Exponential averaging and traveling waves in rapidly varying periodic media

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Abstract

Reaction diffusion systems on cylindrical domains with terms that vary rapidly and periodically in the unbounded direction can be analyzed by averaging techniques. Here, using iterated normal form transformations and Gevrey regularity of bounded solutions, we prove a result on exponential averaging for such systems, i.e., we show that traveling wave solutions can be described by a spatially homogenous equation and exponentially small remainders.

1 Introduction

We consider semilinear reaction diffusion systems on cylindrical domains,

$$\partial_t u = \partial_x^2 u + D\Delta_y u + f(u, \partial_x u, \nabla_y u, y, x/\varepsilon), \tag{1.1}$$

where $u = u(x, y, t) \in \mathbb{R}^p$, $(x, y) \in \mathbb{R} \times \Omega$, $\Omega = [0, L]^d$ with periodic boundary conditions on $\partial\Omega$, and where $D \in \mathbb{R}^{p\times p}$ is a diagonal diffusion matrix with positive entries. The nonlinearity is an entire function, periodic in its last argument, i.e.,

$$f(\cdot, \cdot, \cdot, \cdot, z) = f(\cdot, \cdot, \cdot, \cdot, z + 2\pi),$$

and $0<\varepsilon\ll 1$ is a small parameter. Thus f is rapidly varying in the unbounded direction x.

Systems of the form (1.1) arise for instance in light sensitive chemical reactions with an external space periodic forcing [RGMM$^+$03], in models from physiology [KS98, Kee00], and in a variety of further applications, see [Xin00] and the references therein. These systems are often analyzed by homogenization techniques. Using averaging procedures, the rapid oscillations in x are moved to (formally) higher order terms.

An important question is the existence of traveling or standing waves for (1.1). Here we prove rigorously that traveling wave solutions of (1.1) can be described by an averaged equation with x independent coefficients and an exponentially small remainder. Therefore we consider the spatial dynamics formulation for the traveling wave equation for (1.1) and use iterated normal form transforms on suitable finite dimensional approximations together with Gevrey–type regularity estimates for the remainder. The idea is due to [Nei84] for ODEs and has been transferred to parabolic PDEs with a rapid time periodic forcing in [Mat01]. Elliptic systems in infinite cylinders are considered in [Mat00], while [MW04] treats standing waves for equations of the form (1.1). In these papers the ansatz for the solutions is simpler than in the present paper. From a dynamical systems point of view, the periodic heterogeneity in [Mat00, MW04] can be understood as an interaction of the homogeneous equation with an external one-dimensional oscillator. The homogenization or averaging provides a separation of the traveling wave equation from this fast phase. When considering traveling waves in heterogeneous media, we have to introduce an additional independent variable. From the dynamical systems point of view this introduces a coupling of the main traveling wave equation with an infinite number of rapidly changing phases ($m = 0$ versus $m \neq 0$ in (1.6) below). In the present paper we hence give the first example of the exponential averaging of an infinite dimensional forcing.

We now state our precise result, further remarks and consequences are given below. For notational simplicity henceforth we focus on scalar equations $p = 1$ and let $D = 1$, but see Remark 1.1(i). We are looking for traveling waves, which are periodically modulated with x/ε, i.e.,

$$u(x, y, t) = v(x - ct, y, x/\varepsilon), \tag{1.2}$$

with $v(\cdot, \cdot, z) = v(\cdot, \cdot, z + 2\pi)$ for $z \in \mathbb{R}$. Then $v(\xi, y, z)$ fulfills the traveling wave equation in a periodically varying medium

$$-c\partial_\xi v = (\partial_\xi + \varepsilon^{-1}\partial_z)^2 v + \Delta_y v + f(v, (\partial_\xi + \varepsilon^{-1}\partial_z)v, \partial_y v, y, z). \tag{1.3}$$

As a first order system for

$$V(\xi, y, z) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} v \\ \partial_\xi v \end{pmatrix},$$

this yields

$$\partial_\xi V(\xi) = AV(\xi) + F(V(\xi)), \tag{1.4}$$

where

$$
A = \begin{pmatrix} 0 & 1 \\ -\varepsilon^{-2}\partial_z^2 - \Delta_y & -2\varepsilon^{-1}\partial_z - c \end{pmatrix},
$$

$$
F(V)(y,z) = \begin{pmatrix} 0 \\ -f(v_1, \varepsilon^{-1}\partial_z v_1 + v_2, \nabla_y v_1, y, z) \end{pmatrix}.
$$

For the spatial dynamics formulation (1.4) we choose the phase space

$$
\mathcal{X} = H^s(\Omega_{\mathrm{e}}) \times H^{s-1}(\Omega_{\mathrm{e}}),
$$

$d/2 + 1 < s \in \mathbb{N}$, on the extended cross–section $\Omega_{\mathrm{e}} = \Omega \times S^1$, with the weighted norm

$$
\|V\|_{\mathcal{X}}^2 := \sum_{\alpha+|\beta|\le s} \int |(\varepsilon^{-1}\partial_z)^\alpha \nabla_y^\beta v_1|^2 \,\mathrm{d}z\,\mathrm{d}y + \sum_{\alpha+|\beta|\le s-1} \int |(\varepsilon^{-1}\partial_z)^\alpha \nabla_y^\beta v_2|^2 \,\mathrm{d}z\,\mathrm{d}y, \tag{1.5}
$$

where $\beta \in \mathbb{N}^d$ is a multi index. To separate the rapid changes in the periodic variable z we use the Fourier expansion

$$
V(\xi, y, z) = V_0(\xi, y) + \sum_{m\in\mathbb{Z}\setminus\{0\}} V_m(\xi, y)\mathrm{e}^{\mathrm{i}mz}.
$$

