Validity of the Ginzburg-Landau approximation in pattern forming systems with time periodic forcing

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Abstract

We consider the validity of the Ginzburg-Landau equation in pattern forming systems with time periodic forcing. Beside the proof of an approximation result for a model problem we extend the possibility for the derivation of the Ginzburg-Landau equation to arbitrary frequencies in time by a modified ansatz.

1 Introduction

Our investigations are motivated by electro-convection in nematic liquid crystals, the paradigm for pattern formation in anisotropic systems [10, 27, 6]. In this experiment [4] nematic liquid crystals with negative or only mildly positive dielectric anisotropy are sandwiched between two glass plates with transparent electrodes subject to some external time-periodic electric field, see Figure 1. Liquid crystals are often called the fourth state of matter, because they combine properties of a liquid, like the flow behavior, with such of solids, especially the anisotropy. In nematic liquid crystals the rod-like molecules point at an average in the same direction and can be influenced by an electric field. If the amplitude of the applied alternate current voltage is above a certain threshold the trivial spatially homogenous time periodic solution gets unstable and bifurcates into spatially periodic patterns.

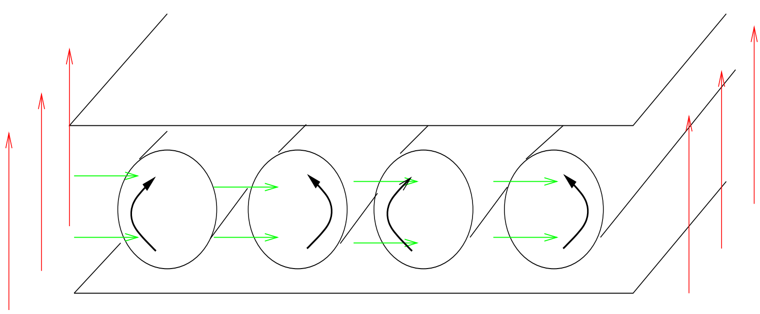

Figure 1: Roll solutions in nematic crystals. The director field is almost parallel to the plates. The external time-periodic electric field is perpendicular to the plates.

The mathematical analysis of the creation and interactions of such patterns is based very often on the reduction of the governing partial differential equations to finite or infinite-dimensional amplitude equations which are expected to capture the essential dynamics near the bifurcation point. The most famous amplitude equation occurring in such a setup is the so called Ginzburg-Landau equation. It is derived by multiple scaling analysis and describes slow modulations in time and space of the amplitude of the linearly most unstable modes. The Ginzburg-Landau equation has been derived for example for reaction-diffusion systems and hydrodynamical stability problems, as the Bénard and the Taylor-Couette problem. For these examples the Ginzburg-Landau equation has been justified as an amplitude equation by a number of mathematical theorems, [3, 29, 8, 19, 21, 11, 12] also including approximation and attractivity results. For an overview see [16]. Hence, the Ginzburg-Landau equation really gives a proper description of these original systems near the bifurcation point.

The Ginzburg-Landau equation has also been used extensively to describe pattern formation in nematic liquid crystals [27, 17, 30, 1]. However, the literature cited above about the mathematical justification of the Ginzburg-Landau equation is restricted so far to autonomous systems and is not covering the situation of nematic liquid crystals due to the time-periodic forcing. Therefore, it is the purpose of this paper to justify the Ginzburg-Landau equation also in case of a time periodic setup.

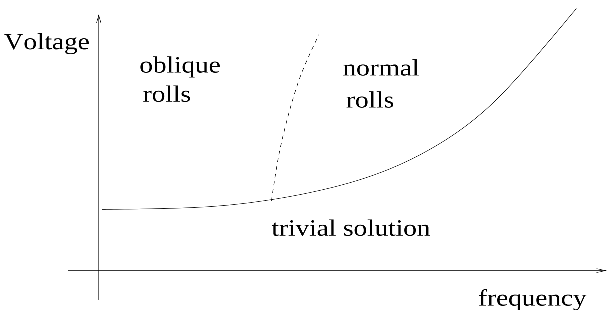

Figure 2: The bifurcation diagram.

The Ginzburg-Landau equation occurs in the parameter region indicated in Figure 2 as "normal rolls". For the derivation of the Ginzburg-Landau equation in this region in the existing literature it is assumed implicitly that the external electric field oscillates with a sufficiently high frequency such that the governing partial differential equations can be replaced by an effective autonomous system such that the usual derivation of the Ginzburg-Landau equation applies.

Our approach is different. By a modified ansatz we remove the assumption of a highly oscillating external electric field, i.e. we extend the possibility for the derivation of a Ginzburg-Landau equation to arbitrary frequencies in time. However, in contrast to the autonomous case the solutions of the Ginzburg-Landau equation then have to be analytic. This is no serious restriction, since this is true for every $t > 0$ by the smoothing properties of the Ginzburg-Landau equation, but it has to be assumed at $t = 0$.

Due to the limited number of pages we restrict ourselves in the discussion of the validity question to a model problem which has the essential features of the nematic liquid crystal problem which are relevant for our purposes. We refer to a forthcoming paper for the application of the theory to electro-convection in nematic liquid crystals.

The plan of the paper is as follows. In Section 2 we present our result in Theorem 2.1 for a scalar valued partial differential equation with time periodic forcing on the real line. We prove that solutions of this original system can be approximated via the solutions of the associated Ginzburg-Landau equation. This is the first time that the Ginzburg-Landau formalism is justified in a non-autonomous situation. The proof of this approximation result is given in a number of steps from Section 3 to Section 7. The general situation is discussed in a number of remarks in Section 2. The proof of Theorem 2.1 goes along the lines of the autonomous case and reviews the basic concepts.

