, from (3.5) depending smoothly on $\widetilde{W}_1=(1-\partial_p^2)^{-1/2}W_1$ and W_2.

4.2 The center manifold reduction

The first step in construction of a center manifold for (4.1) is the examination of the spectrum of the linearization about $W=0$, given by

$$\partial_\xi W=L(\partial_p)W=D\mathcal{M}(\partial_p)W-\frac{\eta_0}{\eta_1}\partial_p W. \tag{4.2}$$

We fix $m\geq 1$ and consider $L(\partial_p)$ as an operator in

$$X=[H^m(\mathcal{T}_{2\pi})]^8 \quad \text{with domain} \quad Y=[H^{m+1}(\mathcal{T}_{2\pi})]^8.$$

To determine the spectrum of $L(\partial_p)$ we write W as a Fourier series $W(\xi,p)=\sum_{l\in\mathbb{Z}}W_l(\xi)\mathrm{e}^{\mathrm{i}lp}$ with $W_l(\xi)\in\mathbb{C}^8$, $W_l(\xi)=\overline{W}_{-l}(\xi)$. Then (4.2) decomposes into a direct sum of systems

$$\partial_\xi W_l=L(\mathrm{i}l)W_l=D\mathcal{M}(\mathrm{i}l)W_l-\frac{\eta_0}{\eta_1}\mathrm{i}lW_l \tag{4.3}$$

For each $l\in\mathbb{Z}$ we obtain 8 eigenvalues $\lambda_{l,j}$, $j=1,\ldots,8$, for $L(\mathrm{i}l)$ and 8 eigenvalues $\mu_{l,j}=\lambda_{l,j}+\frac{\eta_0}{\eta_1}\mathrm{i}l$ for $D\mathcal{M}(\mathrm{i}l)$. The four eigenvalues of the lower block of $D_2\mathcal{M}^{(2)}(\mathrm{i}l)$ are

$$\mu_{5,6}=\big(\gamma_3\pm\mathrm{i}d_3\big)/\eta_1,\quad \mu_{7,8}=\big(\gamma_5\pm\mathrm{i}d_5\big)/\eta_1.$$

They come from the auxiliary variables ϕ_3,ϕ_5 and are independent of the wave-number l. Therefore, the associated eigenvalues $\lambda_{l,j}=\mu_j-\frac{\eta_0}{\eta_1}\mathrm{i}l$ of $L(\mathrm{i}l)$ are strictly bounded away from the imaginary axis.

Tracing back the transformation from (2.4) to (4.3) one finds that λ is an eigenvalue of the first 4×4-block of $L(\mathrm{i}l)$ if and only if the dispersion relation (2.6) holds with

$$\omega=-\nu_1\lambda-\mathrm{i}\nu_0 l \quad\text{and}\quad k=\lambda+\mathrm{i}k_c l.$$

See [EW91, HCS99]. In order to find the central eigenvalues we substitute $\lambda=\mathrm{i}(\tilde\lambda-lk_c)$, $\tilde\lambda\in\mathbb{R}$ and obtain

$$\nu_1\tilde\lambda+(\nu_1 k_c-\nu_0)l\stackrel{!}{=}\omega_j(\tilde\lambda) \tag{4.4}$$

with ω_j from (A1), (A2). The imaginary part of (4.4) gives $0 = \omega_{ji}(\tilde{\lambda})$ and for $\varepsilon = 0$ we obtain $\tilde{\lambda} = \pm k_c$ using (A2), (A3). Then from the real part we obtain $l = \pm 1$.

Thus, for $\varepsilon = 0$ and $l = \pm 1$ we obtain a zero eigenvalue $\lambda_{-1,1} = \lambda_{1,1} = 0$. Due to the quadratic tangency of $\omega_{1i}(k)$ and $\omega_{2i}(k) = -\omega_{1i}(k)$ at the k-axis at k_c, these eigenvalues have geometric multiplicity one but algebraic multiplicity two, cf. [AM95, HCS99]. Thus we obtain Jordan blocks of length 2 for $l = -1$ and $l = +1$. For $\varepsilon = 0$ we denote the associated eigenvectors by $\varphi_{\pm 1,1} \in \mathbb{C}^8$ and the generalized eigenvalues by $\varphi_{\pm 1,2} \in \mathbb{C}^8$. Using perturbation analysis as in [EW91, HCS99] we obtain four distinct eigenvalues with real part of order $\mathcal{O}(\varepsilon)$ for $\varepsilon > 0$.

In order to show that the rest of the spectrum of $L(\mathrm{i}l)$ is strictly bounded away from the imaginary axis, we consider the matrices $D\mathcal{M}(\mathrm{i}l)$ for $|l|$ large. This leads to the following observations. First, for $j = 3, 4$ the limits $\lim_{|l|\to\infty} \mu_{l,j} = \mu_j^*$ exist and satisfy $\mu_j^* \notin \mathrm{i}\mathbb{R}$. Secondly, for $j = 1, 2$ there exists a c^* such that the limits $\lim_{|l|\to\infty} \mu_{l,j} \pm \mathrm{i}c^* l = \mu_j^*$ exist and satisfy $\mu_j^* \notin \mathrm{i}\mathbb{R}$. Moreover, the limits of the associated eigenfunctions $\varphi_{l,j}$ exist, i.e., $\lim_{|l|\to\infty} \varphi_{l,j} = \varphi_j^*$, and the set of vectors $\{\varphi_j^* \mid j = 1, \ldots, 8\}$ forms a basis of $\mathbb{C}^8$.

Figure 5 shows $\lambda_{l,1}, \ldots, \lambda_{l,4}$, $l = -40, \ldots, 40$, calculated numerically for a typical dispersion relation. We summarize our results on the spectrum of $L(\partial_p)$ as follows.

Lemma 4.1 *There exist $\rho_1, \rho_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the operator $L(\partial_p) : Y \to X$ has 4 central eigenvalues $\lambda_{\pm 1,j}$, $j = 1, 2$ of size $\mathcal{O}(\varepsilon)$. All other eigenvalues $\lambda_{l,j}, l \in \mathbb{Z}$ satisfy $\rho_1 \leq |\mathrm{Re}\lambda_{l,j}| \leq \rho_2$.*

By Lemma 4.1 we found a gap of order $\mathcal{O}(1)$ between the central part and the hyperbolic part of the spectrum of $L(\partial_p)$. Therefore, we can define the center, stable, unstable and hyperbolic subspaces X_c, X_s, X_u, and X_h by

$$\begin{aligned} X_c &= \mathrm{span}\{\varphi_{l,j}(\varepsilon)\mathrm{e}^{\mathrm{i}lp} : l = \pm 1, j = 1, 2\}, \\ X_s &= \mathrm{cl}_X \mathrm{span}\{\varphi_{l,j}(\varepsilon)\mathrm{e}^{\mathrm{i}lp} : \mathrm{Re}\lambda_{l,j} \leq -\rho_1\}, \\ X_u &= \mathrm{cl}_X \mathrm{span}\{\varphi_{l,j}(\varepsilon)\mathrm{e}^{\mathrm{i}lp} : \mathrm{Re}\lambda_{l,j} \geq \rho_1\} \quad \text{and} \quad X_h = X_s \oplus X_u. \end{aligned}$$

Due to Lemma 4.1 this splitting is independent of $\varepsilon \in (0, \varepsilon_0)$, and we have $X = X_c \oplus X_h$. Let P_c, P_s, and P_u be the $L(\partial_p)$–invariant projections from X onto X_c, X_s, and X_u, respectively. Moreover, let $P_h = P_s + P_u$.