Denoting by Π_m the projection to the Fourier coefficient of $\mathrm{e}^{\mathrm{i}mz}$, we have equations for the Fourier coefficients V_m

$$
\partial_\xi V_m(\xi) = A_m V_m(\xi) + \Pi_m F(V(\xi)), \quad m \in \mathbb{Z}, \tag{1.6}
$$

with

$$
A_m = \begin{pmatrix} 0 & 1 \\ \varepsilon^{-2}m^2 - \Delta_y & -2\varepsilon^{-1}\mathrm{i}m - c \end{pmatrix},
$$

$$
\Pi_m F(V)(y) = \begin{pmatrix} 0 \\ -\frac{1}{2\pi}\int_0^{2\pi} f(v_1, \varepsilon^{-1}\partial_z v_1 + v_2, \nabla_y v_1, y, z)\mathrm{e}^{-\mathrm{i}mz}\,\mathrm{d}z \end{pmatrix}.
$$

The aim of our analysis is the decoupling of the slow V_0 component from the rapidly changing $V_m\mathrm{e}^{\mathrm{i}mz}$, $m \in \mathbb{Z} \setminus \{0\}$. In other words, we want to find a description of V only depending on V_0, up to an exponentially small remainder.

Notation: $H^s(\Omega)$ is the standard Sobolev space of functions with derivatives up to order s in $L^2(\Omega)$. For $V \in BC(\mathbb{R}, \mathcal{X})$, the space of bounded continuous functions with values in $\mathcal{X}$, we write $\|V\|_{BC(\mathbb{R},\mathcal{X})} = \sup_{\xi\in\mathbb{R}} \|V(\xi)\|_{\mathcal{X}}$. We set

$$
\mathbb{Z}^\star := \mathbb{Z}\setminus\{0\} \quad \text{and} \quad X = H^s(\Omega) \times H^{s-1}(\Omega).
$$

Numerical constants that may vary from place to place are denoted by C if they are independent of ε and $m \in \mathbb{Z}$.

Theorem A *Let $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \Omega \times \mathbb{R} \to \mathbb{R}$ be an entire function, periodic with period 2π in the last argument. For all $R > 0$ there exist $\varepsilon_0 > 0$ and $C > 0$, and smooth functions*

$$T_m : X \times (0, \varepsilon_0) \to X, \quad m \in \mathbb{Z}^\star, \qquad \overline{F} : X \times (0, \varepsilon_0) \to X,$$
$$R_T : \mathcal{X} \times S^1 \times (0, \varepsilon_0) \to \mathcal{X}, \qquad R_E : \mathcal{X} \times (0, \varepsilon_0) \to X,$$

such that the following holds. For all bounded solutions $u(x, y, t)$ of equation (1.1) of the form $u = v(x - ct, y, x/\varepsilon)$ with $v(\cdot, \cdot, z) = v(\cdot, \cdot, z + 2\pi)$ and $\|V\|_{BC(\mathbb{R},\mathcal{X})} \le R$ we have

$$V(\xi, y, z) = V_0(\xi, y) + \varepsilon \sum_{m \in \mathbb{Z}^\star} T_m(V_0(\xi), \varepsilon)(y) \mathrm{e}^{\mathrm{i}mz} + R_T(V(\xi), \varepsilon)(y, z), \tag{1.7}$$

where the transformation is bounded, i.e.

$$\sup_{\xi \in \mathbb{R}} \Big\| \sum_{m \in \mathbb{Z}^\star} T_m(V_0(\xi), \varepsilon)(\cdot) \mathrm{e}^{\mathrm{i}m\cdot} \Big\|_{\mathcal{X}} \le C(\|V_0\|_{BC(\mathbb{R},X)})$$

and the remainder term R_T is exponentially small

$$\sup_{\xi \in \mathbb{R}} \|R_T(V(\xi), \varepsilon)(\cdot, \cdot)\|_{\mathcal{X}} \le C(\|V\|_{BC(\mathbb{R},\mathcal{X})}) \exp(-c_1 \varepsilon^{-1/2}).$$

Moreover, V_0 is separated from $V_m, m \in \mathbb{Z}^\star$, up to exponentially small terms, i.e. it fulfills

$$\partial_\xi V_0(\xi, y) = A_0 V_0(\xi, y) + \overline{F}(V_0(\xi), \varepsilon)(y) + R_E(V(\xi), \varepsilon)(y), \tag{1.8}$$

where

$$\|\Pi_0 F(V(\xi)) - \overline{F}(V_0(\xi), \varepsilon)\|_X \le C(\|V\|_{BC(\mathbb{R},\mathcal{X})})$$

and where the remainder depending on $V_m, m \in \mathbb{Z}^\star$ is exponentially small

$$\|R_E(V(\xi), \varepsilon)\|_X \le C(\|V\|_{BC(\mathbb{R},\mathcal{X})}) \exp(-c_2 \varepsilon^{-1/2}).$$

Remark 1.1 (i) Theorem A also holds for systems of reaction diffusion equations. The main difference to the analysis presented here is that the eigenfunction expansions for the A_m and subsequent definitions (see sec.2) become notationally more elaborate.

(ii) The transformed nonlinearity $\overline{F}$ as well as the functions T_m and the remainder terms R_T and R_E are nonlocal in y, z but local in ξ.