Notation: Throughout this paper many different constants are denoted by the same symbol C if they can be chosen independent of the small perturbation parameter $0 < \varepsilon \ll 1$. Fourier transform is denoted by

$$(\mathcal{F}u)(k) = \hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ikx} dx.$$

2 The model

We consider

$$\partial_t u = -u + \beta \cos(\omega t) - \gamma \partial_x^8 u - u^2 (4\partial_x^2 + \partial_x^4) u + u \partial_x u, \tag{2.1}$$

with $u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, and $t \geq 0$. The three parameters satisfy $\beta \geq 0$, $\omega \neq 0$, and $\gamma > 0$, but small. This model problem has the essential features of the nematic liquid crystal problem which are relevant for our purposes. We have

a) a linear damping $-u$,

b) a time-periodic forcing $\beta \cos(\omega t)$, in which β is the control parameter for the amplitude and $\omega \neq 0$ the fixed temporal wave number,

c) an 8th-order derivative term $\gamma \partial_x^8 u$ in which $\gamma > 0$ is constant and small. This term makes the problem semilinear, and is only added to avoid functional analytic difficulties which are not related to our purposes.

d) the nonlinear term $-u^2(4\partial_x^2 + \partial_x^4)u$ roughly corresponds to nonlinear viscous stress in liquid crystals. This term is responsible for the Turing instability of the trivial solution. Like for the electro-convection problem it will be proportional to β^2.

e) $u\partial_x u$ models the convection term in the Navier-Stokes equations.

For the model problem we find the trivial, spatially homogenous, time-periodic solution

$$u_0(t) = (1 + \omega^2)^{-1} \beta (\cos(\omega t) + \omega \sin(\omega t)).$$

We set $u_0^2(t) = \bar{c} + c(t)$, where $\bar{c} = \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} u_0^2(t) dt$ is the mean value which is proportional to β^2. Note that $u_0(t)$ and $c(t)$ have a vanishing mean value. We do this to separate the equation for the perturbation $v(t) = u(t) - u_0(t)$ into an autonomous and into an oscillating part. We get

$$\begin{aligned} \partial_t v \;=\; & -v - \gamma \partial_x^8 v - \bar{c}(4\partial_x^2 + \partial_x^4) v \\ & -v^2 (4\partial_x^2 + \partial_x^4) v + v \partial_x v \\ & -c(t)(4\partial_x^2 + \partial_x^4) v + u_0(t) \partial_x v \\ & -2u_0(t) v (4\partial_x^2 + \partial_x^4) v. \end{aligned} \tag{2.2}$$

The first line contains the autonomous linear terms, the second line the autonomous nonlinear terms, the third line the non autonomous linear terms, and the fourth line the non autonomous nonlinear terms. To determine the stability of the trivial solution we first consider the linearized autonomous part

$$\partial_t v = -v - \gamma \partial_x^8 v + \bar{c}(4\partial_x^2 + \partial_x^4) v. \tag{2.3}$$

We find the solutions

$$v(x,t) = e^{ikx+\lambda(k,\overline{c})t},$$

where

$$\lambda(k,\overline{c}) = -1 + 4\overline{c}k^2 - \overline{c}k^4 - \gamma k^8.$$

There are a critical $\overline{c}_{\text{crit}}$ and a critical wave number $k_c > 0$, such that $\lambda(k_c, \overline{c}_{\text{crit}}) = 0$, where $\lambda(k,\overline{c}) < 0$, if $\overline{c} < \overline{c}_{\text{crit}}$. We are interested in the case $\overline{c} > \overline{c}_{\text{crit}}$ and introduce the small perturbation parameter $\varepsilon > 0$ by

$$\overline{c} = \overline{c}_{\text{crit}} + \varepsilon^2 .$$

For $\varepsilon > 0$ sufficiently small, we have the typical situation for deriving a Ginzburg-Landau equation. See Figure 3. Adding the non autonomous linear terms from (2.2)

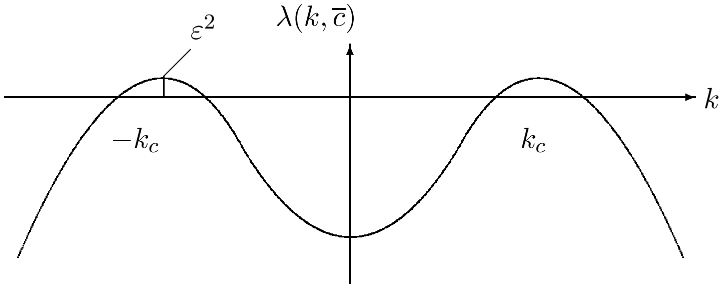

Figure 3: The curve $k \mapsto \lambda(k,\overline{c})$ for $\varepsilon > 0$

to (2.3) we find solutions (see below)

$$v(x,t) = e^{ikx+\lambda(k,\overline{c})t}\hat{v}_k(t),$$

with $\hat{v}_k(t) = \hat{v}_k\left(t + \dfrac{2\pi}{\omega}\right)$, i.e. the Floquet exponents of the linearization about the trivial time periodic solution $u_0(t)$ are also given by $\lambda = \lambda(k,\overline{c})$.