By Lemma 4.1 and the convergence of the associated eigenfunctions to linear independent vectors for $|l| \to \infty$ we have the $\mathcal{O}(1)$-boundedness of the projections P_c, P_s, and P_u. This holds since for instance

$$\|P_s u\|_X \leq \sup_{l \in \mathbb{Z}} \|P_s(l)\|_{\mathbb{C}^8 \to \mathbb{C}^8} \, \|u\|_X \leq C \|u\|_X$$

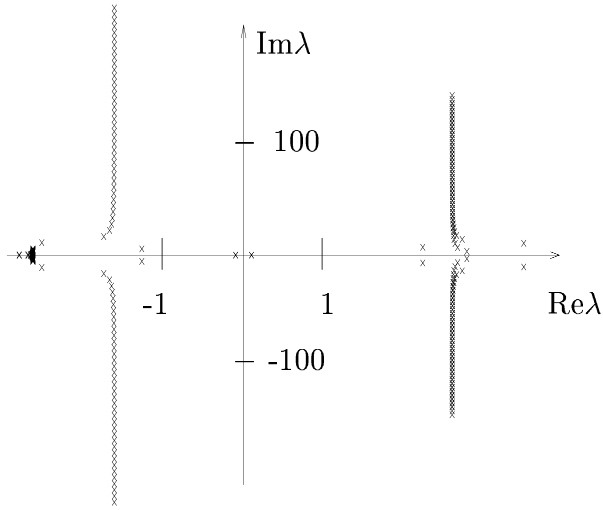

Figure 5: Eigenvalues $\lambda_{l,1}, \ldots, \lambda_{l,4}, l = -40, \ldots, 40$, in the spatial dynamics formulation.

for $P_s = \oplus_{l\in\mathbb{Z}} P_s(l)$, where we do not distinguish in our notation the space X in physical and Fourier space. Note that $\|W\|_X$ is equivalent to $\big(\sum_{l\in\mathbb{Z}, j=1,\ldots 8} |W_{l,j}|^2(1+|l|^2)^m\big)^{1/2}$. For the linear semigroups $P_* \mathrm{e}^{L(\partial_p)t} = \oplus_{l\in\mathbb{Z}} P_*(l)\mathrm{e}^{L(\mathrm{i}l)t}$, $* \in \{s, u, c\}$ we obtain

$$\|P_s \mathrm{e}^{L(\partial_p)t} u\|_X \leq \sup_{l\in\mathbb{Z}} \|P_s(l)\mathrm{e}^{L(\mathrm{i}l)t}\|_{\mathbb{C}^8\to\mathbb{C}^8} \; \|u\|_X \leq C\mathrm{e}^{-\rho_1 t}\|u\|_X \quad \text{for } t \geq 0,$$

$$\|P_u \mathrm{e}^{L(\partial_p)t} u\|_X \leq C\mathrm{e}^{-\rho_1|t|}\|u\|_X \quad \text{for } t \leq 0,$$

and

$$\|P_c \mathrm{e}^{L(\partial_p)t} u\|_X \leq C\mathrm{e}^{\eta|t|}\|u\|_X$$

for all $t \in \mathbb{R}$ for each fixed $0 < \eta < \rho_1$ independent of $\varepsilon \in (0, \varepsilon_0)$.

Since the nonlinearity $N(W_1, W_2)$ is smooth from X to X, all assumptions of the center manifold theorem as stated in [VI92] are satisfied. Using the notation $W_c = P_c W$, $W_h = P_h W$, $L_c = P_c L$, and $F(W_c, W_s) = N(W_1, W_2)$ we thus have proved the following result.

Theorem 4.2 *Fix $r \geq 1$. Then there exists $\varepsilon_0 > 0$ and $\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. For (4.1) there exists a center manifold $\mathcal{M}_c$ of the form $W_h(\xi) = \Phi(W_c(\xi))$ with $\Phi \in C^r(U_\delta^{\mathbb{R}^4}(0), X_h)$ and $\Phi(0) = \partial_{W_c}\Phi(0) = 0$. Moreover, $\mathcal{M}_c$ contains all small bounded solutions of (4.1), and every solution of the reduced equation*

$$\partial_\xi W_c(\xi, p) = L_c W_c(\xi, p) + P_c N(W_c(\xi, p), \Phi(W_c(\xi, p))) \tag{4.5}$$

gives a solution of (4.1) via

$$W = W_c + \Phi(W_h).$$

Remark 4.3 It is also possible to consider the quasilinear case, i.e., the case of an instantaneous polarization $p_{\mathrm{nl}}^{\mathrm{i}}$ as explained in Remark 3.1. In this case the linear operator $L_s = P_s L$ is replaced by a quasilinear operator $L_s(W_c, W_s, \partial_p)$. Similarly to [Ren92] we then can use the iteration scheme

$$\begin{aligned}\partial_\xi W_{c,n} =& L_c W_{c,n} + P_c F(W_{c,n}, W_{s,n}),\\ \partial_\xi W_{s,n} =& L_s(W_{c,n-1}, W_{s,n-1}, \partial_p) W_{c,n} + P_c F(W_{c,n}, W_{s,n})\end{aligned}$$

to prove Theorem 4.2 due to the very explicit formulation in this case.

4.3 Existence of pulses for the reduced equation

The reduced equation (4.5) can be interpreted as an $\mathcal{O}(\varepsilon)$ perturbation of the stationary gGLe (1.9) in a moving frame. In order to see this we introduce coordinates

$$W_c(\xi, p) = \varepsilon^{1/2} A(X) \varphi_{1,1} \mathrm{e}^{\mathrm{i}p} + \varepsilon^{3/2} B(X) \varphi_{1,2} \mathrm{e}^{\mathrm{i}p} + \text{c.c.}, \quad X = \varepsilon\xi$$

in the center subspace and obtain that A, B satisfy

$$\begin{aligned}\partial_X A &= B + \mathcal{O}(\varepsilon),\\ \partial_X B &= \varepsilon\tilde{\nu}_1 B - (\alpha_0 - \mathrm{i}\tilde{\nu}_0)A - c_3|A|^2 A - c_5 |A|^4 A + \mathcal{O}(\varepsilon).\end{aligned} \tag{4.6}$$

This follows from inserting the Ginzburg–Landau ansatz

$$u(t,x) = W_1(\xi, p) = \sum_{l\in\mathbb{Z}} W_{l,1}(\xi)\mathrm{e}^{\mathrm{i}lp} = \sum_{l\in\mathbb{Z}} \varepsilon^{1/2+\gamma_l} A_l(\varepsilon\xi)\mathrm{e}^{\mathrm{i}lp}, \quad \gamma_l = 1 + ||l| - 1|$$

into (2.4). Using formal calculations as in the derivation of the gGLe in App. A.1 we obtain that $A = A_1$ has to satisfy the stationary gGLe with the phase rotation term $\mathrm{i}\tilde{\nu}_0 A$ and the drift term $\varepsilon\tilde{\nu}_1 A_X$, and additional terms of formal higher order $\mathcal{O}(\varepsilon)$. These terms are rigorously of order $\mathcal{O}(\varepsilon)$ in $H^m(\mathbb{R})$ since W lies on M_c.

Thus, for $\varepsilon=0$, $(\alpha_0, c_2, c_3, c_5)\in\mathcal{P}$ and $\tilde{\nu}_0=\omega_+$ we have the pulse $A(\xi)=q_{+,\mathrm{pu},\omega_+}(\xi)$ for (4.6). Due to the spectral properties of $q_{+,\mathrm{pu},\omega_+}(\xi)$ we conclude that $q_{+,\mathrm{pu},\omega_+}(\xi)$ persists for $\varepsilon > 0$ upon tuning $\tilde{\nu}_0, \tilde{\nu}_1$. This is explained in detail in [Sch00] based on the results of [KS98]. See also [Uec01, Section 4.4] for a proof in a similar problem. Thus the proof of Theorem 2.8 is complete. □

5 Stability

We use center manifold theory to prove the stability of the family of modulating pulses. A modulating pulse is time-periodic with period t_0 in a frame moving with the envelope. So Floquet theory has to be used to analyze the spectrum of the linearized system. Using the fact that the dynamics of the extended system (3.4) can be approximated by the dynamics of the Ginzburg-Landau equations (2.11) we will obtain exactly two Floquet multipliers 1 which come from the time-periodicity and the translation invariance. From this we will obtain a two dimensional center-manifold with trivial flow on it. The application of the center manifold theorem to the discrete t_0–map is complicated by the fact that the linearized operator for the time evolution is not sectorial. As a consequence, we cannot argue with the spectrum alone and so we use again the fact that the dynamics of (3.4) can be approximated by the dynamics of (2.11) in order to obtain the estimates which are needed for the linearized system. The results about the Ginzburg–Landau approximation which are used here can be found in Appendix A.2. Therefore, it could be advantageous to have a parallel look to Appendix A.2.