(iii) The smoothness assumptions on f can be relaxed. It is enough that the nonlinearity $F(V, \cdot)$ in (1.4) is analytic in V as map from $\mathcal{X}$ to $\mathcal{X}$ and as map from $\mathcal{Y}$ to $\mathcal{Y}$, where the function space $\mathcal{Y}$ consists of functions analytic in y, see (2.7).

(iv) A first approximation of $\overline{F}$ is given in Remark 3.1.

Remark 1.2 In summary, the result of Theorem A is that up to an exponentially small remainder the fast dynamics for $m \neq 0$ in (1.4) are slaved to the slow dynamics of V_0 described by the truncated averaged equation

$$\partial_\xi V_0(\xi, y) = A_0 V_0(\xi, y) + \overline{F}(V_0(\xi))(y). \tag{1.9}$$

Some consequences and applications of Theorem A are as follows. Assume that, for $\varepsilon = 0$ and some $c = c_0$, (1.9) has a heteroclinic orbit V_0^* to hyperbolic equilibria $V_0^- \in X$ and $V_0^+ \in X$ of (1.9), i.e., $V_0^*(\xi) \to V_0^\pm$ for $\xi \to \pm\infty$ (if $V_0^- = V_0^+$ then V_0^* is homoclinic). Moreover, assume that the orbit is transverse in the extended phase space $(c, V_0) \in (\mathbb{R}, X)$. For $\varepsilon > 0$ sufficiently small and some c_ε with $|c_\varepsilon - c_0| \leq C\varepsilon$, this orbit persists as a heteroclinic orbit $V_0^*(\varepsilon)$ with $V_0^*(\xi, \varepsilon) \to V_0^\pm(\varepsilon)$ as $\xi \to \pm\infty$, where $\|V_0^\pm(\varepsilon) - V_0^\pm\|_X \leq C\varepsilon$. Then, for some $\tilde{c}_\varepsilon$ with $|\tilde{c}_\varepsilon - c_\varepsilon| \leq C\exp(-\varepsilon^{-1/2})$, (1.4) has a heteroclinic orbit $\tilde{V}^*(\varepsilon)$ to equilibria $\tilde{V}^\pm(\varepsilon)$, and

$$\sup_{\xi \in \mathbb{R}} \left\| \tilde{V}^*(\xi, \varepsilon)(.,.) - \Big(V_0^*(\xi, \varepsilon)(.) + \varepsilon \sum_{m \in \mathbb{Z}^\star} T_m(V_0^*(\xi), \varepsilon)(\cdot) \mathrm{e}^{\mathrm{i}m\cdot} \Big) \right\|_{\mathcal{X}} \leq C\exp(-\varepsilon^{-1/2}).$$

See also [HMS88, FS96, Gel99, Mat03] for further discussion of related results and associated phenomena like exponentially small splitting of invariant manifolds and Melnikov analysis.

Remark 1.3 The idea of using spatial dynamics to construct special solutions to PDE on unbounded domains has a long history. In [Kir82] and further work (see, e.g.[IM91, AM95, SU03, FS03] and the references therein), small solutions are constructed by some center manifold reduction, while here we analyze general bounded solutions. Again we also refer to [Xin00, MW04] for further approaches to wave propagation (and propagation failure) in periodic media.

To prove Theorem A we use iterated normal form transforms to obtain the exponentially small remainders R_T and R_E. The basic idea is to use Gevrey regularity in y of bounded solutions (1.4), to perform the iterated normal form transforms for each m on finite dimensional subspaces $P^N X$, and to balance the number of normal form transforms with an exponential estimate for the remainders in the part $(\mathrm{Id} - P^N)\mathcal{X}$ obtained from the high regularity in y. In section 2 we introduce the functional analytic setup. The proof of Theorem A is given in section 3.

2 Function spaces and approximation

Here we introduce the functional analytic set-up and collect several estimates. For (1.4) we use the Hilbert space $\mathcal{X}$ as a phase space. For notational convenience henceforth we assume

$L = 2\pi$ and periodic boundary conditions in y. Thus, the eigenvalues of the operators $A_m : D(A_m) \subset X \to X$ in (1.6) with $D(A_m) = H^{s+1}(\Omega) \times H^s(\Omega)$ can be calculated from the ansatz

$$V_m(\xi, y) = \mathrm{e}^{\mathrm{i}k\cdot y + \lambda_{m,k}\xi}\phi_{m,k}.$$

Then

$$(\lambda_{m,k} - A_{m,k})\phi_{m,k} = 0 \quad \text{with} \quad A_{m,k} = \begin{pmatrix} 0 & 1 \\ \varepsilon^{-2}m^2 + |k|^2 & -2\varepsilon^{-1}\mathrm{i}m - c \end{pmatrix},$$

hence

$$\lambda^{\pm}_{m,k} = -\left(\frac{\mathrm{i}m}{\varepsilon} + \frac{c}{2}\right) \pm \sqrt{\frac{\mathrm{i}cm}{\varepsilon} + \frac{c^2}{4} + |k|^2}, \tag{2.1}$$