In order to handle arbitrary but fixed ω we modify the usual ansatz for the derivation of the Ginzburg-Landau equation to

$$\psi_A(x,t,\varepsilon) = (\varepsilon A(X,T)e^{ik_c x + \varphi_0(t)} + c.c.) + \mathcal{O}(\varepsilon^2), \qquad (2.4)$$

where

$$X = \varepsilon(x + \varphi_1(t)) \quad \text{and} \quad T = \varepsilon^2 t.$$

Because of the oscillating terms we have changed the usual ansatz by introducing functions $\varphi_0 = \varphi_0(t)$ and $\varphi_1 = \varphi_1(t)$. Inserting (2.4) into (2.1) and equating the coefficients in front of the $\varepsilon^m e^{ni(k_c x + \varphi_0(t))}$ to zero shows

$$\begin{aligned}
\varphi_0(t) &= (4k_c^2 - k_c^4)\int_0^t c(\tau)\,d\tau + ik_c\int_0^t u_0(\tau)\,d\tau, \\
\varphi_1(t) &= (8k_c^3 - 4k_c)i\int_0^t c(\tau)\,d\tau + \int_0^t u_0(\tau)\,d\tau,
\end{aligned}$$

and that A has to satisfy a Ginzburg-Landau equation (4.1) with highly oscillating coefficients which can be simplified further to an autonomous Ginzburg-Landau equation

$$\partial_T A = c_1 A + c_2 \partial_X^2 A - c_3 A|A|^2, \tag{2.5}$$

with coefficients $c_j \in \mathbb{C}$.

Note that φ_0 and φ_1 stay bounded, but possess non vanishing imaginary parts. Hence A which is evaluated at $X = \varepsilon(x + \varphi_1(t))$ has to be analytic in a strip in the complex plane. Our approximation theorem is as follows:

Theorem 2.1 *Let $\delta_{GL} > 0$ and $A = A(X,T)$ be a solution of the Ginzburg-Landau equation (2.5) for $T \in [0, T_0]$, analytic in X in a strip $S_{\delta_{GL}} = \{z \in \mathbb{C} \mid |\mathrm{Im} z| < \delta_{GL}\}$ in the complex plane satisfying*

$$\sup_{T\in[0,T_0]} \sup_{z\in S_{\delta_{GL}}} |A(z,T)| < \infty.$$

Then there are $\varepsilon_0 > 0$ and $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions v of (2.2) satisfying

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |v(x,t) - \psi_A(x,t,\varepsilon)| \leq C\varepsilon^2.$$

Remark 2.2 As a consequence of Theorem 2.1 the dynamics known for (2.5) can be found approximately in the original system (2.2), too. The error of order $\mathcal{O}(\varepsilon^2)$ is much smaller than the approximation ψ_A and the solution v which are both of order $\mathcal{O}(\varepsilon)$ for all $T \in [0, T_0]$ or $t \in [0, T_0/\varepsilon^2]$, respectively. This fact should not be taken for granted: there are modulation equations (for an example see [22]) which although derived by reasonable formal arguments do not reflect the true dynamics of the original equations.

Remark 2.3 Like in the autonomous case [23] the approximation theorem as stated above can be improved in a number of directions. However, the proof of an optimized result would be very technical and beyond the scope of this paper.

Remark 2.4 In the autonomous case approximation and attractivity results have been established by a number of authors, cf. [3, 29, 8, 18, 19, 21, 26] for model problems, but also for the general situation including the Navier-Stokes equation. Nowadays the theory is a well established mathematical tool which can be used to prove stability results [28, 25], upper semi-continuity of attractors [15, 24] and global existence results [20, 23].

Remark 2.5 To prove Theorem 2.1 we cannot directly use energy estimates because they do not work on the long timescale of order $\mathcal{O}(1/\varepsilon^2)$ due to quadratic terms in (2.2). We have to use mode filters as in the autonomous case [18, 19] to separate the error function into critical and stable modes.

Surprisingly there are two unexpected points.

Remark 2.6 First, in contrast to the existing literature by the modified ansatz (2.4) we are able to remove the assumption of a highly oscillating external electric field, i.e. we gain the possibility of deriving the Ginzburg-Landau equation for arbitrary frequencies.

Remark 2.7 Secondly, as a consequence of our approach in contrast to the autonomous case the solutions of the Ginzburg-Landau equation have to be analytic. However, this is no serious restriction, since this is true for every $t > 0$ by the smoothing properties of the Ginzburg-Landau equation [26], but it has to be assumed for $t = 0$.

Remark 2.8 The electro-convection problem in liquid crystals possesses two unbounded directions. Due to the anisotropy of the problem the instability takes place at two non zero wave vectors $\pm k_c \in \mathbb{R}^2$. The amplitude equation is then given by a Ginzburg-Landau equation [1]

$$\partial_T A = c_1 A + c_{2,X}\partial_X^2 A + c_{2,Y}\partial_Y^2 A - c_3 A|A|^2,$$

with solutions $A = A(X, Y, T) \in \mathbb{C}$ and coefficients $c_{2,X}, c_{2,Y} \in \mathbb{C}$. Hence the restriction to one unbounded direction is no restriction with respect to our purposes. For the classical isotropic Bénard convection instability occurs at a ring of wave vectors $k \in \mathbb{R}^2$ satisfying $|k| = k_c \in \mathbb{R}$. Hence in two unbounded directions in the isotropic case no longer a Ginzburg-Landau equation occurs.

Remark 2.9 For non small values of ε, i.e. away from the bifurcation point other amplitude equations take the role of the Ginzburg-Landau equation. In general the locally preferred patterns do not fit together globally, and so there will be some phase shifts in the pattern which will be transported or transformed by dispersion and diffusion. For the description of the evolution of the local wavenumber q of these pattern phase diffusion equations, conservation laws and Burgers equation can be derived. Recently, approximation results in the above sense have been proved in [13, 14, 7]. However, there are a number of restrictions. The estimates only hold locally in space and there is a global phase shift which cannot be estimated to be small on the time scale under consideration, i.e. only the form of the solution, but not its position can be approximated by these amplitude equations. These restrictions do not apply for pattern which are perfect for $x \to \pm\infty$.

The rest of the paper contains the proof of Theorem 2.1. It goes along the lines of the autonomous case and reviews the basic concepts.

3 Some preparations

The solution $t \mapsto v(\cdot, t)$ of (2.2) defines a curve in an infinite-dimensional phase space. This means that $x \mapsto v(x, t)$ for fixed t lies in a suitable function space. In order to prove Theorem 2.1 we have to compare the distance between the curve $t \mapsto v(\cdot, t)$ and the associated approximation $t \mapsto \psi_A(\cdot, t, \varepsilon)$ for each fixed t in the norm of the phase space.