For the proof of Theorem 2.9 we write the extended system (3.4) as a first order system in t. Setting

$$U=(U_1,U_2,U_3,U_4,U_5,U_6,U_7,U_8)^T=(u,\partial_t u,\phi_1,\partial_t\phi_1,\phi_3,\partial_t\phi_3,\phi_5,\partial_t\phi_5)^T \tag{5.1}$$

we obtain

$$\begin{aligned} \partial_t U &= A(\partial_x)U + F(U), \\ U|_{t=0} &= (u_0, u_t, \phi_1(0), \partial_t\phi_1(0), \ldots, \partial_t\phi_5(0)), \end{aligned} \tag{5.2}$$

where

$$A(\partial_x) = \begin{pmatrix} A_0(\partial_x) & A_1 \\ \mathbf{0} & A_2 \end{pmatrix},$$

$$A_0(\partial_x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial_x^2-\gamma_0^2-\chi_1'(0)+\beta_{12}\chi_1(0)}{1+\chi_0} & \frac{-2\gamma_0-\chi_1(0)}{1+\chi_0} & -\frac{\beta_{11}}{1+\chi_0} & -\frac{\beta_{12}}{1+\chi_0} \\ 0 & 0 & 0 & 1 \\ \chi_1'(0)-\beta_{12}\chi_1(0) & \chi_1(0) & \beta_{11} & \beta_{12} \end{pmatrix},$$

$$(1+\chi_0)A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\beta_{31} & -\beta_{32} & -\beta_{51} & -\beta_{52} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta_{51} & \beta_{52} \end{pmatrix},$$

and

$$F(U) = \big(0, f_2(U_1,U_2), 0, 0, 0, g_3(U_1,U_2), 0, g_5(U_1,U_2)\big)^T,$$
$$f_2(U_1,U_2) = [-\varepsilon g_3(U_1,U_2) - g_5(U_1,U_2)]/(1+\chi_0).$$

We go into the frame $y{=}x{-}\eta_1 t$ comoving with the modulating pulse $u_{\rm pu}$ and consider

$$\partial_t U = A(\partial_y)U + \eta_1\partial_y U + F(U). \tag{5.3}$$

We fix $m \geq 1$, let

$$X_m = H^{m+1}(\mathbb{R}) \times [H^m(\mathbb{R})]^7, \qquad Y_m = X_{m+1},$$

and show first that (5.3) is well posed in X_m. We start with the linearized system.

Lemma 5.1 *The operator $L_0(\partial_y) = A(\partial_y) + \eta_1\partial_y : Y_m \to X_m$ generates a C_0-semigroup e^{tL_0} in each X_m with*

$$\|\mathrm{e}^{tL_0(\partial_y)}\|_{X_m\to X_m} \leq C\mathrm{e}^{-\alpha_0\varepsilon^2 t} \tag{5.4}$$

with C a constant independent of time $t \geq 0$. Exactly the same results hold if $A(\partial_x)$ is considered instead of $L_0(\partial_y)$.

Proof. We take the Fourier transform $V(t,y) = \int \hat{V}(t,k)\mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}k$ of $\partial_t V = L_0(\partial_y)V$ and obtain $\partial_t\hat{V} = L_0(\mathrm{i}k)\hat{V}$. Denoting the eigenvalues of $L_0(\mathrm{i}k) = A(\mathrm{i}k) + \mathrm{i}\eta_1 k\mathrm{Id}$ by $\lambda_1(k)+\mathrm{i}\eta_1 k, \ldots, \lambda_8(k)+\mathrm{i}\eta_1 k$ we obtain for $\lambda_1,\ldots,\lambda_4$, belonging to $A_0(\mathrm{i}k)$, that

$$\lambda_j(k) = \lambda_{jr}(k) + \mathrm{i}\lambda_{ji}(k) = -\mathrm{i}\omega_j(k) = \omega_{ji}(k) - \mathrm{i}\omega_{jr}(k). \tag{5.5}$$

Thus we have 2 curves $\lambda_{1,2}$ of critical eigenvalues ($\lambda_{1r}(k_c) = \lambda_{2r}(k_c) = -\varepsilon^2\alpha_0$), and 2 curves $\lambda_{3,4}$ belonging to strongly damped modes, cf. Section 2, in particular Fig. 4 and Assumption (A2). The 4 eigenvalues

$$\lambda_{5,6} = -\gamma_3 \pm \mathrm{i}d_3, \;\; \lambda_{7,8} = -\gamma_5 \pm \mathrm{i}d_5 \tag{5.6}$$

of the lower block A_2 belong to damped modes, are independent of k, and come from the auxiliary variables ϕ_3, ϕ_5.

The proof of Lemma 5.1 is based on the following two facts: first, that Fourier transform is an isomorphism from $H^m(\mathbb{R})$ into

$$L^2(m) = \{\hat{u} \in L^2(\mathbb{R}) \mid \|\hat{u}\|_{L^2}^2 = \int |\hat{u}(k)|^2(1+k^2)^m dk < \infty\},$$

and secondly that the two critical eigenvalues $\lambda_{1,2}$ of $L_0(ik)$ are simple in a neighborhood of $\pm k_c$. We start by transforming

$$\tilde{V}(t,k) = G(k)\hat{V}(t,k) \quad \text{with} \quad G(k) = \mathrm{diag}((1+k^2)^{1/2}, 1, \ldots, 1) : \mathbb{C}^8 \to \mathbb{C}^8.$$

Then

$$\partial_t \tilde{V} = B\tilde{V}, \quad \text{where} \quad B = GL_0G^{-1}. \tag{5.7}$$

Note that

$$\|G\|_{L^2(m+1)\times[L^2(m)]^7 \to [L^2(m)]^8}, \|G^{-1}\|_{[L^2(m)]^8 \to L^2(m+1)\times[L^2(m)]^7} \leq C.$$

Below we show that the solution $\tilde{V}(t,k)$ of (5.7) fulfills

$$\|\tilde{V}(t,\cdot)\|_{[L^2(m)]^8} \leq C\mathrm{e}^{-\alpha_0\varepsilon^2 t}\|\tilde{V}(0,\cdot)\|_{[L^2(m)]^8}, \tag{5.8}$$

such that

$$\begin{aligned}\|V(t)\|_{X_m} &\leq C\|\hat{V}(t)\|_{L^2(m+1)\times[L^2(m)]^7} \leq C\|\tilde{V}(t)\|_{[L^2(m)]^8} \leq C\mathrm{e}^{-\alpha_0\varepsilon^2}\|\tilde{V}(0)\|_{[L^2(m)]^8} \\ &\leq C\mathrm{e}^{-\alpha_0\varepsilon^2}\|\hat{V}(0)\|_{L^2(m+1)\times[L^2(m)]^7} \leq C\mathrm{e}^{-\alpha_0\varepsilon^2 t}\|V(0)\|_{X_m},\end{aligned}$$

which shows (5.4).

It remains to show (5.8). Since B has the same eigenvalues as L_0, then due to (A2) and (5.6), for $k \in I_c := [-k_c - \delta, -k_c + \delta] \cup [k_c - \delta, k_c + \delta]$ we can define orthogonal projections $\tilde{P}_j(k) : \mathbb{C}^8 \to \mathrm{span}\{\varphi_j(k)\}$, where $\varphi_{1,2} \in \mathbb{C}^8$ are the normalized eigenvectors of $B(k)$ to the critical eigenvalues $\lambda_{1,2}(k)$. Thus, for $\tilde{V} \in \mathbb{C}^8$, let

$$P_j(k)\tilde{V} = \begin{cases} \tilde{P}_j(k)\tilde{V} & \text{for} \quad k \in I_c \\ 0 & \text{for} \quad k \notin I_c \end{cases}$$

and write

$$\tilde{V}(t,k) = \gamma_1(t,k)\varphi_1(k) + \gamma_2(t,k)\varphi_2(k) + \tilde{V}_s(t,k),$$

where $\tilde{V}_s$ contains only stable modes, i.e., $(\mathrm{Id} - P_1(k) - P_2(k))\tilde{V}_s(t,k) = \tilde{V}_s(t,k)$. Now (5.7) is equivalent to