see fig.1 for a sketch. Since $c \neq 0$ we find that

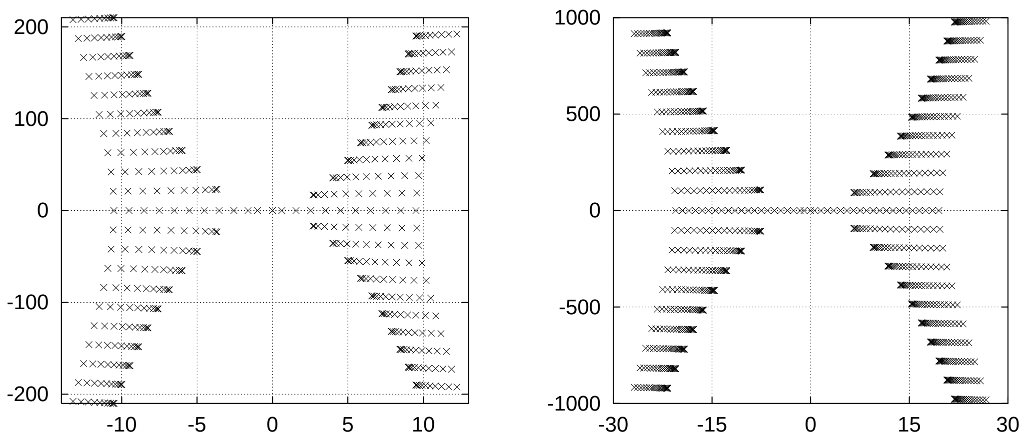

Figure 1: Sketches of the spectrum of A, $c = 1$, $\varepsilon = 0.05$ (left) and $\varepsilon = 0.01$ (right); $\lambda^{\pm}_{m,k}$ for $m = -10, \ldots, 10$, $|k| = 0, \ldots, 10$ (left), $|k| = 0, \ldots, 20$ (right). In particular, the right picture illustrates the increasing spectral gap $|\mathrm{Re}\lambda^{\pm}_{m,k}| \geq C|m/\varepsilon|^{1/2}$, $m \neq 0$, for $\varepsilon \to 0$, used in Lemma 2.5 below.

$$\mathrm{Re}\,\lambda^{\pm}_{m,k} \sim r^{\pm}_{m,k} \quad \text{with} \quad r^{\pm}_{m,k} = \pm\big(|cm/\varepsilon|^{1/2} + |k|\big)\cos\left(\frac{1}{2}\arctan\frac{cm}{\varepsilon|k|^2}\right). \tag{2.2}$$

Here $\sim$ denotes the asymptotics for $|m|, |k| \to \infty$ with ε and c fixed, i.e., $\mathrm{Re}\,\lambda^{\pm}_{m,k}/r^{\pm}_{m,k} \to 1$ as $|m| + |k| \to \infty$. Note that $1/\sqrt{2} \leq \cos(\frac{1}{2}\arctan\frac{cm}{\varepsilon|k|^2}) \leq 1$ in (2.2). More generally we also write $f_{m,k} \sim g_{m,k}$ if there exist $C, c_1, c_2 > 0$ such that

$$c_1 g_{m,k} \leq f_{m,k} \leq c_2 g_{m,k} \quad \text{if} \quad |m| + |k| > C.$$

For complex functions we use the notation $\sim$ in modulus, and similar for the components of vector valued functions. With this notation

$$\phi^{\pm}_{m,k} = \begin{pmatrix} 1 \\ \lambda^{\pm}_{m,k} \end{pmatrix} \sim \begin{pmatrix} 1 \\ |m/\varepsilon| + |k| \end{pmatrix}, \tag{2.3}$$

hence, for

$$V_m(\xi,y)=\sum_{k\in\mathbb{Z}^d} a^{\pm}_{m,k}(\xi)\phi^{\pm}_{m,k}\mathrm{e}^{\mathrm{i}k\cdot z}=\sum_{k\in\mathbb{Z}^d} V_{m,k}\mathrm{e}^{\mathrm{i}k\cdot z}$$

we have

$$V_{m,k}=B_{m,k}\begin{pmatrix}a^+_{m,k}\\ a^-_{m,k}\end{pmatrix},\quad B_{m,k}=\begin{pmatrix}1 & 1\\ \lambda^+_{m,k} & \lambda^-_{m,k}\end{pmatrix},\quad B^{-1}_{m,k}\sim\begin{pmatrix}1 & (|m/\varepsilon|+|k|)^{-1}\\ 1 & (|m/\varepsilon|+|k|)^{-1}\end{pmatrix}. \tag{2.4}$$

Therefore

$$\|V\|^2_{\mathcal{X}}=\sum_{m\in\mathbb{Z},k\in\mathbb{Z}^d}(1+|m/\varepsilon|+|k|)^{2s}(|a^+_{m,k}|^2+|a^-_{m,k}|^2)$$

defines an equivalent norm on $\mathcal{X}$.

For $m\in\mathbb{Z}$ let Π^+_m be the X–orthogonal projection onto $\mathrm{span}\{\phi^+_{m,k}:k\in\mathbb{Z}^d\}$, $\Pi^-_m=\mathrm{Id}-\Pi^+_m$,

$$\Pi^+V=\sum_{m\in\mathbb{Z}}(\Pi^+_mV_m)\mathrm{e}^{\mathrm{i}mz},\quad\text{and}\quad \Pi^-=\mathrm{Id}-\Pi^+.$$

Let $V^m\in X$. Then

$$\mathrm{e}^{\xi A_m}V_m=\sum_{k\in\mathbb{Z}^d}B_{m,k}\,\mathrm{diag}(\mathrm{e}^{\lambda^+_{m,k}\xi},\mathrm{e}^{\lambda^-_{m,k}\xi})\,B^{-1}_{m,k}V_m.$$

