Self–similar decay of localized perturbations in the Integral Boundary Layer Equation

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Abstract

The Integral Boundary Layer equation (IBLe) arises as a long wave approximation for the flow of a viscous incompressible fluid down an inclined plane. The trivial solution of the IBLe is linearly at best marginally stable, i.e., it has essential spectrum at least up to the imaginary axis. Here we show that in the stable case this trivial solution is in fact nonlinearly stable, with a Burgers like self–similar decay of localized perturbations. The proof uses renormalization theory and the fact that in the stable case Burgers equation is the amplitude equation for long small amplitude waves in the IBLe.

Keywords: inclined film flow, Integral Boundary Layer equation, Nusselt solution, nonlinear stability, Burgers equation, renormalization
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1 Introduction

In suitable parameter regimes the Integral Boundary Layer equation (IBLe) can be formally derived as a long wave approximation for the flow of a viscous incompressible fluid down an inclined plane; see [CD96] and the monograph [CD02] for reviews and [LG94] for experiments on inclined film flows. We consider the IBLe in the form

$$\begin{aligned} h_t &= -\, q_x, \\ q_t &= -\,\frac{6}{5}\partial_x\left(\frac{q^2}{h}\right) + \frac{2}{\mathrm{R}}\left(h - \frac{3q}{2h^2} - h_x h \cot\theta\right) + \mathrm{W}h(\partial_x^3 h - \frac{3}{2}\partial_x^3 h h_x^2 - 3h_{xx}^2 h_x) \\ &\quad + \frac{1}{\mathrm{R}}\left(\frac{7}{2}q_{xx} - \frac{9q_x h_x}{h} + \frac{6qh_x^2}{h^2} - \frac{9qh_{xx}}{2h}\right), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}$, $t > 0$, h is the film height, q describes the flow, $0 < \theta \leq \pi/2$ is the inclination angle, R is the Reynolds number, W is the Weber number, and the equation is written after rescaling to the original (dimensionless) time and space scales t, x of the underlying Navier–Stokes equations. See [Uec03] for the derivation of (1.1), which due to the term $\frac{7}{2\mathrm{R}} q_{xx}$ is a parabolic system in contrast to the classic Shkadov model [Shk67]. In this derivation it is assumed that the Weber number W is large, while $\mathrm{R} = \mathcal{O}(1)$ and $\cot\theta = \mathcal{O}(1)$. The latter means, that the plane must not be close to horizontal.

There exists a trivial solution $u = (h, q) = u_{\mathrm{N}} = (1, 2/3)$ to (1.1) which in the Navier–Stokes problem corresponds to the so called Nusselt solution U_{N} with a constant film height and a laminar flow profile. It turns out that u_{N} is unstable due to a long wave instability for R larger than the critical Reynolds number, i.e.,

$$\mathrm{R} > \mathrm{R}_c = \frac{5}{4} \cot\theta. \tag{1.2}$$

For the Navier–Stokes system this instability criterion for U_{N} has already been derived in [Ben57]. In the unstable case, the dynamics of long waves with small amplitude in the IBLe are described by the Kuramoto–Sivasinsky equation, in the limit $\mathrm{W} \to \infty$; in [Uec03] the approximation properties of this long wave/small amplitude approximation are established.

Here we are interested in the stable case

$$\mathrm{R} < \mathrm{R}_c. \tag{1.3}$$

Then Burgers equation serves as amplitude equation for (1.1) and we show that small localized perturbations of u_{N} decay in a Burgers–like self–similar way. Therefore we write (1.1) as $\partial_t u = F(u)$, set $h = 1 + \eta, q = 2/3 + \tilde{q}$, i.e., $u = u_{\mathrm{N}} + \tilde{u}$ with $\tilde{u} = (\eta, \tilde{q})$, go into a comoving frame $x = x - 2t$, and rewrite (1.1) as

$$\tilde{u}_t = A\tilde{u} + B(\tilde{u}, \tilde{u}) + H(\tilde{u}). \tag{1.4}$$

Here

$$A = \begin{pmatrix} 2\partial_x & -\partial_x \\ \frac{6}{\mathrm{R}} + (\frac{4}{5} - \frac{2}{\mathrm{R}}\cot\theta)\partial_x - \frac{3}{\mathrm{R}}\partial_x^2 + \mathrm{W}\partial_x^3 & -\frac{3}{\mathrm{R}} + \frac{2}{5}\partial_x + \frac{7}{2\mathrm{R}}\partial_x^2 \end{pmatrix},$$

$$B(\tilde{u}, \tilde{u}) = \begin{pmatrix} 0 \\ 6(\eta\tilde{q} - \eta^2)/\mathrm{R} \end{pmatrix}, \quad H(\tilde{u}) = \begin{pmatrix} 0 \\ h(\tilde{u}) \end{pmatrix} := F(u_N + \tilde{u}) - A\tilde{u} - B(\tilde{u}, \tilde{u}), \tag{1.5}$$

and we consider $B(\tilde{u}, \tilde{u})$ as a bilinear form in the obvious way. The reason for this splitting of $F(u_{\mathrm{N}} + \tilde{u})$ is that only terms from $\tilde{u}_t = A\tilde{u} + B(\tilde{u}, \tilde{u})$ contribute to the description of long small amplitude waves for (1.4) by Burgers equation. In what follows we rename $\tilde{u}$ to u and $\tilde{q}$ to q. In components, (1.4) then reads

$$\begin{aligned} \eta_t &= 2\eta_x - q_x, &(1.6)\\ q_t &= \big(\tfrac{6}{\mathrm{R}}+(\tfrac{4}{5}-\tfrac{2}{\mathrm{R}}\cot\theta)\partial_x-\tfrac{3}{\mathrm{R}}\partial_x^2+\mathrm{W}\partial_x^3\big)\eta+\big(-\tfrac{3}{\mathrm{R}}+\tfrac{2}{5}\partial_x+\tfrac{7}{2\mathrm{R}}\partial_x^2\big)q &(1.7)\\ &\quad+\tfrac{6}{\mathrm{R}}(\eta q-\eta^2)+h(\eta,q). \end{aligned}$$