$$\begin{aligned} \partial_t \gamma_1(t,k) &= \lambda_1(k)\gamma_1(t,k), \\ \partial_t \gamma_2(t,k) &= \lambda_2(k)\gamma_2(t,k), \\ \partial_t \tilde{V}_s(t,k) &= S^{-1}(k)\Lambda(k)S(k)\tilde{V}_s(t,k), \end{aligned}$$

where $\Lambda(k)$ is the Jordan normal form of $B(k)$. Since $B(k){=}G(k)A(\mathrm{i}k)G(k)^{-1}{+}\mathrm{i}\eta_1 k\mathrm{Id}$ with

$$G(k)A(\mathrm{i}k)G(k)^{-1} = \begin{pmatrix} B_0(k) & A_1 \\ \mathbf{0} & A_2 \end{pmatrix},$$

$$B_0(k) = \begin{pmatrix} 0 & (1+k^2)^{1/2} & 0 & 0 \\ -\frac{k^2}{(1+\chi_0)(1+k^2)^{1/2}} & \frac{-2\gamma_0-\chi_1(0)}{1+\chi_0} & -\frac{\beta_{11}}{1+\chi_0} & -\frac{\beta_{12}}{1+\chi_0} \\ 0 & 0 & 0 & 1 \\ \frac{\chi_1'(0)-\beta_{12}\chi_1(0)}{(1+k^2)^{1/2}} & \chi_1(0) & \beta_{11} & \beta_{12} \end{pmatrix},$$

it follows that the eigenvectors $\varphi_1(k), \ldots, \varphi_8(k)$ of $G(k)A(\mathrm{i}k)G(k)^{-1}$ converge to fixed vectors $\varphi_1^*, \ldots, \varphi_8^*$ that form a basis of $\mathbb{C}^8$. Therefore the transformations $S(k), S^{-1}(k) : \mathbb{C}^8 \to \mathbb{C}^8$ can be bounded with a constant independent of k. Now let $\hat{W}(k) = S(k)\tilde{V}_s(k)$. Then $\partial_t \hat{W} = \Lambda \hat{W}$, and hence

$$|\hat{W}(t,k)|_{C^8} \le C(1+t^l)\mathrm{e}^{-\sigma t}|\hat{W}(0,k)|_{\mathbb{C}^8} \le C\mathrm{e}^{-\sigma t/2}|\hat{W}(0,k)|_{\mathbb{C}^8} \tag{5.9}$$

with $\sigma > 0$ due to (A2) and (5.6), and with $0 \le l \le N$ due to possible Jordan blocks in Λ, with N from (A2). Here $N = 2$ due to (3.1), but of course (5.9) holds true for any number of resonances in (2.2). Therefore

$$\begin{aligned} \|\tilde{V}(t)\|^2_{[L^2(m)]^8} &\le \int \left|\gamma_1(t,k)\varphi_1(k) + \gamma_2(t,k)\varphi_2(k) + \tilde{V}_s(t,k)\right|^2_{\mathbb{C}^8} (1{+}k^2)^m \,\mathrm{d}k \\ &\le C \int \left[|\gamma_1(t,k)|^2 + |\gamma_2(t,k)|^2 + |\hat{W}(t,k)|^2_{\mathbb{C}^8}\right](1{+}k^2)^m \,\mathrm{d}k \\ &\le C \int \left[\mathrm{e}^{-2\alpha_0\varepsilon^2 t}(|\gamma_1(0,k)|^2 + |\gamma_2(0,k)|^2) + \mathrm{e}^{-\sigma t/2}|\hat{W}(t,0)|^2_{\mathbb{C}^8}\right](1{+}k^2)^m \,\mathrm{d}k \\ &\le C\mathrm{e}^{-2\alpha_0\varepsilon^2 t}\|\tilde{V}(0)\|^2_{[L^2(m)]^8}. \end{aligned}$$

This is (5.8). □

Since the nonlinearity $F : X_m \to X_m$ is locally Lipschitz continuous we can apply a fixed point argument to the variation of constant formula and obtain the following local existence and uniqueness theorem for our semilinear system (5.2).

Lemma 5.2 *Fix $m \geq 1$. For all $t_0 > 0$ there exists a $\rho > 0$ such that for all $U_0 \in X_m$ with $\|U_0\|_{X_m} \leq \rho$ there exists a unique solution $U \in C([0, t_0], X_m)$ to (5.2) with $U|_{t=0} = U_0$.*

In order to prove stability of the modulated pulse u_{pu} we set

$$U(t, y + \eta_1 t) = U_{\text{pu}}(y, k_c y - (\eta_0 - k_c \eta_1)t) + V(t, y),$$

where U_{pu} is obtained from transfering u_{pu} into (3.2) and (5.1), respectively. Then

$$\partial_t V = LV + N(V), \quad \text{where} \tag{5.10}$$

$$L = A(\partial_y) + \eta_1 \partial_y + DF(U_{\text{pu}}), \quad N(V) = F(V + U_{\text{pu}}) - F(U_{\text{pu}}) - DF(U_{\text{pu}}).$$

Since $u_{\text{pu}}(y, k_c y - (\eta_0 - k_c \eta_1)t) = \varepsilon^{1/2} q_{+,\text{pu},\omega_+}(\varepsilon y) \mathrm{e}^{\mathrm{i}(k_c y - (\eta_0 - k_c \eta_1)t)} + \text{c.c.} + \text{h.o.t}$ is periodic in t with period $t_0 = 2\pi/(k_c \eta_1 - \eta_0)$, so is the linear operator L and the nonlinearity N.

Thus, the idea is to construct a center manifold for the Floquet–map, i.e., the time–t_0–map for (5.10). Using the Ginzburg–Landau approximation we will obtain the spectrum of the linearized time–t_0–map and estimates for its iterates.

We define the linear flow $\Psi_{t,s} : X_m \to X_m$ by the solution $V(t) = \Psi_{t,s} V_0$ of the linear problem

$$\partial_t V = LV, \quad V|_{t=s} = V_0. \tag{5.11}$$

Lemma 5.3 *The linear flow $\Psi_{t,s} : X_m \to X_m$ is well–defined for all $t \geq s \geq 0$ and the behavior of the time–t_0–map $\Lambda = \Psi_{t_0,0}$, $t_0 = 2\pi/(k_c \eta_1 - \eta_0)$, is as follows. There are two simple eigenvalues $\mu_{1,2} = 1$, and there exist $\varepsilon_0, b, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the complementary part Λ_s of Λ is exponentially damping, i.e., for all $m \in \mathbb{N}$ we have*

$$\|\Lambda_s^m\|_{X_m \to X_m} \leq C \mathrm{e}^{-b\varepsilon^2 m}. \tag{5.12}$$

Proof. The operator $L_1 = DF(U_{\text{pu}}) : X_m \to X_m$ is clearly bounded. Using Lemma 5.1 and, e.g., [Paz83, Theorem 3.1.1] it follows that $L = A(\partial_y + \eta_1 \partial_y) + L_1 = L_0(\partial_y) + L_1$ generates a C_0–semigroup in X_m and hence $\Psi_{t,s}$ is well defined. In order to control the evolution generated by Λ and its Floquet-spectrum we use the Ginzburg–Landau formalism, now for the linear equation (5.11). As already said the results about the Ginzburg–Landau formalism which are used here can be found in Appendix A.2.

For (5.11) we obtain by Lemma A.3 that after a time $\mathcal{O}(1/\varepsilon^2)$ every solution to an initial condition of order $\mathcal{O}(1)$ in X_m can be written as

$$V_0(y) = \varepsilon^{1/2} q_{+,0}(\varepsilon y) \mathrm{e}^{\mathrm{i}k_c y} \varphi_+(k_c) + \varepsilon^{1/2} q_{-,0}(\varepsilon y) \mathrm{e}^{\mathrm{i}k_c y} \varphi_-(k_c) + \text{c.c.} + \varepsilon R(y) \tag{5.13}$$

where $\|q_j\|_{H^{m+4}}, \|R\|_{X_m} = \mathcal{O}(1)$, and where $\varphi_\pm(k_c) \in \mathbb{C}^8$ are the normalized eigenvectors of $A(\mathrm{i}k_c)$ to the two eigenvalues with real part $-\varepsilon^2\alpha_0$. As a consequence all iterations Λ^n with $n = \mathcal{O}(1/\varepsilon^2)$ applied to order $\mathcal{O}(1)$ initial conditions in X_m are of this form. This is a linear version of the so called attractivity of the set of modulated pattern.