Hence, for $\xi<0$ and $V\in P^+\mathcal{X}$,

$$\begin{aligned}\|\mathrm{e}^{\xi A}V\|^2_{\mathcal{X}}\le C&\sum_{m\in\mathbb{Z},\,k\in\mathbb{Z}^d}\exp\big(2(|cm/\varepsilon|^{1/2}+|k|)\xi\big)\times\\ &\qquad\left|\begin{pmatrix}\Big(|V^1_{m,k}|+\frac{1}{|m/\varepsilon|+|k|}|V^2_{m,k}|\Big)\,(1{+}|m/\varepsilon|{+}|k|)^s\\ \Big(|V^2_{m,k}|{+}(|m/\varepsilon|{+}|k|)|V^1_{m,k}|\Big)\,(1{+}|m/\varepsilon|{+}|k|)^{s-1}\end{pmatrix}\right|_{\mathbb{C}^2}\\ \le C&\sum_{m\in\mathbb{Z},\,k\in\mathbb{Z}^d}\exp\big(-2(|cm/\varepsilon|^{1/2}+|k|)|\xi|\big)\times\\ &\qquad\Big(|V^1_{m,k}|^2(1{+}|m/\varepsilon|{+}|k|)^{2s}+|V^2_{m,k}|^2(1{+}|m/\varepsilon|{+}|k|)^{2(s-1)}\Big)\,,\end{aligned}\tag{2.5}$$

and similar for $V\in P^-\mathcal{X}$ and $\xi>0$. This formula shows that: (i) $\|\cdot\|_{\mathcal{X}}$ is the natural norm for the solutions of (1.4); (ii) we have exponential dichotomies in the sense of a forward smoothing in y on $\Pi^-\mathcal{X}$ by $\mathrm{e}^{\xi A_-}$, $\xi>0$, and a backward smoothing in y on $\Pi^+\mathcal{X}$ by $\mathrm{e}^{\xi A_+}$, $\xi<0$, together with an exponential decay of V_m for $m\neq0$ with rate $\exp(-|cm/\varepsilon|^{1/2}|\xi|)$.

Motivated by the smoothing in y we define Gevrey spaces, similar to those used in the regularity analysis of parabolic equations, see e.g. [FT89, Pro91, TBD+96, FT98]. We let

$$\begin{aligned} G^s_{m,\sigma} &= \{V =: \sum_{k\in\mathbb{Z}^d} a^\pm_{m,k}\phi^\pm_{m,k}\mathrm{e}^{\mathrm{i}k\cdot y} :\ \|V\|_{G^s_{m,\sigma}} < \infty\}, \\ \|V\|^2_{G^s_{m,\sigma}} &= \sum_{k\in\mathbb{Z}^d}(1+|k|)^{2s}\mathrm{e}^{\sigma|k|}(|a^+_{k,m}|^2+|a^-_{k,m}|^2), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathcal{G}^s_\sigma &= \{V = \sum_{m\in\mathbb{Z},\,k\in\mathbb{Z}^d} a^\pm_{m,k}\phi^\pm_{m,k}\mathrm{e}^{\mathrm{i}mz+\mathrm{i}k\cdot y} :\ \|V\|_{\mathcal{G}^s_\sigma} < \infty\}, \\ \|V\|^2_{\mathcal{G}^s_\sigma} &= \sum_{m\in\mathbb{Z},\,k\in\mathbb{Z}^d}(1+|m/\varepsilon|+|k|)^{2s}\mathrm{e}^{\sigma|k|}(|a^+_{k,m}|^2+|a^-_{k,m}|^2). \end{aligned} \tag{2.7}$$

For fixed $\sigma > 0$ we also use the abbreviation

$$\mathcal{Y} = \mathcal{G}^s_\sigma.$$

Lemma 2.1 *Let f fulfill the assumptions of Theorem A. Then*

$$F(V)(y,z) = \begin{pmatrix} 0 \\ f(v_1, \varepsilon^{-1}\partial_z v_1 + v_2, \nabla_y v_1, y, z) \end{pmatrix}$$

is analytic as a mapping $\mathcal{X} \to \mathcal{X}$ and as a mapping $\mathcal{Y} \to \mathcal{Y}$.

Proof. The first statement follows from standard Sobolev embeddings, noting that F lives in the second component. The second statement follows as, e.g., [Mat00, Lemma 4.2]. The key observation is that the Gevrey classes form an algebra under point–wise multiplication as in [FT98]. ∎

The iterated normal form transforms will be done on Galerkin type subspaces $P^N\mathcal{X}$: for $N \in \mathbb{N}$ let

$$\mathcal{H}^N = \{V^N = \sum_{m\in\mathbb{Z},\,|k|\le N} a^\pm_{m,k}\phi^\pm_{m,k}\mathrm{e}^{\mathrm{i}mz+\mathrm{i}k\cdot y}\} \subset \mathcal{X}, \tag{2.8}$$

and let P^N be the A invariant projection of $\mathcal{X}$ onto $\mathcal{H}^N$. To estimate the error in the complement of $P^N\mathcal{X}$ we shall use exponential decay of the Fourier coefficients in $|k|$.

Lemma 2.2 *(i) $\|P^N A_0 V\|_Z \le CN\|V\|_Z$ for all $V \in Z$ with $Z = X$ and $Z = Y$.*

(ii) $\|A_m^{-1}V_m\|_Z \le C_A|\varepsilon/m|\,\|V\|_Z$ for all $m \in \mathbb{Z}^\star$ and all $V \in Z$ with $Z = X$ and $Z = Y$.