Sobolev spaces H^m equipped with the norm

$$\|u\|_{H^m} = \sum_{j=0}^{m} \|\partial_x^j u\|_{L^2}, \qquad \text{with} \qquad \|u\|_{L^2}^2 = \int_{\mathbb{R}} |u(x)|^2 dx,$$

which turned out to be a good choice for the handling of partial differential equations on finite domains are too small for our purposes. Fundamental solutions like constant functions, spatially periodic functions, and fronts are not contained in H^m. It turned out that it is advantageous [19, 15] to work with the $H^m_{l,u}$-space of uniform local Sobolev functions equipped with the norm

$$\|u\|_{H^m_{l,u}} = \sup_{x\in\mathbb{R}} \|u\|_{H^m(x,x+1)}$$

satisfying

$$\|T_y u - u\|_{H^m_{l,u}} \to 0 \quad \text{for} \quad y \to 0,$$

where $(T_y u)(x) = u(x+y)$. This space contains the missing functions and easily allows to use Fourier transform. In Fourier space linear differential operators and fundamental solutions of linear partial differential equations are multiplication operators. In order to control these operators in the $H^m_{l,u}$-spaces we use the following multiplier theorem.

Lemma 3.1 *Let W_1, W_2 be some Hilbert spaces, $m \in \mathbb{Z}$ and $k \mapsto (1+k^2)^{m/2}\hat{M}(k) \in C^2_b(\mathbb{R}, L(W_1, W_2))$. Then the linear operator $M_{l,u} : H^q_{l,u}(W_1) \mapsto H^{q+m}_{l,u}(W_2)$ is bounded for all $q \in \mathbb{N}_0$ with $q+m \geq 0$ by*

$$\leq c(q,m)\|(1+|\cdot|^2)^{m/2}\hat{M}\|_{C^2_b(\mathbb{R},L(W_1,W_2))} ,$$

where $c(q,m)$ is independent of $\hat{M}$.

Proof. See page 441/442 of [19]. ■

We mainly use multiplier theory to separate the critical and non critical modes by so called mode filters.

Fix $\delta > 0$ smaller than $k_c/8$ independent of $0 < \varepsilon \ll 1$. Let χ_c be a C^∞_0 cut off function with values in $[0,1]$ and

$$\chi_c(k) = \begin{cases} 1, & \text{for } k \in I_c = [-k_c-\delta\,, -k_c+\delta] \bigcup [k_c-\delta\,, k_c+\delta], \\ 0, & \text{for } k \in \mathbb{R}\setminus([-k_c-2\delta\,, -k_c+2\delta] \bigcup [k_c-2\delta\,, k_c+2\delta])\,. \end{cases}$$

Then we define the mode filter for the critical modes by

$$E_c v = \mathcal{F}^{-1}\chi_c\mathcal{F}v\,.$$

The mode filter for the stable modes is defined by

$$E_s = 1 - E_c\,.$$

According to the fact that E_c and E_s are not projections we define auxiliary mode filters E^h_c and E^h_s satisfying $E^h_c E_c = E_c$ and $E^h_s E_s = E_s$ by

$$E^h_c v = \mathcal{F}^{-1}\chi^h_c\mathcal{F}v,$$

where χ^h_c is a C^∞_0-cut off function with values in $[0,1]$ and

$$\chi^h_c(k) = \begin{cases} 1, & \text{for } k \in I_c = [-k_c-2\delta\,, -k_c+2\delta] \bigcup [k_c-2\delta\,, k_c+2\delta], \\ 0, & \text{for } k \in \mathbb{R}\setminus([-k_c-3\delta\,, -k_c+3\delta] \bigcup [k_c-3\delta\,, k_c+3\delta]) \end{cases}$$

and by

$$E^h_s v = \mathcal{F}^{-1}(1-\chi^h_s)\mathcal{F}v,$$

where χ_s^h is a C_0^∞-cut off function with values in $[0,1]$ and

$$\chi_s^h(k) = \begin{cases} 1, & \text{for } k \in I_c = [-k_c - \delta/2\,, -k_c + \delta/2] \bigcup [k_c - \delta/2\,, k_c + \delta/2], \\ 0, & \text{for } k \in \mathbb{R} \setminus ([-k_c - \delta\,, -k_c + \delta] \bigcup [k_c - \delta\,, k_c + \delta])\,. \end{cases}$$

We will use that the critical part of the quadratic interaction of critical modes vanishes identically due to disjoint supports in Fourier space, i.e.

$$E_c((E_c u) \cdot (E_c v)) = 0\,. \tag{3.1}$$

With Lemma 3.1 we conclude

Lemma 3.2 *The operators E_c and E_c^h are linear bounded mappings from $L^2_{l,u}$ into $H^m_{l,u}$, i.e. for all $m \geq 0$ there exist $C_m \geq 0$, such that*

$$\|E_c u\|_{H^m_{l,u}} + \|E_c^h u\|_{H^m_{l,u}} \leq C_m \|u\|_{L^2_{l,u}}\,.$$

Proof: Using Lemma 3.1 shows

$$\begin{aligned} \|E_c u\|_{H^m_{l,u}} &\leq C\|(1+|\cdot|^2)^{m/2}\hat{E}_c(\cdot)\|_{C_b^2}\|u\|_{L^2_{l,u}} \\ &\leq C\|u\|_{L^2_{l,u}} \end{aligned}$$

using the compact support of E_c. ∎

As a direct consequence we have

Lemma 3.3 *The operators E_s and E_s^h are bounded mappings in $H^m_{l,u}$, i.e. for all $m \geq 0$ there exist $C_m \geq 0$ such that*

$$\|E_s u\|_{H^m_{l,u}} + \|E_s^h u\|_{H^m_{l,u}} \leq C_m \|u\|_{H^m_{l,u}}\,.$$