After this time we can use the approximation property of the system of coupled generalized Ginzburg–Landau equations (2.11). In detail, let $q = (q_1, q_2) \in C([0,T_0],[H^{m+4}(\mathbb{R})]^2)$ be the solution of the linearization $q_T = L_{\mathrm{pu}}^{\mathrm{GL}} q$ of (2.11) around the pulses with the initial conditions $q|_{T=0} = (q_{+,0}, q_{-,0})$ from (5.13), and let V be the solution to (5.11), $V|_{t=0} = V_0$. Then due to Lemma A.4 and Remark A.5 there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{0 \le t \le T_0/\varepsilon^2} \|V(t,\cdot) - \varepsilon^{1/2}\psi_q(t,\cdot)\|_{X_m} \le C\varepsilon,$$

where here

$$\psi_q(t,y) = q_+(\varepsilon^2 t, \varepsilon y)\mathrm{e}^{\mathrm{i}(k_c y + (k_c\eta_1 - \eta_0)t)}\varphi_+(k_c) + q_-(\varepsilon^2 t, \varepsilon y)\mathrm{e}^{\mathrm{i}(k_c y + (k_c\eta_1 + \eta_0)t)}\varphi_-(k_c) + \text{c.c.}.$$

Since, due to Theorem A.8, $\mathrm{Re}(\lambda) < -b_0$ for $\lambda \in \mathrm{spec}(L_{\mathrm{pu}}^{\mathrm{GL}})$, except for the two simple eigenvalues $\lambda = 0$, we find for some $m = M_0/\varepsilon^2$ that

$$\|\Lambda^m V_0\|_{X_m} \le C_2 \mathrm{e}^{-b_0\varepsilon^2 m t_0}\|V_0\|_{X_m} + \mathcal{O}(\varepsilon) \le \|V_0\|_{X_m}/2$$

for $\varepsilon > 0$ sufficiently small and $M_0 = \mathcal{O}(1)$ sufficiently large, except of V_0 in a two-dimensional subspace. Hence, except of two eigenvalues, Λ^m has no spectral values of modulus bigger than 1/2.

This two-dimensional subspace is controlled with the help of the translation invariance of (5.2). It follows that Λ has two simple Floquet multipliers $\mu_{1,2} = 1$ with associated eigenfunctions

$$V_1(y) = \partial_\xi U_{\mathrm{pu}}(y, k_c y) = \partial_1 U_{\mathrm{pu}}(y, k_c y) \quad \text{and} \quad V_2(y) = \partial_p U_{\mathrm{pu}}(y, k_c y) = \partial_2 U_{\mathrm{pu}}(y, k_c y)$$

and so we have also controlled the two zero eigenvalues coming from the amplitude equation (2.11).

Hence, we define some Λ–invariant projection P_c onto $\mathrm{span}\{V_1, V_2\}$, i.e. $\Lambda P_c = P_c \Lambda$. Then we define $\Lambda_s = (1 - P_c)\Lambda$ and proceed as above. We obtain with Lemma A.4, Remark A.5 and Theorem A.8 again with some $m = M_0/\varepsilon^2$ that

$$\|\Lambda_s^m\|_{X_m \to X_m} \le C_2 \mathrm{e}^{-b_0\varepsilon^2 m t_0} + \mathcal{O}(\varepsilon) \le 1/2$$

for $\varepsilon > 0$ sufficiently small and $M_0 = \mathcal{O}(1)$ sufficiently large. Thus the proof of Lemma 5.3 is complete. □

Proof of Theorem 2.9. We define the nonlinear flow $\Phi_{t,s} : X_m \to X_m$ by the solution $V(t) = \Phi_{t,s}V$ of (5.10) with $V|_{t=s} = V_0$. The long time dynamics of (5.10) for small V_0 can be described by iteration of the nonlinear Floquet operator $\Gamma = \Phi_{t_0,0} : \mathcal{U} \to X_m$, where $\mathcal{U}$ is a neighborhood of 0 in X_m. The mapping Γ exists for $\mathcal{U}$ sufficiently small since $N : X_m \to X_m$ is locally Lipschitz and hence (5.10) has a local strong solution V; cf. Lemma 5.2. Thus we consider the discrete dynamical system

$$V^{(n+1)}(y) = \Gamma V^{(n)}(y). \tag{5.14}$$

The linearization of Γ is given by $V^{(n+1)} = \Lambda V^{(n)}$. From Lemma 5.3 it follows that there exists a two–dimensional center manifold $\mathcal{M}_c$ for (5.14) which is of size $\mathcal{O}(\varepsilon^{1/2})$ since $\|N(V)\|_{X_m} = \mathcal{O}(\varepsilon^{5/2})$ for $\|V\|_{X_m} = \mathcal{O}(\varepsilon^{1/2})$. This manifold is tangential to $\mathrm{span}\{V_1, V_2\}$ and contains all small solutions of (5.14). Hence it coincides with the family

$$\{U_{\mathrm{pu}}(\cdot - y_0, k_c \cdot -\theta_0) - U_{\mathrm{pu}}(\cdot,\cdot) : y_0 \in \mathbb{R}, \theta_0 \in [0, 2\pi)\}$$

of fixed points of Γ and the flow on $\mathcal{M}_c$ is trivial. Now assume that

$$\|V^{(0)}(\cdot) - U_{\mathrm{pu}}(\cdot - y_0, k_c \cdot -\theta_0)\|_{X_m} \le C_1\varepsilon^{1/2}$$

for some $C_1 > 0$ sufficiently small. Then for $t = mt_0$ and some $y_1 \in \mathbb{R}, \theta_1 \in [0, 2\pi)$ we obtain

$$\|V^{(m+1)}(\cdot) - (U_{\mathrm{pu}}(\cdot - y_1, k_c \cdot -\theta_1) - U_{\mathrm{pu}}(\cdot,\cdot))\|_{X_m} \le C_2\varepsilon^{1/2}\mathrm{e}^{-b\varepsilon^2 m t_0}$$

for $m \to \infty$. Hence

$$\|U(t,\cdot) - U_{\mathrm{pu}}(\cdot - \eta_1 t - y_1, k_c \cdot -\eta_0 t - \theta_1)\|_{X_m} \le C_2\varepsilon^{1/2}\mathrm{e}^{-b\varepsilon^2 t}. \tag{5.15}$$

Since (5.10) has a local solution we obtain (5.15) for all $t\in[mt_0, (m+1)t_0)$ and all $m\in\mathbb{N}$. Considering only the first two components of U, the proof of Theorem 2.9 is complete. □

6 Extensions and variants

Here we discuss how our results can be extended to optical fibers and comment on the periodically forced Nonlinear Schrödinger equation as modulation equation and on modulating multi-pulse solutions.

6.1 Optical fibers

Under suitable assumptions our results also hold for the technologically relevant problem of pulses in optical fibers. We give a brief overview of the changes in the analysis compared to linearly polarized light in bulk material.

We model the optical fiber as a cylindrical domain $\Omega = \mathbb{R} \times \Sigma$ of infinite extend where Σ is a bounded cross–section. The essential difference between (isotropic and centrosymmetric) bulk material and an optical fiber is that the refractive index $\hat{n}(x_\perp, \omega) = \sqrt{1 + \hat{\chi}_1(x_\perp, \omega)}$ depends on the transverse coordinates $x_\perp = (y, z)$. For circular symmetric fibers we have $\hat{n} = \hat{n}(r, \omega)$, $r^2 = y^2 + z^2$. The optical fiber consists of a core and a cladding with refractive indices $\hat{n}(r, \omega) = \hat{n}_0(\omega)$ in the core and $\hat{n}(r, \omega) = \hat{n}_1(\omega)$ in the cladding, where $\operatorname{Re} \hat{n}_0(\omega) > \operatorname{Re} \hat{n}_1(\omega)$. Typically $\operatorname{Re}(\hat{n}_0 - \hat{n}_1)/\operatorname{Re} \hat{n}_0 \ll 1$, and the transition between core and cladding may be sharp and even discontinuous.