Plugging the ansatz

$$\eta(t,x)=\delta\eta_1(\tau,y),\quad q(t,x)=\delta q_1(\tau,y)+\delta^2 q_2(\tau,y),\quad \tau=\delta^2 t,\quad y=\delta x, \tag{1.8}$$

into (1.4) yields

$$\begin{aligned} \mathcal{O}(\delta(1.7)):&\quad q_1=2\eta_1,\\ \mathcal{O}(\delta^2(1.7)):&\quad q_2=(\tfrac{8\mathrm{R}}{15}-\tfrac{2}{3}\cot\theta)\partial_X\eta_1+2\eta_1^2, \end{aligned} \tag{1.9}$$

and at $\mathcal{O}(\delta^3(1.6))$ we obtain

$$\partial_\tau\eta_1=\alpha\partial_y^2\eta_1+\beta\partial_y(\eta_1^2),\qquad\text{with}\qquad \alpha=(\frac{2}{3}\cot\theta-\frac{8\mathrm{R}}{15}),\quad \beta=-2. \tag{1.10}$$

Note that $\alpha>0$ due to (1.3). It can be checked that the terms in $h(u)$ only enter this long wave/small amplitude expansion at higher orders in δ. However, we did not remove high order δ terms from the linear part Au since the full linear operator will be needed for the local existence theory for the quasilinear system (1.4).

Hence small amplitude long waves are governed by Burgers equation (1.10). This could be rescaled to the more standard form $\eta_\tau=\eta_{yy}+\partial_y(\eta^2)$. Here we don't do this in order to keep track of α and β. The Cole–Hopf transformation

$$\psi(t,x)=\exp\left(\frac{\beta}{\alpha}\int_{-\infty}^{\sqrt{\alpha}x}\eta(t,\xi)\,\mathrm{d}\xi\right),\quad \eta(t,x)=\frac{\sqrt{\alpha}}{\beta}\frac{\psi_y(t,y)}{\psi(t,y)},\quad y=x/\sqrt{\alpha},$$

transforms (1.10) to the linear diffusion equation $\psi_t=\psi_{xx}$. With $\psi(-\infty)=1$ and setting $\psi(\infty)=z+1$, i.e., $\ln(z+1)=\frac{\beta}{\alpha}\int_{\mathbb{R}}\eta(t,\xi)\,\mathrm{d}\xi$, it is well known that

$$1+ze(x/\sqrt{t})\quad\text{with}\quad e(x)=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{x}\mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi$$

is an exact solution, and, moreover, that for initial conditions ψ_0

$$\psi(t,x)=\frac{1}{\sqrt{4\pi t}}\int \mathrm{e}^{-(x-y)^2/(4t)}\psi_0(y)\,\mathrm{d}y\to 1+ze(x/\sqrt{t})\text{ as }t\to\infty,$$

with rate $\mathcal{O}(t^{-1})$. It follows that

$$\eta^{(z)}(t,x)=t^{-1/2}f_z(x/\sqrt{t})\quad\text{with}\quad f_z(y)=\frac{\sqrt{\alpha}}{\beta}\frac{ze'(y/\sqrt{\alpha})}{1+ze(y/\sqrt{\alpha})} \tag{1.11}$$

is a self–similar solution of Burgers equation. This is illustrated in fig.1, taking into account that for $\int u(1,x)\,\mathrm{d}x > 0$ we have $-1 < z < 0$. Moreover, for localized initial conditions η_0 it follows that the so called renormalized solution satisfies

$$\lim_{t\to\infty} t^{1/2}\eta(t,t^{1/2}x) = f_z(x), \quad \text{with rate} \quad \mathcal{O}(t^{-1/2}),$$

i.e., it converges towards a non-Gaussian limit. This is not true for spatially non-localized initial conditions since Burgers equation has front solutions $\eta(t,x) = h(x-ct)$ with $|h(\xi)| \not\to 0$ as $|\xi| \to \infty$.

The calculations so far have been formal, i.e., we ignored all terms that are formally of higher order in δ, hence corresponding to higher power nonlinearities or higher order derivatives. However, in [BKL94] is has been shown that the self–similar decay in Burgers equation is stable under perturbation by terms which (in the language of renormalization theory) are "asymptotically irrelevant" (see section 2.1).

Here we follow a similar approach. We take initial data for (1.4) in the space Y of functions $u(x) = (\eta(x), q(x))$ with $\hat{u} \in C^1(\mathbb{R}, \mathbb{C}^2)$, $\hat{u}(k) = \mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}kx} u(x)\,\mathrm{d}x$, with norm

$$\|u\|_Y = \sup_{k\in\mathbb{R}} \Big((1+k^5)(|\hat{\eta}(k)| + |\partial_k\hat{\eta}(k)|) + (1+k^4)(|\hat{q}(k)| + |\partial_k\hat{q}(k)|) \Big). \tag{1.12}$$

The different weights (corresponding to smoothness in x–space) of the components of $\hat{u}$ take care of the different orders of differentiation. The term $|\partial_k\hat{u}|$ gives decay in x space. Note that convergence in $\|\cdot\|_Y$ implies convergence in $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$ due to $\|u\|_\infty \le \|\hat{u}\|_1 \le C\|u\|_Y$ and $\|u\|_1 \le \|(1+|x|)u\|_2 \le \|u\|_Y$. For convenience we take the initial data at $t = 1$. Our result is as follows:

Theorem 1.1 *Fix some small $\delta > 0$. There exists $\varepsilon, C > 0$ such that the following holds. If $\|u_0\|_Y \le \varepsilon$ then there exists a unique solution $u \in C([1,\infty), Y)$ of (1.4) with $u|_{t=0} = u_0$. Moreover,*

$$\left\| u(t,\cdot) - t^{-1/2} f_z(t^{-1/2}\cdot) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|_\infty \le Ct^{-1+\delta} \tag{1.13}$$

with f_z as in (1.11) and z defined by $\ln(z+1) = \frac{\beta}{\alpha}\int \eta(1,x)\,\mathrm{d}x$.

Remark 1.2 The vector $(1,2)$ in (1.13) is the eigenvector of $\hat{A}(k)$ to the eigenvalue $\lambda_1 = 0$ at $k = 0$. The fact that z in (1.13) can be explicitly given is due to the special structure of (1.1) that $\partial_t\eta = \partial_x(2\eta - q)$ is a total derivative, which accounts for the conservation of mass in the underlying inclined film problem.