Our model (2.2) and the associated Ginzburg-Landau equation (2.5) are semilinear equations, i.e. the local existence and uniqueness of the solutions follows by a standard fixed point argument with the help of the variation of constant formula [9]. Thus, for given $A|_{T=0} \in H^m_{l,u}$ there exists a $T_1 > 0$ and a solution $A \in C([0,T_1], H^m_{l,u})$ of the Ginzburg-Landau equation (2.5) for all $m \geq 1$. Moreover, for given $v|_{t=0} \in H^m_{l,u}$ there exists a $t_1 > 0$ and a solution $v \in C([0,t_1], H^m_{l,u})$ of the model (2.2) for all $m \geq 4$. As a consequence the solutions of the Ginzburg-Landau equation (2.5) exist as a long as they can be bounded in $H^m_{l,u}$ for a $m \geq 1$ and the solutions of the model (2.2) exist as a long as they can be bounded in $H^m_{l,u}$ for a $m \geq 4$. By the smoothing properties of the linearized system the solutions are analytic for every $T > 0$ and $t > 0$, respectively.

Finally, we introduce the space

$$C_a^\omega = \{u : S_a \mapsto \mathbb{C} \mid u \text{ analytic in } S_a\},$$

where $S_a = \{z \in \mathbb{C} \mid |\mathrm{Im} z| < a\}$ equipped with the norm

$$\|u\|_{C_a^\omega} = \sup_{z \in S_a} |u(z)| < \infty.$$

4 Derivation of the Ginzburg-Landau-equation

The so called residual

$$\begin{aligned}\mathrm{Res}(v) &= -\partial_t v - v - \gamma\partial_x^8 v - \overline{c}(4\partial_x^2 + \partial_x^4)v \\ &\quad -v^2(4\partial_x^2 + \partial_x^4)v + v\partial_x v \\ &\quad -c(t)(4\partial_x^2 + \partial_x^4)v + u_0(t)\partial_x v \\ &\quad -2u_0(t)v(2\partial_x^2 + \partial_x^4)v\end{aligned}$$

contains all terms which do not cancel after inserting the approximation into the model (2.2). Hence, $\mathrm{Res}(v) = 0$ if and only if v is a solution of (2.2). In this section we mainly estimate the residual for the approximation (2.4). We recall $X = \varepsilon(x + \varphi_1(t))$ and $T = \varepsilon^2 t$ and refine the ansatz to

$$\begin{aligned}\psi_A(x,t,\varepsilon) &= \varepsilon A_1(X,T)e^{ik_c x + \varphi_0(t)} + c.c. \\ &\quad + \frac{\varepsilon^2}{2}A_0(X,T) + \varepsilon^2 A_2(X,T)e^{2(ik_c x + \varphi_0(t))} + c.c.\ .\end{aligned}$$

Equating the coefficient in front of $\varepsilon e^{ik_c x}$ to zero shows

$$\varphi_0(t) = (4k_c^2 - k_c^4)\int_0^t c(\tau)d\tau + ik_c\int_0^t u_0(\tau)d\tau.$$

The integrals stay bounded due to the vanishing mean values of u_0 and c. Equating the coefficient in front of $\varepsilon^2 e^{ik_c x}$ to zero shows

$$\varphi_1(t) = (4k_c^3 - 8k_c)\, i\int_0^t c(\tau)d\tau + \int_0^t u_0(\tau)d\tau.$$

Again the integrals stay bounded due to the vanishing mean values of u_0 and c. Especially, we have the $\mathcal{O}(1)$-boundedness of the imaginary part of φ_1.

Remark 4.1 The solutions of the Ginzburg-Landau-equation are analytic in a strip of width δ_{GL}. Since $|\mathrm{Im}(X)| \le \mathcal{O}(\varepsilon) \ll \delta_{GL}$ the solution can be evaluated at the position $X = \varepsilon(X + \varphi_1(t))$.

Equating the coefficient in front of $\varepsilon^3 e^{ik_c x}$ to zero gives an equation of the form

$$\begin{aligned}\partial_T A_1 &= d_1(\frac{T}{\varepsilon^2})A_1 + d_2(\frac{T}{\varepsilon^2})\partial_X^2 A_1 + d_3(\frac{T}{\varepsilon^2})A_1 A_0 \\ &\quad + d_4(\frac{T}{\varepsilon^2})A_2 A_{-1} + d_5(\frac{T}{\varepsilon^2})A_1 A_1 A_{-1},\end{aligned}$$

with time-dependent coefficients $d_j = d_j(\frac{T}{\varepsilon^2})$. In order to get an equation for A_0 we equate the coefficient in front of $\varepsilon^2 e^{0k_c x}$ to zero. We obtain an equation of the form

$$d_6(\frac{T}{\varepsilon^2})A_0 = d_7(\frac{T}{\varepsilon^2})A_{-1}A_1,$$

with

$$d_6(\frac{T}{\varepsilon^2}) = \lambda(0, \overline{c}_{\mathrm{crit}}) = -1 \neq 0.$$

In order to get on equation for A_2 we equate the coefficient in front of $\varepsilon^2 e^{2ik_cx}$ to zero. We obtain an equation of the form

$$d_8(\frac{T}{\varepsilon^2})A_2 = d_9(\frac{T}{\varepsilon^2})A_1A_1,$$

with

$$d_6(\frac{T}{\varepsilon^2}) = \lambda(2k_c) - c(t)(-16k_c^2 + 16k_c^4) + 2u_0(t)ik_c \neq 0,$$

which can be computed explicitly. Eliminating A_0 and A_2 in the equation of A_1 shows that A_1 satisfies an equation of the form

$$\partial_T A_1 = d_1(\frac{T}{\varepsilon^2})A_1 + d_2(\frac{T}{\varepsilon^2})\partial_x^2 A_1 + d_{10}(\frac{T}{\varepsilon^2})A_1|A_1|^2. \tag{4.1}$$

Then for a given solution A of (4.1), we can construct A_0 and A_2 such that the residual for the critical modes is of order $\mathcal{O}(\varepsilon^4)$ and for the stable modes of order $\mathcal{O}(\varepsilon^3)$. This can be made rigorous with the help of Lemma 3.1, cf. [19].