We thus consider Maxwells equation (1.1) in the form

$$\begin{aligned} &\triangle \vec{E}(t, \vec{x}) - \nabla(\nabla \cdot \vec{E}(t, \vec{x})) - \partial_t^2 \vec{E}(t, \vec{x}) \\ &= -\partial_t^2 \int_{-\infty}^{t} \chi_1(x_\perp, t-\tau) \vec{E}(\tau, \vec{x}) \, \mathrm{d}\tau - \partial_t^2 \vec{P}_{\mathrm{nl}}(t, \vec{x}). \end{aligned} \tag{6.1}$$

For (6.1) typically one either seeks solutions that decay exponentially as $|x_\perp| \to \infty$, or chooses a number R_0 larger than the diameter of the fiber and prescribes Dirichlet boundary conditions at $|x_\perp| = R_0$, i.e. one considers (6.1) with

$$\vec{E}(t, x, x_\perp) = 0 \text{ for } x_\perp \in \partial\Sigma.$$

Then we proceed as above for bulk material and define some constitutive laws for the polarization, where for instance $\hat{\chi}_1(x_\perp, \omega)$ is given by (2.2) but with coefficients $a_j, b_j, \ldots$ depending on $x_\perp$. In the modeling usually a so called mono–mode fiber is considered where from the infinitely many solutions $k_j(\omega)$ of the dispersion relation exactly one function is chosen. See [Gow93, Chapter 10] for concrete examples of mono–mode fibers.

After such a choice we can proceed as above with the respective changes in the function spaces. See, e.g., [NM92, Section 3b] for the derivation of the Nonlinear Schrödinger equation for optical fibers, and [HCS99] for an example of how our setup of spatial dynamics and center–manifold reduction is translated to vector–valued problems over cylindrical domains.

6.2 Periodically forced systems

Another interesting situation arises when in the original system there is a spatially periodic linear response of order $\mathcal{O}(\varepsilon^2)$ with twice the spatial and temporal wavenumber of the critical spatial and temporal wavenumbers k_c and $\omega_r(k_c)$. In optical fibers this can be achieved using an external supporting electromagnetic wave, see, e.g, [MS99]. Then assuming a coefficient in front of the cubic terms which is of order $\mathcal{O}(1)$ and no longer $\mathcal{O}(\varepsilon)$, and scaling the solutions with ε instead of $\varepsilon^{1/2}$, a so called parametrically forced Nonlinear Schrödinger equation (pfNLSe)

$$\partial_T q_1 = c_0 q_1 + c_2 \partial_X^2 q_1 + c_1 \overline{q}_1 + c_3 q_1 |q_1|^2$$

with coefficients $c_j \in \mathbb{C}$ can be derived as the modulation equation for wave packets.

The pfNLS is another example for a perturbed Nonlinear Schrödinger equation that has exponentially stable pulse solutions, cf. [KS98]. Formally these again lead to modulating pulse solutions in the original system. For a phenomenological model this analysis has been carried rigorously in [Uec01]. From the above analysis it is clear that the results also hold for the equations of nonlinear optics considered in the present paper.

Finally, we remark that instead of a single pulse, multi–pulse solutions can be considered. The existence and stability for the amplitude equations is also guaranteed by the results of [KS98].

A Appendix

A.1 Derivation of the generalized Ginzburg–Landau equation

The derivation of the amplitude equations by multiple scaling perturbation analysis from the ansatz (2.10), i.e.,

$$\begin{aligned} u(t,x) &= \varepsilon^{1/2}\psi_q(t,x) \\ &= \varepsilon^{1/2} q_+(\varepsilon^2 t, \varepsilon(x-\nu_1 t)) e_+(t,x) + \varepsilon^{1/2} q_-(\varepsilon^2 t, \varepsilon(x+\nu_1 t)) e_-(t,x) + \text{c.c.} \end{aligned}$$

with $q_\pm = q_\pm(T,X) \in \mathbb{C}$ and $e_\pm = e_\pm(t,x) = \mathrm{e}^{\mathrm{i}(k_c x \mp \nu_0 t)}$ is standard and based on relations of the form

$$\begin{aligned} \partial_t u &= \varepsilon^{1/2}\big((-\mathrm{i}\nu_0 - \varepsilon\nu_1\partial_X + \varepsilon^2\partial_T) q_+\big) e_+ + \varepsilon^{1/2}\big((\mathrm{i}\nu_0 + \varepsilon\nu_1\partial_X + \varepsilon^2\partial_T) q_-\big) e_- + \text{c.c.}, \\ \partial_x u &= \varepsilon^{1/2}\big((\mathrm{i}k_c + \varepsilon\partial_X) q_+\big) e_+ + \varepsilon^{1/2}\big((\mathrm{i}k_c + \varepsilon\partial_X) q_-\big) e_- + \text{c.c.}. \end{aligned}$$

As an example for the somewhat less standard treatment of the memory terms consider

$$
\begin{aligned}
&\partial_t^2 \int_0^\infty \chi(\tau)(q(\varepsilon^2(t-\tau),\varepsilon(x-\nu_1 t+\nu_1\tau))\mathrm{e}^{\mathrm{i}(k_c x-\nu_0(t-\tau))})\,\mathrm{d}\tau \qquad \text{(A.1)}\\
&\quad=\int_0^\infty \Big\{\chi_1(\tau)\big[-\nu_0^2+2\mathrm{i}\varepsilon\nu_0\nu_1\partial_X+\varepsilon^2\nu_1^2\partial_X^2-2\mathrm{i}\nu_0\varepsilon^2\partial_T+\mathcal{O}(\varepsilon^3)\big]\\
&\qquad\qquad \big(q+\varepsilon\nu_1\tau q_X+\frac{1}{2}\varepsilon^2\nu_1^2\tau^2 q_{XX}-\varepsilon^2\tau q_T+\mathcal{O}(\varepsilon^3)\big)\mathrm{e}^{\mathrm{i}\nu_0\tau}\Big\}\,\mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}(k_c x-\nu_0 t)}\\
&\quad=\Big\{-\nu_0^2\hat\chi(\nu_0)q+\varepsilon\big[2\mathrm{i}\nu_0\nu_1\hat\chi(\nu_0)+\mathrm{i}\nu_0^2\nu_1\hat\chi'(\nu_0)\big]q_X\\
&\qquad+\varepsilon^2\big(\nu_1^2\hat\chi(\nu_0)-2\mathrm{i}\nu_0\nu_1^2\hat\chi'(\nu_0)+\frac{1}{2}\nu_0\nu_1^2\hat\chi''(\nu_0)\big)q_{XX}\\
&\qquad-\varepsilon^2\big(2\mathrm{i}\nu_0\hat\chi(\nu_0)+\mathrm{i}\nu_0^2\hat\chi'(\nu_0)\big)q_T\Big\}\mathrm{e}^{\mathrm{i}(k_c x-\nu_0 t)},
\end{aligned}
$$

where in the third and the following lines q, q_X mean $q(T,X), q_X(T,X)$ etc., and where for example we used

$$
\int_0^\infty \chi(\tau)\tau\mathrm{e}^{\mathrm{i}\nu_0\tau}\,\mathrm{d}\tau=\int_{-\infty}^\infty \chi(\tau)\tau\mathrm{e}^{\mathrm{i}\nu_0\tau}\,\mathrm{d}\tau=-\mathrm{i}\hat\chi'(\nu_0).
$$

Applying these formal calculations to the nonlinear terms gives for instance

$$
\begin{aligned}
\varepsilon\partial_t^2\int_0^\infty \chi_3(t-\tau)u^3(\tau,x)\,\mathrm{d}\tau =&\varepsilon^{5/2}\big[-3\nu_0^2\hat\chi^{(3)}(\nu_0)(|q_+|^2+|\tau_{-2\nu_1 T/\varepsilon}q_-|^2)q_+e_+\\
&-(3\nu_0)^2\hat\chi_3(-3\nu_0)(|q_-|^2+|\tau_{2\nu_1 T/\varepsilon}q_+|^2)q_+e_-\big]\\
&+\text{c.c.}+h_3(t,x)
\end{aligned}
$$

where $h_3(t,x)=\mathcal{O}(\varepsilon^{5/2}e_+^m e_-^n)+\mathcal{O}(\varepsilon^{7/2})$ with $(m,n)\notin\{(\pm1,0),(0,\pm1)\}$. Equating the coefficients of $\varepsilon^{5/2}e_+$ and $\varepsilon^{5/2}e_-$ to zero gives the amplitude equations (2.11).