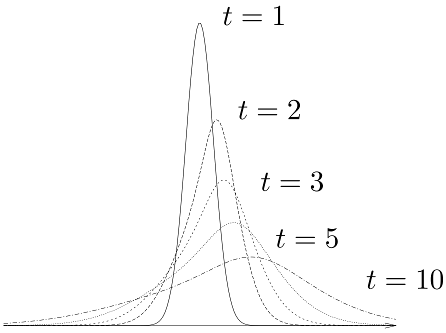

Figure 1: Sketch of self–similar decay in Burgers equation

The paper continues work where the renormalization approach by Bricmont and Kupiainen for the proof of diffusive behavior in nonlinear diffusion equations [BKL94] has been transfered to more complicated systems, as the Ginzburg–Landau equation [BK92, CEE92, BK94, GM98] or pattern forming systems [Sch96, Sch98, Uec99, ES00, GSU03, SU03]. In contrast to these latter works our system is quasilinear and the renormalized solution has a non Gaussian limit.

In section 2 we explain the idea of renormalization, consider (1.4) in Fourier space and provide the functional analytic frame. In section 3 we use renormalization theory to show that the higher order terms ignored so far are asymptotically irrelevant and thus prove Theorem 1.1.

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2 Preliminaries

2.1 The idea of renormalization

For convenience, we briefly repeat the ideas from [BK92, BKL94]. Consider

$$u_t = u_{xx} + f(u, u_x, u_{xx}), \quad u(t,x) \in \mathbb{R}, \quad u(1,x) = u_0(x), \tag{2.1}$$

with $f(a,b,c) = a^{d_1} b^{d_2} c^{d_3}$ a monomial. For $L > 0$ define the rescaling operators

$$\mathcal{R}_L u(x) = u(Lx),$$

and for $L > 1$ sufficiently large to be chosen below and $n \in \mathbb{N}$ let $u_n(\tau) = L^n \mathcal{R}_{L^n} u(L^{2n}\tau)$, i.e., $u_n(\tau, y) = L^n u(L^{2n}\tau, L^n y)$. Then

$$\partial_\tau u_n = \partial_y^2 u_n + f_n(u_n, \partial_y u_n, \partial_y^2 u_n), \tag{2.2}$$

$$f_n(a,b,c) = L^{nd_f} a^{d_1} b^{d_2} c^{d_3}, \quad d_f = 3 - d_1 - 2d_2 - 3d_3. \tag{2.3}$$

and solving (2.1) on $t \in [1,\infty)$ is equivalent to iterating the renormalization process

$$\text{solve (2.2) on } \tau \in [L^{-2},1] \text{ with initial data } u_n(L^{-2}) = L\mathcal{R}u_{n-1}(1). \tag{2.4}$$

If $d_f < 0$ then the factor L^{nd_f} in (2.2) goes to 0 as $n \to \infty$ and in the limit we obtain $\partial_\tau u_n = \partial_y^2 u_n$ with $u_n(L^{-2}) = L\mathcal{R}_L u_{n-1}(1)$. This problem has the line of (Gaussian) fixpoints $z\mathrm{e}^{-y^2/4}$, $z \in \mathbb{R}$, which, moreover, is attractive in suitable spaces. For instance, similar to (1.12) let

$$\|u\|_{\tilde{X}} = \sup_{k\in\mathbb{R}} \Big((1+k^4)(|\hat{u}(k)| + |\partial_k \hat{u}(k)| \Big). \tag{2.5}$$

The weight in k yields smoothness in x and the derivatives in k are used to show contraction properties of $\mathrm{e}^{(1-L^{-2})\partial_y^2} L\mathcal{R}_L f$ when acting on functions with $\hat{f}(0) = 0$. One more particular feature of the norm (2.5) is that it allows to use directly the variation of constant formula to solve the quasilinear or fully nonlinear problems (2.2).

Hence the basic idea is that by a power–counting argument one can easily identify nonlinearities f that are "asymptotically irrelevant" ($d_f < 0$). Note that by (2.2) derivatives in the nonlinearity give higher powers of L^{-n}. Burgers case $f = u\partial_x u$ with $d_f = 0$ is called marginal and yields the non–Gaussian fixed point (1.11), while a nonlinearity with $d_f > 0$ would be called relevant. Relevant nonlinearities and also the marginal case $f = u^3$ may lead to finite–time blow up of the solution, see, e.g., [Wei81]. The advantage of the discrete renormalization is that the large time behaviour of (2.1) is split into the sequence (2.4) of finite time problems and that it uses only few special features of the equation. Hence it can be applied to a variety of problems; see the references in the introduction. A related method is the continuous rescaling to similarity coordinates used in [Way97].

Below we show that $B(u,u)$ in (1.4) is marginal while $H(u)$ is irrelevant. This is just another way of expressing that only $B(u,u)$ contributes to the long–wave expansion (1.8)–(1.10). However, by simple power counting we obtain $d_B = 1$ and $d_H = 0$. To show and exploit that $B(u,u)$ has a "derivative–like" structure (and hence $d_B = 0$) and that $H(u)$ is in fact irrelevant ($d_H = -1$) we shall consider (1.4) in Fourier space and apply so called mode filters to extract the relevant terms.

2.2 The IBLe in Fourier space

Let

$$\hat{u}_t = \hat{A}\hat{u} + \hat{B}(\hat{u},\hat{u}) + \hat{H}(\hat{u}) \tag{2.6}$$

be the Fourier transform of (1.4). Here

$$\hat{A} = \begin{pmatrix} 2\mathrm{i}k & -\mathrm{i}k \\ \frac{6}{\mathrm{R}} + (\frac{4}{5} - \frac{2}{\mathrm{R}}\cot\theta)\mathrm{i}k + \frac{3}{\mathrm{R}}k^2 - \mathrm{Wi}k^3 & -\frac{3}{\mathrm{R}} + \frac{2}{5}\mathrm{i}k - \frac{7}{2\mathrm{R}}k^2 \end{pmatrix}, \tag{2.7}$$

and the eigenvalues of $\hat{A}$ are

$$\begin{aligned}\lambda_{1,2}(k) = \ & -\tfrac{1}{2}\big(\tfrac{7}{2\mathrm{R}}k^2+\tfrac{8}{5}\mathrm{i}k+\tfrac{3}{\mathrm{R}}\big)\\ & \pm\sqrt{\tfrac{1}{4}\big(\tfrac{7}{2\mathrm{R}}k^2+\tfrac{8}{5}\mathrm{i}k+\tfrac{3}{\mathrm{R}}\big)^2-\tfrac{6}{\mathrm{R}}\mathrm{i}k+\big(\tfrac{4}{5}-\tfrac{2}{\mathrm{R}}\cot\theta\big)k^2-\tfrac{3}{\mathrm{R}}\mathrm{i}k^3-\mathrm{W}k^4}.\end{aligned} \tag{2.8}$$