Notation. If the approximation is constructed via the solutions of the autonomous Ginzburg-Landau equation (2.5) it will be denoted with $\varepsilon\psi_A$ in the following. If the approximation is constructed via the solutions of the non autonomous Ginzburg-Landau equation (4.1) it will be denoted with $\varepsilon\psi_B$ in the following.

Lemma 4.2 *Let $\delta_1 > 0$ and $C_1 > 0$. For all $\varepsilon \in (0,1)$ let $A_1 = A_1(X,T) \in C([0,T_0], C^\omega_{\delta_1})$ be a solution of (4.1) with $\sup_{T\in[0,T_0]} \|A_1(\cdot,T)\|_{C^\omega_{\delta_1}} \leq C_1$. Then we have a $C_2 > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\begin{aligned} \sup_{t\in[0,T_0/\varepsilon^2]} \|E_s(\mathrm{Res}(\psi_B))\|_{H^m_{l,u}} &= \mathcal{O}(\varepsilon^3), \\ \sup_{t\in[0,T_0/\varepsilon^2]} \|E_c(\mathrm{Res}(\psi_B))\|_{H^m_{l,u}} &= \mathcal{O}(\varepsilon^4). \end{aligned}$$

5 The non autonomous case

As a major step of the proof of Theorem 2.1 we show here that the solutions of (2.2) can be approximated via the solutions of the non autonomous Ginzburg-Landau equation (4.1).

Theorem 5.1 *Let $\delta_1 > 0$ and $C_1 > 0$. For all $\varepsilon \in (0,1)$ let $A_1 = A_1(X,T) \in C([0,T_0], C^\omega_{\delta_1})$ be a solution of (4.1) with $\sup_{T\in[0,T_0]} \|A_1(\cdot,T)\|_{C^\omega_{\delta_1}} \leq C_1$. Then there are $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions v of (2.2) satisfying*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|v(\cdot,t) - \psi_B(\cdot,t,\varepsilon)\|_{H^m_{l,u}} \leq C\varepsilon^2.$$

Proof. We write (2.2) as

$$\partial_t v = \Lambda v + B(v,v) + C(v,v,v), \tag{5.1}$$

with

$$\begin{aligned} \Lambda v &= -v - \gamma\partial_x^8 v - \overline{c}(4\partial_x^2 + \partial_x^4)v - c(t)(4\partial_x^2 + \partial_x^4)v + u_0(t)\partial_x v, \\ B(v,v) &= v\partial_x v - 2u_0(t)v(4\partial_x^2 + \partial_x^4)v, \\ C(v,v,v) &= -v^2(4\partial_x^2 + \partial_x^4)v. \end{aligned}$$

Inserting

$$v = \varepsilon\psi_c + \varepsilon^2\psi_s + \varepsilon^2 R_c + \varepsilon^3 R_s ,$$

with $R_c = E_c^h R_c$, $R_s = E_s^h R_s$, $\psi_c = E_c^h \psi_c$, and $\psi_s = E_c^h \psi_s$ gives, by using (3.1),

$$\begin{aligned} \partial_t R_c &= \Lambda R_c + \varepsilon^2 L_c(R) + \varepsilon^3 N_c(R) + \varepsilon^2 \mathrm{Res}_c , \\ \partial_t R_s &= \Lambda R_s + L_s(R_c) + \varepsilon N_s(R) + \mathrm{Res}_s , \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \mathrm{Res}_c &= \varepsilon^{-4} E_c(\mathrm{Res}(\psi_B)) , \\ \mathrm{Res}_s &= \varepsilon^{-3} E_s(\mathrm{Res}(\psi_B)) , \\ L_c(R) &= 2E_c(B(R_s, \psi_c) + B(R_c, \psi_s)) , \\ L_s(R_c) &= 2E_s B(R_c, \psi_c) , \end{aligned}$$

and where $N_c(R)$ and $N_s(R)$ satisfy

$$\begin{aligned} \|N_c(R)\|_{H^m_{l,u}} &\leq C(D_c, D_s)(\|R_c\|_{H^m_{l,u}} + \|R_s\|_{H^m_{l,u}})^2, \\ \|N_s(R)\|_{H^{m-4}_{l,u}} &\leq C\|R_s\|_{H^m_{l,u}} + C(D_c, D_s)(\|R_c\|_{H^m_{l,u}} + \|R_s\|_{H^m_{l,u}})^2 \end{aligned}$$

as long as

$$\|R_c\|_{H^m_{l,u}} \leq D_c \quad \text{and} \quad \|R_s\|_{H^m_{l,u}} \leq D_s , \tag{5.3}$$

where $C(D_c, D_s)$ is a constant depending on D_c and D_s independent of $0 \leq \varepsilon \ll 1$. The constants D_c and D_s will be chosen later on independent of $0 \leq \varepsilon \leq 1$. This system is solved with initial datum $(R_c(0), R_s(0)) = (0, 0)$. The solution of

$$\partial_t R = \Lambda R, \quad R|_{t=\tau} = R_0$$

is denoted with $R(t) = \mathcal{K}(t,\tau)R_0$ which defines a linear evolution operator $\mathcal{K}(t,\tau)$ satisfying $\mathcal{K}(t,\tau) = \mathcal{K}(t + \frac{2\pi}{\omega}, \tau + \frac{2\pi}{\omega})$. In Fourier space we have