Remark A.1 Note that (A.1) makes sense if, e.g., $q\in C_b^j((-\infty,T),H^{5-2j}(\mathbb{R}))$, $j=0,1,2$. Then for example in the third line of (A.1), $\mathcal{O}(\varepsilon^3)$ means the leading order term in the expression

$$
\frac{\varepsilon^3}{6}\nu_1^3\tau^3 q_{XXX}(T,\tilde X)-\varepsilon^3\tau^2\nu_1\nu_0 q_{TX}(T,\tilde X)+\mathcal{O}(\varepsilon^4),\quad \tilde X\in(X,X+\varepsilon\nu_1\tau)
$$

for the remainder in the Taylor expansion of $q(T-\varepsilon^2\tau,X+\varepsilon\nu_1\tau)$. This is crucial for the approximation properties of the Ginzburg–Landau formalism, see Lemma A.4 and Corollary A.6 below.

Remark A.2 From (A1), (A2) we have the so called non resonance conditions

$$0 \neq n\hat{\chi}(\nu_0) - \hat{\chi}(n\nu_0) = \mathcal{O}(1), n \neq 1.$$

Due to these we may define an improved approximation

$$\varepsilon^{1/2}\widetilde{\psi}_q(t,x) = \varepsilon^{1/2}\psi_q + \sum_{m,n\in\mathbb{Z}} \varepsilon^{1/2+\gamma_{m,n}} q_{m,n} e_+^m e_-^n, \quad \gamma_{m,n} = 1 + ||l+n|-1|,$$

where the sum goes over indices with $(m,n) \notin \{(\pm 1, 0), (0, \pm 1)\}$, and where the $q_{m,n}$ are given by algebraic expressions in q_+, q_- such that the so called residual

$$\begin{aligned} &\mathrm{Res}(\varepsilon^{1/2}\widetilde{\psi}_q) = \\ &\quad -\varepsilon^{1/2}(1+\chi_0)\partial_t^2\widetilde{\psi}_q + \varepsilon^{1/2}\left[\partial_x^2\widetilde{\psi}_q - \partial_t^2(\chi_1 * \widetilde{\psi}_q) - 2\gamma_0\partial_t\widetilde{\psi}_q - \gamma_0^2\widetilde{\psi}_q\right] - \partial_t^2 p_{\mathrm{nl}}(\varepsilon^{1/2}\widetilde{\psi}_q) \end{aligned}$$

is (formally) of order $\mathcal{O}(\varepsilon^{7/2})$. This is the first step in the proof of an approximation result, but we omit these lengthy calculations.

A.2 The validity result

The derivation of the gGLe (2.11) from Maxwell's equations (2.4), (2.5) is purely formal. There are counterexamples [Sch95, GS01], where a formally derived amplitude equation makes wrong predictions, i.e., where the solutions of the original system cannot be approximated via the solutions of the associated modulation equation. However, for many relevant systems the approximation by amplitude equations makes sense. This has been shown for instance for different problems in [Cra85, Kal88, CE90, vH91, Sch94]. Error estimates in Sobolev spaces have been considered for instance in [MS95].

In the following we state the so called attractivity and a pproximation result for our situation. Due to the special form of the nonlinearity ("$\varepsilon u^3 + u^5$ ") the proofs are standard and trivial adaptions of existing proofs in the literature. Hence they will be omitted or only briefly sketched. We only use these results in Section 5 for the description of the dynamics generated by the linearization around the pulse solutions.

We mainly work with the extended system (3.4) in the first order formulation (5.2) and use the following notations. As in Section 5 we fix $m \geq 1$ and let $X_m = H^{m+1}(\mathbb{R}) \times [H^m(\mathbb{R})]^7$. By $\varphi_\pm(k_c) \in \mathbb{C}^8$ we denote the normalized eigenvectors of $A(\mathrm{i}k_c)$ to the two eigenvalues with real part $-\varepsilon^2\alpha_0$ and analogous to (2.10) we define the formal approximation

$$\varepsilon^{1/2}\psi_q(t,x) = \varepsilon^{1/2}[q_+(\varepsilon^2 t, \varepsilon(x-\nu_1 t))e_+\varphi_+(k_c) + q_-(\varepsilon^2 t, \varepsilon(x+\nu_1 t))e_-\varphi_-(k_c)] + \text{c.c.},$$

$e_\pm(t,x) = \mathrm{e}^{\mathrm{i}(k_c x \mp \nu_0 t)}$. Since the formulation of a nonlinear attractivity result ([Eck93]) for the set of modulated pattern would need the introduction of additional spaces this result is only formulated for the linearized system (5.11).

Lemma A.3 *For all $C_1 > 0$ there exist $C_2, T_0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let $V_0 \in X_m$ with $\|V_0\|_{X_m} \le C_1$. Then there exist $q_+, q_- \in H^{m+4}(\mathbb{R})$ and $R \in X_m$ such that for the solution V of (5.11) with $V|_{t=0} = V_0$ we have*

$$V(y, T_0/\varepsilon^2) = \varepsilon^{1/2}\big[q_+(\varepsilon y)\mathrm{e}^{\mathrm{i}k_c y}\varphi_+(k_c) + q_-(\varepsilon y)\mathrm{e}^{\mathrm{i}k_c y}\varphi_-(k_c) + \text{c.c.}\big] + \varepsilon R(y)$$

with $\|q_\pm\|_{H^{m+4}} \le C_2$ and $\|R\|_{X_m} \le C_2$.

Proof. Similar to the proof of Lemma 5.1 but somewhat more complicated, the proof of Lemma A.3 is based on the locally parabolic shape of $\lambda_1(k)$ near $k = k_c$, on the fact that all other eigenvalues of $A(\partial_y)$ are strongly exponentially damped, and on the fact that the perturbation $DF(U_{\text{pu}})$ does not disturb the desired mode structure in Fourier space. For a proof of the analogous result in a similar problem see [Uec01, Appendix B]. □

In the set of modulated functions the dynamics can be controlled via the associated amplitude equations. For completeness this result is also formulated for the nonlinear case.

Lemma A.4 *For all $C_1, T_0 > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let $(q_+, q_-) \in C([0, T_0], [H^{m+4}(\mathbb{R}, \mathbb{C})]^2)$ be a solution of (2.11) with $\sup_{T\in[0,T_0]} \|q_j(t,\cdot)\|_{H^{m+4}} \le C_1$ and let U_0 satisfy*

$$\|U_0 - \varepsilon^{1/2}\psi_q|_{t=0}\|_{X_m} \le C_1\varepsilon. \tag{A.2}$$

Then the associated solution U of (5.2) with $U|_{t=0} = U_0$ can be approximated by $\varepsilon^{1/2}\psi_q$, i.e.

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|U(t) - \varepsilon^{1/2}\psi_q(t)\|_{X_m} \le C_2\varepsilon. \tag{A.3}$$

Proof. To prove the result one first defines an improved approximation $\varepsilon^{1/2}\widetilde{\psi}_q$ such that the residual is of order $\mathcal{O}(\varepsilon^{7/2})$, cf. Remark A.2. Due to the form of the nonlinearity the result then follows by a simple application of Gronwall's inequality to the equation for the error $\varepsilon R = U - \varepsilon^{1/2}\widetilde{\psi}_q$ as in [KSM92], using Lemma 5.1 to estimate the linear semigroup generated by $A(\partial_x)$. □

Remark A.5 It is easy to see that the above theorem also holds if the linearization around the modulating pulse is considered instead of the full nonlinear system.

For the sake of completeness we reformulate Lemma A.4 for the Maxwell-equations (2.4), (2.5) in integro-differential form. The crucial point is that in order to obtain a good approximation of solutions u of (2.4) to (2.5) via (2.10) it is necessary that not only the initial conditions $(u, u_t)|_{t=0}$ match with $\varepsilon^{1/2}(q_+, q_-)|_{T=0}$ but also the initial polarization. This is somewhat hidden in Lemma A.4 in the 3rd to 8th component of U_0. It is formulated in the following corollary as a condition on the history u_h from (2.5). We therefore consider the corollary also to be important in its own right.