From λ_1 we recover the instability criterion (1.2): for $\mathrm{R} > \mathrm{R}_c$ we have a long wave instability with maximum growth rate $\mathrm{Re}\lambda_1(k_c) = \mathcal{O}(\mathrm{W}^{-1})$, $k_c = \mathcal{O}(\mathrm{W}^{-1/2})$; note that W is typically very large [Uec03]. For $\mathrm{R} < \mathrm{R}_c$ we have $\lambda_1(k) = -\alpha k^2 + \mathcal{O}(k^3)$ with $\alpha = (\frac{2}{3}\cot\theta - \frac{8\mathrm{R}}{15}) > 0$. In any case, for $|k| \to \infty$ we have

$$\begin{aligned}\lambda_{1,2}(k) &= \left(-\frac{7}{4\mathrm{R}} \pm \sqrt{\left(\frac{7}{4\mathrm{R}}\right)^2 - \mathrm{W}}\right) k^2 + \mathcal{O}(|k|^{3/2})\\ &= \left(-\frac{7}{4\mathrm{R}} \pm \mathrm{i}(\sqrt{\mathrm{W}}+\mathcal{O}(\mathrm{W}^{-1}))\right) k^2+\mathcal{O}(|k|^{3/2})\\ &= -\alpha_2 k^2 \pm \beta_2 \mathrm{i}k^2+\mathcal{O}(|k|^{3/2}), \quad \text{with} \quad \alpha_2 = \frac{7}{4\mathrm{R}},\end{aligned} \tag{2.9}$$

where in the second equality of (2.9) we assumed for simplicity that $(\frac{7}{4\mathrm{R}})^2 < \mathrm{W}$. This shows the parabolic damping of the high wavenumber modes. In [Uec03] this has been used to construct an analytic semigroup e^{tA} in the phase space $H^3(\mathbb{R}) \times H^2(\mathbb{R})$ and to show local existence for the quasilinear problem (1.4) using maximal regularity methods.

Here we shall use a more direct approach in Fourier space. We have

$$\hat{A}(k) = M(k)\Lambda(k)M^{-1}(k) \quad \text{where} \quad \Lambda(k) = \mathrm{diag}(\lambda_1(k), \lambda_2(k)),$$

and where $M(k) = (\phi^1(k), \phi^2(k))$ contains the eigenvectors of $\hat{A}(k)$. It follows from (2.7), (2.8) that

$$M(k) = \begin{pmatrix} \mathcal{O}(\frac{1}{1+|k|}) & \mathcal{O}(\frac{1}{1+|k|}) \\ 1 & 1 \end{pmatrix} \text{ with } |\mathrm{det} M(k)| \geq C/(1+|k|) \text{ as } |k| \to \infty,$$

hence

$$M(k)^{-1} = \begin{pmatrix} \mathcal{O}(1+|k|) & 1 \\ \mathcal{O}(1+|k|) & 1 \end{pmatrix} \text{ as } |k| \to \infty. \tag{2.10}$$

The reason for the different weights in $\|\cdot\|_Y$ can be seen in estimating

$$\begin{aligned}\|\mathrm{e}^{(t-1)\hat{A}}\hat{f}\|_Y &= \|M\mathrm{e}^{(t-1)\Lambda}M^{-1}\hat{f}\|_X\\ &\leq C \left\| \mathrm{e}^{-\alpha_2(t-1)k^2} \begin{pmatrix} |f_1|+\frac{1}{1+|k|}|f_2| \\ (1+|k|)|f_1|+|f_2| \end{pmatrix} \right\|_Y \leq Ct^{1/2}\|\hat{f}\|_Y,\end{aligned} \tag{2.11}$$

due to $\sup_k |2\alpha_2 k t \mathrm{e}^{-\alpha_2 k^2 t}| \leq Ct^{1/2}$.

2.3 The mode filters

Let $\rho > 0$ be sufficiently small, and let χ be a smooth cutoff function with $\chi(k) = 1$ for $|k| < \rho$, $\chi(k) = 0$ for $2\rho < |k|$ and $\chi(k) \in [0,1]$ elsewhere. Write

$$\phi^1(k) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k \begin{pmatrix} 1 \\ 2-\mathrm{i}\alpha \end{pmatrix} + \mathcal{O}(k^2), \quad \psi^1(k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 2\mathrm{iR}/3-1 \\ \mathrm{iR}/3 \end{pmatrix} + \mathcal{O}(k^2) \quad (2.12)$$

for the eigenvector of $\hat{A}(k)$ to $\lambda_1(k)$ and for the associated eigenvector of $\hat{A}^H(k)$, and let $\langle u, v \rangle = u \cdot \overline{v}$. Then

$$P_c(k)\hat{u}(k) = c(k)\chi_c(k)\langle \hat{u}(k), \psi^1(k) \rangle \phi^1(k)$$

with $c(k) = 1/\langle \phi^1(k), \psi^1(k) \rangle = 1 + \mathcal{O}(k)$ defines the so called central modefilter with $(\hat{A}P_c\hat{u})(k) = (P_c\hat{A}\hat{u})(k) = \lambda_1(k)\hat{u}(k)$ and $\sup_k |P_c(k)| \leq C$. Similarly define the stable modefilter $P_s = \mathrm{Id} - \mathrm{P_c}$ and the auxiliary modefilters

$$P_c^h \hat{u}(k) = c(k)\chi_c(k/2)\langle \hat{u}(k), \psi^1(k) \rangle \phi^1(k),$$
$$P_s^h \hat{u}(k) = \hat{u}(k) - c(k)\chi_c(2k)\langle \hat{u}(k), \psi^1(k) \rangle \phi^1(k).$$