(iii) $\|P^N V - V\|_{\mathcal{X}} \le \|V\|_{\mathcal{Y}}\mathrm{e}^{-\sigma N}$ for all $V \in \mathcal{Y}$.

Proof. (i) follows directly from (2.2). (ii) follows from the explicit representation

$$\begin{aligned}
\left| A_{m,k}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|_{\mathbb{C}^2} &= \left| B_{m,k}^{-1} \operatorname{diag}(1/\lambda_{m,k}^{+}, 1/\lambda_{m,k}^{+})\, B_{m,k} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right|_{\mathbb{C}^2} \\
&\le C \left| \begin{pmatrix} 1 & \frac{1}{|m/\varepsilon|+|k|}) \\ 1 & \frac{1}{|m/\varepsilon|+|k|} \end{pmatrix} \begin{pmatrix} |\lambda_{m,k}^{+}|^{-1}(|v_1|+|v_2|)| \\ |\lambda_{m,k}^{-}|^{-1}(|m/\varepsilon|+|k|)(|v_1|+|v_2|) \end{pmatrix} \right|_{\mathbb{C}^2} \\
&\le C(|1/\lambda_{m,k}^{+}|^{-1} + |1/\lambda_{m,k}^{+}|^{-1})(|v_1|+|v_2|)
\end{aligned}$$

using $|1/\lambda_{m,k}^{\pm}|^{-1} \le C|\varepsilon/m|$. (iii) follows from definitions (2.7) and (2.8). ∎

Lemma 2.3 *Let f fulfill the assumptions of Theorem A. Then all bounded solutions of (1.4) are highly regular. In detail, for all $R > 0$ there exist $\sigma_0, C > 0$ such that for all $0 < \sigma \le \sigma_0$ and all solutions $V(\xi)$ of (1.4) with $\sup_{\xi\in\mathbb{R}} \|V(\xi)\|_X \le R$ we have*

$$\sup_{\xi\in\mathbb{R}} \|V(\xi)\|_{\mathcal{Y}} \le C \sup_{\xi\in\mathbb{R}} \|V(\xi)\|_{\mathcal{X}}. \tag{2.9}$$

Proof. A detailed proof of a similar result is given in [Mat00, Proposition 4.1], so here we only sketch the main steps. Consider (1.6), i.e., $\partial_\xi V_m(\xi) = A_m V_m(\xi) + F_m(\xi)$, $m \in \mathbb{Z}$, with

$$F_m(\xi, y) = \Pi_m F(V(\xi, y)) = \begin{pmatrix} 0 \\ \frac{1}{2\pi}\int_0^{2\pi} f(v_1, \varepsilon^{-1}\partial_z v_1 + v_2, \nabla_y v_1, y, z) \mathrm{e}^{-imz}\, \mathrm{d}z \end{pmatrix}.$$

Let $V_{N,m}^{\pm} = P^N \Pi_m^{\pm} V$. Then $V_{N,m}^{\pm}$ fulfill

$$\partial_\xi V_{N,m}^{+} = A_m U_{N,m}^{+} + P^N \Pi_m^{+} F_m(V), \tag{2.10}$$

$$\partial_\xi V_{N,m}^{-} = A_m U_{N,m}^{-} + P^N \Pi_m^{-} F_m(V). \tag{2.11}$$

For given $\xi_0 \in \mathbb{R}$ we want to estimate $\|V_{N,m}^{\pm}(\xi_0)\|_{G_\sigma^s}$. Without loss of generality let $\xi_0 = 0$. Using (2.2) we can use backward "smoothing" by $\mathrm{e}^{-\xi A_m}\Pi_m^{+}$ (start at $\xi > 0$ in (2.10)) and forward "smoothing" by $\mathrm{e}^{\xi A_m}\Pi_m^{-}$ (start at $-\xi < 0$ in (2.11)) and continuous dependence on initial data in the finite dimensional system (2.10,2.11) to obtain

$$\|V_m^N(0)\|_{G_\xi^s} \le C(\|V_m\|_{BC(\mathbb{R},X)} + \|F_m\|_{BC(\mathbb{R},X)}),$$

where C, ξ depend on R but not on N, m. Letting $N \to \infty$ and summing over m yields the result. ∎

We denote complex extensions of the Hilbert spaces X, Y by $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$. For a general Hilbert space Z the complexification is

$$Z_{\mathbb{C}} = Z \times Z$$

with norm $\|(U_1,U_2)\|_{Z_\mathbb{C}} = \sqrt{\|U_1\|_Z^2 + \|U_2\|_Z^2}$. Linear operations are extended to $Z_\mathbb{C}$ as

$$\overline{(U_1,U_2)} = (U_1,-U_2), \qquad (a+bi)(U_1,U_2) = (aU_1 - bU_2, bU_1 + aU_2),$$
$$L(U_1,U_2) = (LU_1, LU_2) \text{ for } L \in L(Z,Z), \qquad L(U_1,U_2) = LU_1 + iLU_2 \text{ for } L \in L(Z,\mathbb{R}).$$

As complex extensions we define for some open set $\mathcal{D}$ contained in $Z = X$ respectively $Z = Y$,

$$\mathcal{N}_{\delta,Z}(\mathcal{D}) = \{U \in Z_\mathbb{C} \ : \ \inf_{V\in\mathcal{D}}\|U - V\|_{Z_\mathbb{C}} < \delta\},$$

see fig.2 for a sketch. For such extensions we have a Cauchy estimate for $\mathcal{X}$-norms and