$$\partial_t \hat{R}(k) = \Lambda(k)R(k) = \lambda(k)\hat{R}(k) - d_1(k)c(t)\hat{R}(k) + d_2(k)u_0(t)\hat{R}(k),$$

with constants $d_j = d_j(k)$ which is solved by

$$\begin{aligned} \hat{v}(k,t) &= e^{\int_0^t \lambda(k) + d_1(k)c(\tau) + d_2(k)u_0(\tau)\, d\tau} \hat{v}(k,0) \\ &= e^{\lambda(k)t} e^{d_1(k)\int_0^t c(\tau)d\tau} e^{d_2(k)\int_0^t u_0(\tau)d\tau} \hat{v}(k,0) \end{aligned}$$

leading to the Floquet multipliers $e^{\lambda(k)\frac{2\pi}{\omega}}$ due to the vanishing mean values of c and u_0. As a direct consequence it follows: [19]

Lemma 5.2 *There exist C, $\sigma > 0$ independent of $0 < \varepsilon \ll 1$ such that we have for the stable part*

$$\|\mathcal{K}(t,\tau)E_s^h\|_{L(H^m_{l,u}, H^m_{l,u})} \leq C e^{-\sigma(t-\tau)}$$

and

$$\|\mathcal{K}(t,\tau)E_s^h\|_{L(H^{m-4}_{l,u}, H^m_{l,u})} \leq C \max(1, (t-\tau)^{-1/2}) e^{-\sigma(t-\tau)}.$$

Moreover, we have for the critical part

$$\|\mathcal{K}(t,\tau)E_c^h\|_{L(H^m_{l,u}, H^m_{l,u})} \leq C e^{C\varepsilon^2(t-\tau)}.$$

We apply the variation of constant formula to (5.3) and obtain

$$\begin{aligned} R_c(t) &= \int_0^t \mathcal{K}(t,\tau) E_c^h(\varepsilon^2 L_c(R) + \varepsilon^3 N_c(R) + \varepsilon^2 \mathrm{Res}_c)(\tau) d\tau \;, \\ R_s(t) &= \int_0^t \mathcal{K}(t,\tau) E_s^h(L_s(R_c) + \varepsilon N_s(R) + \mathrm{Res}_s)(\tau) d\tau \;. \end{aligned}$$

With $S_i(s) := \sup_{0 \le t \le s} \|R_i(t)\|_{H^m_{l,u}}$, $(i = s, c)$ we find, by using (5.2) and

$$\Big(\int_0^t C \max(1, \tau^{-1/2}) e^{-\sigma\tau} d\tau \Big) = \mathcal{O}(1)$$

independent of $t > 0$, that

$$\begin{aligned} S_s(t) &\le CS_c(t) + \varepsilon(CS_s(t) + C_s(D_c, D_s)(S_c(t) + S_s(t))^2) + C_{Res}, \\ &\le CS_c(t) + 1 + C_{Res}, \end{aligned}$$

if

$$\varepsilon(CD_s + C_s(D_c, D_s)(D_c + D_s)^2) \le 1. \tag{5.4}$$

Similarly, we find

$$\begin{aligned} S_c(t) &\le \varepsilon^2 \int_0^t C(S_c(\tau) + S_s(\tau)) + \varepsilon C_s(D_c, D_s)(S_c(\tau) + S_s(\tau))^2 + C_{Res} d\tau, \\ &\le \varepsilon^2 \int_0^t C(S_c(\tau) + S_s(\tau)) + 1 + C_{Res} d\tau, \end{aligned}$$

if

$$\varepsilon C_s(D_c, D_s)(D_c + D_s)^2 \le 1. \tag{5.5}$$

Using the above estimate for $S_s(t)$ finally shows

$$S_c(t) \le \varepsilon^2 \int_0^t C(S_c(\tau) + 1 + C_{Res}) d\tau.$$

Gronwall's inequality yields

$$S_c(t) \le C(1 + C_{Res}) T_0 e^{CT_0} =: D_c$$

for all $t \in [0, T_0/\varepsilon^2]$. Then by the estimate for $S_s(t)$

$$S_s(t) \le CD_c + 1 + C_{Res} =: D_s.$$

We are done, if we choose $\varepsilon_0 > 0$ so small that for all $\varepsilon \in (0, \varepsilon_0)$ the conditions (5.4) and (5.5) are satisfied. ■

6 Comparison of the Ginzburg-Landau equations

Obviously the Ginzburg-Landau equation (4.1) is useless for constructing approximations for the solutions of (2.2) since it still contains the small parameter ε in a singular way. We expect that for $\varepsilon \to 0$ only the average of the d_j will play a role and that it is sufficient to consider an autonomous Ginzburg-Landau equation. Therefore, it is the purpose of this section to compare the solutions of the autonomous Ginzburg-Landau-equation

$$\partial_T A = c_1 \partial_X^2 A + c_2 A + c_3 A|A|^2, \tag{6.1}$$

with the solutions of the non-autonomous Ginzburg-Landau-equation

$$\partial_T B = (c_1 + c_4(\frac{T}{\varepsilon^2}))\partial_X^2 B + (c_2 + c_5(\frac{T}{\varepsilon^2}))B + (c_3 + c_6(\frac{T}{\varepsilon^2}))B|B|^2, \tag{6.2}$$

with highly oscillating coefficients c_j for $j = 4, 5, 6$ satisfying

$$c_j(t) = c_j(t + 2\pi/\omega) \qquad \text{and} \qquad \int_0^{2\pi/\omega} c_j(t)dt = 0.$$