Then $P_c^h P_c = P_c$ and $P_s^h P_s = P_s$ which is used to replace the missing projection properties of P_c, P_s. Let $(\hat{u}_c, \hat{u}_s)$ solve

$$\partial_t \hat{u}_c = \hat{A}\hat{u}_c + \hat{B}_c(\hat{u}, \hat{u}) + \hat{H}_c(\hat{u}), \quad \partial_t \hat{u}_s = \hat{A}\hat{u}_s + \hat{B}_s(\hat{u}, \hat{u}) + \hat{H}_s(\hat{u}) \quad (2.13)$$

where $\hat{u} = \hat{u}_c + \hat{u}_s$, and $\hat{B}_c = P_c\hat{B}, \hat{H}_c = P_c\hat{H}, \hat{B}_s = P_s\hat{B}, \hat{H}_s = P_s\hat{H}$. Then, by construction, $\hat{u}$ solves (2.6).

The idea of this splitting into central modes $\hat{u}_c$ and stable (exponentially damped) modes $\hat{u}_s$ is as follows. By construction, the function

$$w_z(t, k) = \hat{f}_z(t^{1/2}k)\chi(k)\phi^1(k)$$

with f_z from (1.11) fulfills

$$\partial_t w_z = \hat{A}w_z + \hat{B}_c(w_z, w_z) + \mathcal{O}(|k|^2).$$

This holds since $\hat{u}^{(z)}(t, x) = \hat{f}_z(t^{1/2}k)$ fulfills $\partial_t \hat{u}^{(z)} = -\alpha k^2 \hat{u}^{(z)} + \beta \mathrm{i}k(\hat{u}^{(z)} * \hat{u}^{(z)})$, since $\hat{A}w_z = (-\alpha k^2 + \mathcal{O}(k^3))w_z$, and since

$$\begin{aligned} &\hat{B}_c(w_z, w_z)(k) \\ &= c(k)\chi(k)\left\langle \begin{pmatrix} 0 \\ \frac{6}{\mathrm{R}}(\hat{f}_z * (2\hat{f}_z - \hat{f}_z)(k)(1+\mathcal{O}(|k|)) \end{pmatrix}, \begin{pmatrix} 1+\mathcal{O}(|k|) \\ -\mathrm{i}k\mathrm{R}/3 \end{pmatrix} + \mathcal{O}(k^2) \right\rangle \phi^1(k) \\ &= (-2\mathrm{i}k + \mathcal{O}(k^2))(\hat{f}_z * \hat{f}_z)\chi(k)\phi^1(k). \end{aligned} \quad (2.14)$$

This shows the "derivative–like" structure of B_c. Then splitting $\hat{u}_c(t,k) = w_z(t,k) + \hat{v}(t,k)$ with $\hat{v}|_{(t,k)=(1,0)} = 0$ we will obtain $\hat{v}(t) \to 0$. On the other hand, there exists a $\gamma > 0$ such that

$$\mathrm{Re}\lambda^s_{1,2}(k) < -\gamma \tag{2.15}$$

for all $k \in \mathbb{R}$ for the eigenvalues of $\lambda^s_{1,2}$ of $\mathcal{L}_s$. Hence $\hat{u}_s$ is linearly exponentially damped. Also note that reasoning as in (2.14) the whole nonlinearity $\hat{B}^c + \hat{H}^c$ locally at $k = 0$ has the form of a derivative, which is why z with $\ln(1+z) = \frac{\beta}{\alpha}\hat{\eta}(1,0)$ in Theorem 1.1 can be explicitly given. In a nutshell, these are the reason why $u^{(z)}(t,x)\begin{pmatrix}1\\2\end{pmatrix}$ emerges as the asymptotic solution of (1.4). These arguments will now be made rigorous.

3 The renormalization process

3.1 The rescaled systems

To set up a renormalization process for (2.13) similar to (2.4) note that $\mathcal{F}(L\mathcal{R}_L u) = \mathcal{R}_{L^{-1}}\hat{u}$. Hence, for $L > 1$ sufficiently large, to be chosen later, let

$$u_{c,n}(\tau,\ell) = \hat{u}_c(L^{2n}\tau, \ell/L^n), \quad u_{s,n}(\tau,\ell) = \hat{u}_s(L^{2n}\tau, \ell/L^n). \tag{3.1}$$

These are rescaled variables in Fourier space, but we omit the ˆ since the rest of the analysis will be almost entirely in Fourier space. Then $(u_{c,n}, u_{s,n})$ fulfill

$$\begin{aligned}\partial_\tau u_{c,n}(\tau,\ell) &= \mathcal{L}_n P^h_{c,n} u_{c,n} + B_{c,n}(u_n,u_n) + H_{c,n}(u_n),\\ \partial_\tau u_{s,n}(\tau,\ell) &= \mathcal{L}_n P^h_{s,n} u_{s,n} + B_{s,n}(u_n,u_n) + H_{s,n}(u_n),\end{aligned} \tag{3.2}$$

where $u_n = u_{c,n} + u_{s,n}$ and, with $\star \in \{c,s\}$,

$$\begin{aligned}&\mathcal{L}_n = L^{2n}\mathcal{R}_{L^{-n}}\hat{A}\mathcal{R}_{L^n}, \quad P^h_{\star,n} = \mathcal{R}_{L^{-n}} P^h_* \mathcal{R}_{L^n},\\ &B_{\star,n}(u_n,u_n) = L^{2n}\mathcal{R}_{L^{-n}}\hat{B}_*(\mathcal{R}_{L^n}u_n, \mathcal{R}_{L^n}u_n),\\ &H_{\star,n}(u_n) = L^{2n}\mathcal{R}_{L^{-n}}\hat{H}_\star(\mathcal{R}_{L^n}u_n).\end{aligned} \tag{3.3}$$

As before, the idea is that solving (2.13), or equivalently (1.4), on $t \in [1,\infty)$ is equivalent to iterating the renormalization process

$$\text{solve (3.2) on } \tau \in [L^{-2}, 1] \text{ with initial data } \begin{pmatrix}u_{c,n}(L^{-2})\\ u_{s,n}(L^{-2})\end{pmatrix} = \mathcal{R}_{L^{-1}}\begin{pmatrix}u_{c,n-1}(1)\\ u_{s,n-1}(1)\end{pmatrix}. \tag{3.4}$$