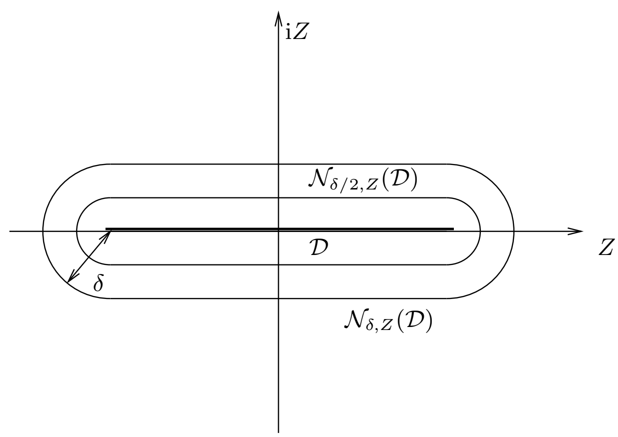

Figure 2: Sketch of the complex extensions $\mathcal{N}_{\delta/2,Z}(D)$ and $\mathcal{N}_{\delta,Z}(D)$.

$\mathcal{Y}$-norms.

Lemma 2.4 *Let $F : \mathcal{N}_{\delta,Z}(\mathcal{D}) \to Z$ be analytic, $Z = \mathcal{X}$ or $Z = \mathcal{Y}$, then*

$$\sup_{V\in\mathcal{N}_{\delta-\eta,Z}(\mathcal{D})} \|DF(V)\|_{L(Z_\mathbb{C},Z_\mathbb{C})} \le \frac{1}{\eta} \sup_{V\in\mathcal{N}_{\delta,Z}(\mathcal{D})} \|F(V)\|_{Z_\mathbb{C}}.$$

Proof. The lemma follows directly from the usual one-dimensional Cauchy formula. For $V \in \mathcal{N}_{\delta-\eta,Z}(\mathcal{D})$ we take a circle in the complex plane defined by $W = V + \zeta U, \zeta \in \mathbb{C}, |\zeta| = \eta$, letting without restriction $\|U\|_Z = 1$. Then we have with the one-dimensional Cauchy formula

$$\begin{aligned} F(V) &= \frac{1}{2\pi\mathrm{i}} \oint_{W=V+\zeta U,\zeta\in\mathbb{C},|\zeta|=\eta} \frac{F(V+\zeta U)}{\zeta}\mathrm{d}\zeta, \\ F(V+hU) &= \frac{1}{2\pi\mathrm{i}} \oint_{W=V+\zeta U,\zeta\in\mathbb{C},|\zeta|=\eta} \frac{F(V+\zeta U)}{\zeta - h}\mathrm{d}\zeta. \end{aligned}$$

Therefore

$$\frac{1}{h}(F(V+hU) - F(V)) = \frac{1}{2\pi\mathrm{i}} \oint_{W=V+\zeta U,\zeta\in\mathbb{C},|\zeta|=\eta} \frac{F(V+\zeta U)}{\zeta(\zeta - h)}\mathrm{d}\zeta.$$

For $h \to 0$ and $V \in \mathcal{N}_{\delta-\eta,Z}(D)$ this yields the Gateaux derivative

$$\|D_U F(V)\|_{L(Z_{\mathbb{C}},Z_{\mathbb{C}})} \le \frac{1}{2\pi} 2\pi\eta \frac{\sup_{W\in\mathcal{N}_{\delta,Z}(\mathcal{D})}\|F(W)\|}{\eta^2} = \frac{\sup_{W\in\mathcal{N}_{\delta,Z}(\mathcal{D})}\|F(W)\|}{\eta}$$

and the lemma is proved. ■

Using the variation of constants formula and the spectral gap $|\operatorname{Re}\lambda_{m,0}| \ge C|m/\varepsilon|^{1/2}$ we obtain for solutions of systems like (1.6) an additional factor $|\varepsilon/m|^{1/2}$.

Lemma 2.5 *There exists a $C > 0$ such that for all $m \in \mathbb{Z}^\star$ the following holds. If $\|G_m\|_{BC(\mathbb{R},X)} \le R$ and $V_m(\xi)$ is a bounded solution of*

$$\partial_\xi V_m(\xi) = A_m V_m(\xi) + G_m(\xi) \tag{2.12}$$

then

$$\|V_m(\xi)\|_{BC(\mathbb{R},X)} \le C|\varepsilon/m|^{1/2}\|G_m\|_{BC(\mathbb{R},X)}.$$

The corresponding estimates hold for solutions of the Galerkin approximated equations.

Proof. Let $\xi \in \mathbb{R}$. Using the projections Π_m^+ and Π_m^- write the solution of (2.12) as

$$\begin{aligned} V_m(\xi) &= V_m^+(\xi) + V_m^-(\xi) \\ &= \mathrm{e}^{(\xi-\xi_+)A_m}\Pi_m^+ V(\xi_+) + \mathrm{e}^{(\xi-\xi_-)A_m}\Pi_m^- V(\xi_-) \\ &\quad + \int_{\xi_+}^{\xi} \mathrm{e}^{(\xi-\tau)A_m}\Pi_m^+ G_m(\tau)\,\mathrm{d}\tau + \int_{\xi_-}^{\xi} \mathrm{e}^{(\xi-\tau)A_m}\Pi_m^- G_m(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{2.13}$$