Corollary A.6 *Let $(q_+, q_-) \in C([0, T_0), [H^5(\mathbb{R})]^2)$ be a solution of the gGLe (2.11). Then for all $C_1 > 0$ there exist $\varepsilon_0, C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Assume that*

$$\|(u_0 - \varepsilon^{1/2}\psi_q|_{t=0}, u_1 - \varepsilon^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\psi_q|_{t=0})\|_{H^2\times H^1} \leq C_1\varepsilon$$

and that for some functions $q_{\pm,h} : (-\infty, 0) \to \mathbb{C}$ that can be chosen in such a way that the functions $\tilde{q}_\pm : (-\infty, T_0]$ defined by $\tilde{q}_\pm(T) = q_{\pm,h}(T)$ for $T < 0$ and $\tilde{q}_\pm(T) = q_\pm(T)$ for $T \geq 0$ fulfill

$$\tilde{q}_\pm \in C_b^j((-\infty, T_0], H^{5-2j}(\mathbb{R})), \quad j = 0, 1, 2,$$

we have

$$\sup_{t<0} \mathrm{e}^{-\gamma|t|}\|u_h - \varepsilon^{1/2}\psi_{q_h}\|_{H^2} \leq C_1\varepsilon \text{ where } \gamma = \max_{j=1,3,5}\{\gamma_{i,j},\ \delta_{i,j} : i = 1 \ldots n_j\}.$$

Then there exist a unique solution u of Maxwell's equation on $t \in [0, T_0/\varepsilon^2]$ and this solution fulfills

$$\sup_{0\leq t\leq T_0/\varepsilon^2} \|(u - \varepsilon^{1/2}\psi_q, \partial_t u - \varepsilon^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\psi_q)\|_{H^2\times H^1} \leq C_2\varepsilon. \tag{A.4}$$

Remark A.7 Note that we do not assume $q_{\pm,h}$ to be solutions of the gGLe (2.11). Moreover, note that also in the derivation of the gGLe from Maxwell's equation in integro-differential form we need to assume that we can continue (q_+, q_-) smoothly to $T < 0$, cf. Remark A.1.

Proof. The assumptions of Corollary A.6 imply (A.2) with $m = 1$. From (A.3) we obtain (A.4) by considering only the first two components of $U(t)$. □

A.3 Stability of pulses in the modulation equations

It is the purpose of this section to prove the existence and stability of pulses in the system of nonlinear coupled gGLe which we write as

$$\partial_T q_+ = L_+ q_+ + N_+(q_+) + F_+(q_+, q_-), \quad \partial_T q_- = L_- q_- + N_-(q_-) + F_-(q_+, q_-), \quad \text{(A.5)}$$

where

$$L_+ q_+ = c_2 \partial_X^2 q_+ + c_0 q_+, \quad N_+(q_+) = c_3 |q_+|^2 q_+ + c_5 |q_+|^4 q_+,$$
$$F_+(q_+, q_-) = c_4 |\tau_{2\varepsilon^{-1}\nu_1 T} q_2|^2 q_1 + c_6 |\tau_{2\varepsilon^{-1}\nu_1 T} q_2|^4 q_1,$$

and similarly for L_-, N_- and F_-. Moreover, we write $Q = (q_+, q_-)$, $L = (L_+, L_-)$, $N = (N_-, N_+)$, and $F = (F_+, F_-)$.

Theorem A.8 *There exist an open set $\mathcal{P} \subset \mathbb{R}^7$ such that for $(\alpha_0, c_2, c_3, c_5) \in \mathcal{P}$ the following holds. There exists an $\omega_+ = \omega_+(\alpha_0, c_2, c_3, c_5) > 0$ such that (A.5) has a two-parameter family*

$$\mathcal{M}_q = \{(q_{+,\mathrm{pu},\omega_+}(X - X_0)\mathrm{e}^{\mathrm{i}\theta_0}, 0) : X_0 \in \mathbb{R}, \theta_0 \in [0, 2\pi)\} \qquad \text{(A.6)}$$

of pulse solutions. Moreover, for $\varepsilon > 0$ sufficiently small these pulses are exponentially orbital stable, i.e., there exists a constant $b_0 > 0$ such that the following holds. For all $C_2 > 0$ there exists a $C_1 > 0$ such that from

$$\|Q(0, \cdot) - (q_{+,\mathrm{pu},\omega_+}(\cdot - X_0)\mathrm{e}^{\mathrm{i}\theta_0}, 0)\|_{H^m} \leq C_1$$

for some $X_0 \in \mathbb{R}$, $\theta_0 \in [0, 2\pi)$, it follows that

$$\|Q(T, \cdot) - (q_{+,\mathrm{pu},\omega_+}(\cdot - X_1)\mathrm{e}^{\mathrm{i}\theta_1}, 0)\|_{H^m} \leq C_2 \mathrm{e}^{-b_0 T}$$

for some $X_1 \in \mathbb{R}$, $\theta_1 \in [0, 2\pi)$.

Proof. The existence of solutions to (A.5) is obvious. For the stability we consider the linearization around a pulse solution $Q_{\mathrm{pu}} = (q_{+,\mathrm{pu},\omega_+}, 0)$, i.e., we set $Q = Q_{\mathrm{pu}} + V$ and obtain

$$\partial_t V = LV + DN(Q_{\mathrm{pu}})V + DF(Q_{\mathrm{pu}})V.$$

From Theorem 2.5 we know that for the V_+–part we have two zero eigenvalues and that the rest of the spectrum is strictly in the left half plane. If we ignore $DF(Q_{\mathrm{pu}})V$ the system decouples and the V_-–part reduces to $\partial_t V_- = L_- V_-$ and so the exponential stability of $V_- = 0$ follows since $\mathrm{Re}\, c_0 = \alpha_0 < 0$.

In order to estimate the term $DF(Q_{\rm pu})V$ we consider the variation of constant formula

$$V(t) = \mathrm{e}^{Mt}V(0) + \int_0^t \mathrm{e}^{M(t-s)} DF(Q_{\rm pu}(s))V(s)\,\mathrm{d}s,$$

where e^{Mt} is the analytic semigroup generated by $M = L + DN(Q_{\rm pu})$. A typical term which comes up in the computation of the time–1–map is given by

$$\begin{aligned}&\int_0^1 \mathrm{e}^{L_-(1-s)} |\tau_{2\varepsilon^{-1}\nu_1 s} q_{+,\mathrm{pu},\omega_+}|^2 V_-(s)\,\mathrm{d}s \\ &= \int_0^1 \int_{\mathbb{R}} G(x-y, 1-s) |q_{+,\mathrm{pu},\omega_+}(y-2\nu_1 s/\varepsilon)|^2 V_-(y,s)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned}$$

where G is the associated Greens function to L_-. This expression is estimated by

$$\begin{aligned}&\int_{\mathbb{R}} \left(\int_0^1 \int_{\mathbb{R}} G(x-y, 1-s) |q_{+,\mathrm{pu},\omega_+}(y-2\nu_1 s/\varepsilon)|^2 V_-(y,s)\,\mathrm{d}y\,\mathrm{d}s \right)^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}} \left(\int_0^1 \int_{\mathbb{R}} G(z, 1-s) |q_{+,\mathrm{pu},\omega_+}(x-z-2\nu_1 s/\varepsilon)|^2 \,\mathrm{d}z\,\mathrm{d}s \right)^2 \mathrm{d}x \sup_{y\in\mathbb{R}, s\in[0,1]} |V_-(y,s)|^2 \\ &\leq \int_{\mathbb{R}} \left(\int_0^1 f(x-2\nu_1 s/\varepsilon)\,\mathrm{d}s \right)^2 \mathrm{d}x \sup_{y\in\mathbb{R}, s\in[0,1]} |V_-(y,s)|^2 \\ &\leq C\varepsilon^2 \sup_{y\in\mathbb{R}, s\in[0,1]} |V_-(y,s)|^2\end{aligned}$$

for $\varepsilon \to 0$ since f is a smooth function due to the spatial localization of the pulse and the exponential decay of the Green's function in space.

For $\varepsilon > 0$ sufficiently small the nonlinear stability then immediately follows by using the smoothing properties of the linear semigroup with the help of the center manifold theorem. □

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