We solve (3.2) in $Y_n \times Y_n$ with

$$\|u\|_{Y_n} = \sup_{\ell\in\mathbb{R}}(1+\ell^4)\Big((1+|\ell/L^n|)(|\hat{\eta}(\ell)| + |\partial_\ell\hat{\eta}(\ell)|) + |\hat{q}(\ell)| + |\partial_\ell\hat{q}(\ell)|\Big). \tag{3.5}$$

Hence, though $\|\cdot\|_{Y_n}$ is still equivalent to $\|\cdot\|_Y$ we loose a factor L^{-n} in the control of the highest derivative of η. But this is no problem since a derivative ∂_x^j yields a factor L^{-n}, cf. sec.2.1. On the other hand, the norm $\|\cdot\|_{Y_n}$ is convenient in solving (3.2).

Henceforth, many constants which are independent of L are denoted by C. We set

$$R_n = \sup_{\tau\in[L^{-2},1]} (\|u_{c,n}(\tau)\|_{Y_n} + \|u_{s,n}(\tau)\|_{Y_n})$$

and start with the following crucial lemma.

Lemma 3.1 *There exist $L_0 > 1$ and $C > 0$ such that for all $L > L_0$, all $\tau \in [L^{-2}, 1]$ and all $\hat{f} \in Y_n$ the following holds,*

$$\|\mathrm{e}^{(\tau-L^{-2})\mathcal{L}_n} P^h_{c,n}\hat{f}\|_{Y_n} \le C\|\hat{f}\|_{Y_n}, \tag{3.6}$$

$$\|\mathrm{e}^{(\tau-L^{-2})\mathcal{L}_n} P^h_{s,n}\hat{f}\|_{Y_n} \le C\mathrm{e}^{-\gamma L^{2n}(\tau-L^{-2})}\|\hat{f}\|_{Y_n}, \tag{3.7}$$

with $\gamma > 0$ from (2.15). Moreover, let $\delta > 0$ be sufficiently small and $R_n \le \delta$. Then

$$\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} B_{c,n}(u_n(s),u_n(s))\,\mathrm{d}s\right\|_{Y_n} \le CR_n^2, \tag{3.8}$$

$$\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} H_{c,n}(u_n(s))\,\mathrm{d}s\right\|_{Y_n} \le CL^{-n}R_n^2, \tag{3.9}$$

$$\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} B_{s,n}(u_n(s),u_n(s))\,\mathrm{d}s\right\|_{Y_n} \le CL^{-n}R_n^2, \tag{3.10}$$

$$\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} H_{s,n}(u_n(s))\,\mathrm{d}s\right\|_{Y_n} \le CL^{-n}R_n^2, \tag{3.11}$$

Proof. Similar to (2.11), (3.6) follows from $\mathcal{L}_n(\ell) = L^{2n}M_n(\ell)\Lambda_n(\ell)M_n(\ell)^{-1}$ with $M_n(\ell) = M(\ell/L^n)$ and $\Lambda_n(\ell) = \Lambda(\ell/L^n) = \mathrm{diag}(\lambda_1(\ell/L^n), \lambda_2(\ell/L^n))$. Note that

$$M_n(\ell) = \begin{pmatrix} \frac{1}{1+|\ell/L^n|} & \frac{1}{1+|\ell/L^n|} \\ 1 & 1 \end{pmatrix} \quad \text{as} \quad |\ell| \to \infty$$

which explains why we use the norm $\|\cdot\|_{Y_n}$. From $|\mathrm{e}^{t\hat{A}(k)}P^h_s(k)\hat{f}(k)| \le C\mathrm{e}^{(-\gamma-\alpha k^2)t}|\hat{f}(k)|$ we obtain (3.7).

For the nonlinear terms, first note that $f(u,v) = (\partial_x^{d_1}u)(\partial_x^{d_2}v)$ in Fourier space becomes $\hat{f}(\hat{u},\hat{v}) = ((\mathrm{i}k)^{d_1}\hat{u}) * ((\mathrm{i}k)^{d_2}\hat{v})$, and by rescaling

$$\mathcal{R}_{L^{-n}}((\mathrm{i}\ell)^{d_1}\mathcal{R}_{L^n}u_n) * ((\mathrm{i}\ell)^{d_2}\mathcal{R}_{L^n}v_n))(\ell) = L^{-nd_f}(((\mathrm{i}\ell)^{d_1}u_n) * ((\mathrm{i}\ell)^{d_2}v_n))(\ell), \tag{3.12}$$

with $d_f = (1+d_1+d_2)$, cf. (2.3). Clearly $d_B = 1$ and (3.8) cannot be established by such power counting. Hence we need to use the definition of B_c^n, i.e.,

$$
\begin{aligned}
&\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} B_{c,n}(u_n(s),u_n(s))\,\mathrm{d}s\right\|_{Y_n} \\
&\quad = \frac{6}{\mathrm{R}} L^{2n} \left\| M_n \int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)L^{2n}\Lambda_n} \left\langle \begin{pmatrix} 0 \\ L^{-n} u_{n,1} * (u_{n,2}-u_{n,1}) \end{pmatrix}, \begin{pmatrix} 1+\mathcal{O}(\ell/L^n) \\ \mathcal{O}(\ell/L^n) \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}s \right\|_{Y_n} \\
&\quad \leq C\|u_n\|_{Y_n}^2 \left\| \frac{1}{1+|\ell/L^n|}\, \ell \left(\frac{1}{1+\ell^4}\right)^{*2} \begin{pmatrix} \int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)L^{2n}\lambda_1(\ell/L^n)}\,\mathrm{d}s \\ 0 \end{pmatrix} \right\|_{Y_n} \\
&\quad \leq C\|u_n\|_{Y_n}^2 \sup (1+\ell^4)(1+\ell^2)^{-2} \int_{L^{-2}}^{\tau} |\mathrm{e}^{sL^{2n}\lambda_1(\ell/L^n)}| + \left|\partial_k \mathrm{e}^{sL^{2n}\lambda_1(\ell/L^n)}\right| \mathrm{d}s \\
&\quad \leq C\|u_n\|_{Y_n}^2
\end{aligned}
$$

where $u_{n,j}$, $j=1,2$, denotes the components of u_n.