We prove

Theorem 6.1 *Let $\delta > 0$ and let $A \in C([0, T_0], C^\omega_{2\delta})$ be a solution of (6.1). Then there exist $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ equation (6.2) possesses solutions B satisfying*

$$\sup_{T\in[0,T_0]} \|A(\cdot, T) - B(\cdot, T)\|_{C^\omega_\delta} \leq C\varepsilon^2 .$$

Proof. We introduce the error function R by $B = A + \varepsilon^2 R$ which satisfies

$$\begin{aligned} \partial_T R &= \Lambda R + 3C(A, A, R) + 3\varepsilon^2 C(A, R, R) \\ &\quad + \varepsilon^4 C(R, R, R) + \varepsilon^{-2}\mathrm{Res}(A) , \end{aligned}$$

where

$$\begin{aligned} \Lambda R &= (c_1 + c_4(\frac{T}{\varepsilon^2}))\partial_X^2 R + (c_2 + c_5(\frac{T}{\varepsilon^2}))R , \\ 3C(R_1, R_2, R_3) &= (c_3 + c_6(\frac{T}{\varepsilon^2}))(R_1 R_2 \bar{R}_3 + R_1 \bar{R}_2 R_3 + \bar{R}_1 R_2 R_3) , \\ \mathrm{Res}(A) &= c_4(\frac{T}{\varepsilon^2})\partial_X^2 A + c_5(\frac{T}{\varepsilon^2})A + c_6(\frac{T}{\varepsilon^2})A|A|^2 . \end{aligned}$$

We define the linear evolution operator $S(T, \tau)$ by $R(T) = S(T, \tau)R(\tau)$, where $R(t)$ solves

$$\partial_T R = \Lambda R \quad , \quad R|_{T=\tau} = R(\tau) .$$

Then, we consider

$$\begin{aligned} R(T) &= \int_0^T S(T, \tau)(3C(A, A, R) + 3\varepsilon^2 C(A, R, R) \\ &\qquad + \varepsilon^4 C(R, R, R) + \varepsilon^{-2}\mathrm{Res}(A))(\tau)d\tau . \end{aligned}$$

We estimate

$$\sup_{T\in[0,T_0]} \left\| \int_0^T S(T, \tau)\varepsilon^{-2}\mathrm{Res}(A)(\tau)d\tau \right\|_{C^\omega_\delta} \leq C_{Res},$$

with $C_{Res} > 0$ a constant independent of $0 < \varepsilon \ll 1$. This follows for instance from

$$\begin{aligned}
& \| \int_0^T S(T,\tau) c_4(\frac{\tau}{\varepsilon^2}) \partial_X^2 A(\tau) d\tau \|_{C_\delta^\omega} \\
\leq \; & \| \varepsilon^2 S(T,\tau) \tilde{c}_4(\frac{\tau}{\varepsilon^2}) \partial_X^2 A(\tau)|_{\tau=0}^T \|_{C_\delta^\omega} \\
& + \| \int_0^T \varepsilon^2 S(T,\tau) \tilde{c}_4(\frac{\tau}{\varepsilon^2}) \partial_X^2 \partial_\tau A d\tau \|_{C_\delta^\omega} \\
& + \| \int_0^T \varepsilon^2 (-S(T,\tau)\Lambda(\tau)) \tilde{c}_4(\frac{\tau}{\varepsilon^2}) \partial_X^2 A \, d\tau \|_{C_\delta^\omega} \\
\leq \; & C\varepsilon^2 ,
\end{aligned}$$

where $\partial_\tau \tilde{c}_4 = c_4$ and $\partial_\tau S(T,\tau) = -S(T,\tau)\Lambda(\tau)$. The function $\tilde{c}_4$ stays $\mathcal{O}(1)$-bounded due to the vanishing mean value of c_4. The rest of the theorem then follows by a simple application of Gronwall's inequality. ■

Remark 6.2 It is easy to see that for every $0 \leq \delta' < \delta$ we have $\|\partial_T B\|_{C_{\delta'}^\omega} = \mathcal{O}(1)$ by expressing $\partial_T B$ by the right hand side of (6.2), but $\partial_T^2 B = \mathcal{O}(\varepsilon^{-2})$.

7 The final step

It remains to conclude Theorem 2.1 from Theorem 5.1 and Theorem 6.1. Let ψ_A be the approximation constructed via the solution A of the autonomous Ginzburg-Landau equation (2.5). Let ψ_B be the approximation constructed via the solution $B = A_1$ of the non-autonomous Ginzburg-Landau equation (4.1). Moreover, let v be the solution of the model (2.2). From Theorem 5.1 we have

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_B(\cdot,t,\varepsilon) - v(\cdot,t)\|_{H_{l,u}^m} = \mathcal{O}(\varepsilon^2).$$

From Theorem 6.1 and $C_\delta^\omega \subset H_{l,u}^m$ we have

$$\sup_{T\in[0,T_0]} \|A(\cdot,T) - B(\cdot,T)\|_{H_{l,u}^m} = \mathcal{O}(\varepsilon^2)$$

which implies

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_B(\cdot,t,\varepsilon) - \psi_A(\cdot,t,\varepsilon)\|_{H_{l,u}^m} = \mathcal{O}(\varepsilon^2).$$

Hence, by the triangle inequality and Sobolev's embedding theorem we have

$$\begin{aligned}
& \sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |\psi_A(x,t,\varepsilon) - v(x,t)| \\
\leq \; & C \sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_A(\cdot,t,\varepsilon) - v(\cdot,t)\|_{H_{l,u}^m} \\
\leq \; & C(\sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_A(\cdot,t,\varepsilon) - \psi_B(\cdot,t,\varepsilon)\|_{H_{l,u}^m} \\
& \quad + \sup_{t\in[0,T_0/\varepsilon^2]} \|\psi_B(\cdot,t,\varepsilon) - v(\cdot,t)\|_{H_{l,u}^m}) \\
= \; & \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Therefore, we are done. ■

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