Similar estimates yield (3.9). First note that $\|\mathcal{F}^{-1}(u_n)\|_{L^\infty} \leq C\|u\|_{Y_n}$ such that for δ sufficiently small the fractions in $H(u)$ can be expanded in power series. Then for instance the cubic terms without derivatives in $H(u)$ coming from $\frac{3}{\mathrm{R}}\frac{q}{h^2}$ in (1.1) work just as above. Next consider a typical high order term in $H(u)$, for instance $G(\hat{u}) = (0, \mathrm{W}\hat{\eta} * (-\mathrm{i}k^3\hat{\eta}))$. This yields

$$
\begin{aligned}
&\left\|\int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} \mathcal{R}_{L^{-n}} E_c L^{2n} G(\mathcal{R}_{L^n} u_n(s))\,\mathrm{d}s\right\|_{Y_n} \\
&\quad = \mathrm{W} L^{-n} \left\| \int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} \left\langle \begin{pmatrix} 0 \\ u_{n,1} * (-\mathrm{i}\ell^3 u_{n,1}) \end{pmatrix}, \psi^1(\ell/L^n) \right\rangle \mathrm{d}s \right\|_{Y_n} \\
&\quad \leq C L^{-n} \|u_n\|_{Y_n}^2 \left\| \frac{1}{1+|\ell/L^n|} \left(\frac{1}{(1+|\ell|)(1+|\ell/L^n|)} * \frac{1}{1+\ell^4} \right) \begin{pmatrix} \int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)L^{2n}\lambda_1(\ell/L^n)}\,\mathrm{d}s \\ 0 \end{pmatrix} \right\|_{Y_n} \\
&\quad \leq C\|u_n\|_{Y_n}^2 .
\end{aligned}
$$

Here we used $\sup_{|k|<2\rho} |\psi^1(k)| \leq C$, hence the needed factors of L^{-n} come from the derivatives in the nonlinearity itself.

The estimates (3.10),(3.11) for the nonlinearity in the stable part are obtained as follows. First, let $G(u) = (0, g(u))$ with $g(u) = u_i u_j$ be quadratic without derivatives. Then

$$
\begin{aligned}
\|N_g^n(u_n)\|_{Y_n} &:= \left\| \int_{L^{-2}}^{\tau} \mathrm{e}^{(\tau-s)\mathcal{L}_n} \mathcal{R}_{L^{-n}} E_s L^{2n} G(\mathcal{R}_{L^n} u_n(s))\,\mathrm{d}s \right\|_{Y_n} \\
&\leq C L^n \|u_n\|_{Y_n}^2 \left\| \int_{L^{-2}}^{\tau} \mathrm{e}^{-(L^{2n}\gamma + \alpha_2 l^2)(\tau-s)} \begin{pmatrix} \frac{1}{1+|\ell/L^n|}\left(\frac{1}{1+\ell^4} * \frac{1}{1+\ell^4}\right) \\ \left(\frac{1}{1+\ell^4} * \frac{1}{1+\ell^4}\right) \end{pmatrix} \mathrm{d}s \right\|_{Y_n} \\
&\leq C L^n \|u_n\|_{Y_n}^2 \sup_l \left((1+\ell^4)(1+\ell^4)^{-1} \frac{1}{L^{2n}\gamma + \alpha_2 \ell^2} \right) \leq C L^{-n} \|u_n\|_{Y_n}^2 .
\end{aligned}
$$

Here we don't use the smoothing properties of $e^{\tau\mathcal{L}_n}$. Derivatives in $g(u)$, i.e., $g(u) = u_i\partial_x^m u_j$, yield a factor $|\ell|^m L^{-mn}$ where the $|\ell|^m$ must be compensated by smoothing by $e^{\tau\mathcal{L}_n}$. For $m = 1$ this yields

$$\|N_g^n(u_n)\|_{Y_n} \leq R_n^2 \sup_\ell \left((1+\ell^4)(1+|\ell|^3)^{-1} \frac{1}{L^{2n}\gamma + \alpha_2\ell^2} \right) \leq CL^{-n}R_n^2$$

since $\sup_\ell L^n(1+|\ell|)/(L^{2n} + \alpha_2\ell^2) \leq C$. For $m > 1$ we obtain enough L^{-n} from the derivatives in the nonlinearity itself and $|\ell|^m$ must (and can) be controlled using smoothing by $e^{\tau\mathcal{L}_n}$. □

Now let

$$\rho_{n,c} = \|u_{c,n}\|_{Y_n}, \quad \rho_{n,c} = \|u_{c,n}\|_{Y_n}, \quad \rho_n = \rho_{n,c} + \rho_{n,s},$$

and note that

$$\|\mathcal{R}_{L^{-n}} u_{n-1}\|_{Y_n} \leq CL^4 \|u_{n-1}\|_{Y_{n-1}}.$$

Combining this with Lemma 3.1 yields the local existence and estimates for (3.2).

Lemma 3.2 *There exist $L_0 > 1$ and $C_1, C_2 > 0$ such that for all $L > L_0$ the following holds. If $\rho_{n-1} \leq C_1 L^{-4}$ then there exists a unique solution $u_n \in C([L^{-2}, 1], Y_n)$ of (3.2) with $u_n(1/L^2, \ell) = u_{n-1}(1, \ell/L)$, $R_n \leq \delta$ with δ from Lemma 3.1, and*

$$R_n \leq C_2(L^4\rho_{n-1} + R_n^2). \tag{3.13}$$

Proof. This follows by a standard application of the contraction mapping theorem. □

3.2 Splitting, iteration, and conclusion

Due to the loss of L^4 in (3.13) we need better control of ρ_n to iterate (3.4). Therefore we split

$$u_{c,n}(\tau, \ell) = w_{z,n}(\tau, \ell) + v_n(\tau, \ell),$$

where

$$w_{z,n}(\tau, \ell) = \hat{u}^{(z)}(\tau, \ell)\chi(\ell/L^n)\phi^1(\ell/L^n), \quad \hat{u}^{(z)}(\tau, \ell) = \hat{f}_z(\tau^{1/2}\ell),$$

with z defined by

$$\ln(z+1) = \frac{\beta}{\alpha} \int \eta(1, x)\, dx = \frac{\beta}{\alpha} \hat{\eta}(1, 0).$$

Then

$$\partial_\tau v_n = \mathcal{L}_n v_n + B_{c,n}(u_n, u_n) - B_{c,n}(w_{z,n}, w_{z,n}) + H_c^n(u_n) + \mathrm{Res}_n$$

where

$$\mathrm{Res}_n = -\partial_\tau w_{z,n} + \mathcal{L}_n w_{z,n} + B_{c,n}(w_{z,n}, w_{z,n}).$$

Lemma 3.3 *Let $|z|<1$. There exists a $C>0$ such that* $\sup_{\tau\in[L^{-2},1]}\|\mathrm{Res}_n\|_{Y_n}\le CL^{-n}|z|$.

Proof. By construction, $\mathcal{L}_n w_{z,n} = L^{2n}\lambda_1(\ell/L^n)w_{z,n} = (-\alpha\ell^2+\mathcal{O}(l^3/L^n))w_{z,n}$ as $|\ell|\to 0$. Moreover,

$$\begin{aligned} &B_{c,n}(w_{z,n},w_{z,n})(\ell)\\ &\quad = L^n\left\langle\begin{pmatrix}0\\ \frac{6}{\mathrm{R}}(u_z*u_z+\mathcal{O}(|\ell/L^n|)u_z*u_z+(\mathcal{O}(|\ell/L^n|)u_z)^{*2})\end{pmatrix},\psi^1(\ell/L^n)\right\rangle\phi^1(\ell/L^n)\\ &\quad = (\mathrm{i}\beta\ell u_z*u_z)(1+\mathcal{O}(|\ell/L^n|))\phi^1(\ell/L^n).\end{aligned}$$

Combining with $\partial_\tau\hat u^{(z)} = -\alpha\ell^2\hat u^{(z)}+\mathrm{i}\beta\ell(\hat u^{(z)}*\hat u^{(z)})$ yields

$$\mathrm{Res}_n(\ell) = CL^{-n}(\mathcal{O}(\ell^3)w_{z,n}+\mathcal{O}(\ell^2)(\hat u^{(z)}*u^{(z)})\phi^1(\ell/L^n)),$$

which can be estimated in Y_n by $CL^{-n}|z|$ since $u^{(z)}$ is an analytic and exponentially decaying function. □

To proceed we write

$$u_{c,n}(1,\ell) = w_{z,n}(1,\ell)+g_{n,c}(\ell),\quad u_{s,n}(1,\ell) = g_{n,s}(\ell).$$

By construction we have $v_0(1,0)=0$, and $B_{c,n}(u_n)$, $H_{c,n}(u_n)$ and Res_n locally at $\ell=0$ have the form of a total derivative, i.e., $\partial_\tau v_n(\tau,0)=0$, hence

$$v_n(\tau,0)=0\quad\forall\tau\in[L^{-2},1],\quad\text{hence}\quad g_{n,c}(0)=0\quad\forall n\in\mathbb{N}.$$

Remark 3.4 This is the reason why z in Theorem 1.1 can be explicitly given in terms of the initial conditions. However, even if $H_{c,n}$ were no derivative (but asymptotically irrelevant) a result similar to Theorem 1.1 can be shown, with z then given by some constant with complicated dependence on the initial data. To do so, we would define $u_{c,n}(1,\ell) = w_{z_n,n}(1,\ell)+v_n(1,\ell)$ with z_n defined in such a way that $v_n(1,0)=0$ and show that the sequence z_n converges; see [BKL94]. This is not necessary here.

The penultimate estimate are the contraction properties of $\mathrm{e}^{(1-L^{-2})\mathcal{L}_n}P^h_{c,n}\mathcal{R}_{L^{-1}}$ when acting on functions g with $g(0)=0$, i.e.

$$\begin{aligned}\left\|\mathrm{e}^{(1-L^{-2})\mathcal{L}_n}P^h_c g(\cdot/L)\right\|_{Y_n} &\le C\left\|M\mathrm{e}^{(1-L^{-2})L^{2n}\lambda_1(l/L^n)}|\ell/L|(1+\ell^4)^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right\|_{Y_n}\|g\|_{Y_{n-1}}\\ &\le CL^{-1}\|g\|_{Y_{n-1}}\end{aligned}$$

where we used $g_j(l/L) = (l/L)\partial_k g_j(\tilde l)$ for some $\tilde l\in[0,l]$. Combining this with Lemma 3.1 we obtain, for L sufficiently large,

$$\rho_n\le CL^{-1}\rho_{n-1}+C\big(|z|L^4\rho_{n-1}+(L^4\rho_{n-1})^2+L^{-n}(L^4\rho_{n-1})^2+|z|L^{-n}\big). \tag{3.14}$$

Proof of Theorem 1.1. Let $\rho_0 \leq L^{-m_0} =: \varepsilon$, hence also $|z| \leq CL^{-m_0}$, and let $L \geq L_0$ with L_0 sufficiently large such that $CL^{-1} \leq L^{-(1-\delta)}$. Then (3.14) implies $\rho_n \leq L^{-(m_n - n\delta)}$ with

$$m_n = \min\{m_{n-1}+1, m_0+m_{n-1}-4, 2m_{n-1}-8, m_0+n\}.$$

Choosing, for instance, $m_0 = 9$ yields $m_1 = 10, m_2 = 11, \ldots$, hence $\rho_n \leq L^{-n(1-\delta)}$. Therefore,

$$\|u_n(1) - w_{z,n}\|_{Y_n} = \|\hat{u}(t,\ell/L^n) - \hat{f}_z(\ell)\chi(\ell/L^n)\phi^1(\ell/L^n)\|_{Y_n} \leq L^{-n(1-\delta)}. \qquad (3.15)$$

Using $\|\hat{f}_z(\ell)\big(\phi^1(0) - \chi(\ell/L^n)\phi^1(\ell/L^n)\|_Y \leq CL^{-n}$ and $\|u\|_{L^\infty} \leq C\|u\|_{Y_n}$ this yields (1.13) and hence completes the proof of Theorem 1.1. □

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