Letting $\xi_+ \to \infty$ and $\xi_- \to -\infty$ the first two terms vanish, cf.(2.5), while the first integral can be estimated as

$$\begin{aligned} \left\| \int_{\xi_+}^{\xi} \mathrm{e}^{(\xi-\tau)A_m}\Pi_m^+ G_m(\tau)\,\mathrm{d}\tau \right\|_X &\le C \int_{-\infty}^{\xi} \exp(-|m/\varepsilon|^{1/2}(\xi-\tau))\,\mathrm{d}\tau \sup_{\tau\in\mathbb{R}} \|\Pi_m^+ G_m(\tau)\| \\ &\le C|\varepsilon/m|^{1/2} \sup_{\tau\in\mathbb{R}} \|G_m(\tau)\|, \end{aligned}$$

and similar for $\int_{\xi_-}^{\xi} \mathrm{e}^{(\xi-\tau)A_m}\Pi_m^- G_m(\tau)\,\mathrm{d}\tau$. ■

3 Iterated normal forms

The proof of Theorem A consists of three steps. In Step 1 we describe a number of iterated normal form transformations on the Galerkin approximation of (1.4). To make this rigorous, we need careful estimates coupling ε, N and the number of normal form transformations in Step 2. Finally in Step 3 we will estimate the remainder term using the high regularity in y of solutions of (1.4) obtained in Lemma 2.3.

Step 1: Normal form transformations in Galerkin space

With $V^N = P^N V$ we write the Galerkin approximation of the infinite system of equations (1.6) as

$$\begin{aligned} \partial_\xi V_m^N(\xi, y) &= A_m V_m^N(\xi) + P^N \Pi_m F(V^N(\xi))(y) \\ &:= A_m V_m^N(\xi) + F_m(V_0^N(\xi))(y) + G_m(V_0^N(\xi), (V_\ell^N(\xi))_{\ell\in\mathbb{Z}^\star})(y) \end{aligned} \tag{3.1}$$

with $G_m(V_0^N(\xi), 0) = 0$. For notational convenience we suppress the dependence of the nonlinearities on N. We will iteratively transform the oscillatory modes V_m^N for $m \neq 0$. The transformations will have the form

$$V_m^N = \Psi_m(V_0^N, W_m^N) \text{ for } m \in \mathbb{Z}^\star.$$

In the first transformation we let $V_m^N = \Psi_{m,1}(V_0^N, W_m^N)$, $m \in \mathbb{Z}^\star$, which defines the new variables $(W_m^N)_{m\in\mathbb{Z}^\star}$. After renaming W_m^N again as V_m^N, we let $V_m^N = \Psi_{m,2}(V_0^N, W_m^N)$, $m \in \mathbb{Z}^\star$, or both transformations together $V_m^N = \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, W_m^N)), m \in \mathbb{Z}^\star$. Iteratively this gives

$$V_m^N = \Psi_m(V_0^N, W_m^N) = \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, \Psi_{m,3}(V_0^N, \ldots, \Psi_{m,j}(V_0^N, W_m^N)\ldots)))$$

after j transformations. Finally we want to achieve, that $\Psi_m(V_0^N, W_m^N) - \Psi_m(V_0^N, 0)$ is small to show (1.7). In the first transformation we let

$$V_m^N = \Psi_{m,1}(V_0^N, W_m^N) = -(P^N A_m)^{-1} F_m(V_0^N) + \frac{1}{K} W_m^N, \quad m \in \mathbb{Z}^\star,$$

where $K \geq 2$ is a fixed constant. The transformed equations are

$$\begin{aligned} \partial_\xi V_0^N(\xi) &= A_0 V_0^N(\xi) + F_{0,1}(V_0^N(\xi)) + G_{0,1}(V_0^N(\xi), (W_\ell^N)_{\ell\in\mathbb{Z}^\star}), \\ \partial_\xi W_m^N(\xi) &= A_m W_m^N(\xi) + F_{m,1}(V_0^N(\xi)) + G_{m,1}(V_0^N(\xi), (W_\ell^N)_{\ell\in\mathbb{Z}^\star}), \quad m \in \mathbb{Z}^\star, \end{aligned}$$

with $G_{\ell,1}(V_0^N, 0) = 0$ for all $\ell \in \mathbb{Z}$. We rename W_m^N as V_m^N and repeat for $m \in \mathbb{Z}^\star$, i.e.,

$$V_m^N = \Psi_{m,j}(V_0^N, W_m^N) = -(P^N A_m)^{-1} F_{m,j-1}(V_0^N) + \frac{1}{K} W_m^N. \tag{3.2}$$

For $m \in \mathbb{Z}^\star$ this yields

$$\begin{aligned} \partial_\xi V_0^N(\xi) &= A_0 V_0^N(\xi) + F_{0,j}(V_0^N(\xi)) + G_{0,j}(V_0^N(\xi), (W_\ell^N)_{\ell\in\mathbb{Z}^\star}(\xi)), \\ \partial_\xi W_m^N(\xi) &= A_m W_m^N(\xi) + F_{m,j}(V_0^N(\xi)) + G_{m,j}(V_0^N(\xi), (W_\ell^N)_{\ell\in\mathbb{Z}^\star}(\xi)), \end{aligned} \tag{3.3}$$

with $G_{\ell,j}(V_0^N, 0) = 0$ for all $\ell \in \mathbb{Z}$ and all j. Here the nonlinearities are given explicitly for $m = 0$ by

$$\begin{aligned} F_{0,j}(V_0^N) &= F_{0,j-1}(V_0^N) + G_{0,j-1}(V_0^N(\xi), (\Psi_{\ell,j}(V_0^N, 0))_{\ell\in\mathbb{Z}^\star}), \\ G_{0,j}(V_0^N(\xi), (W_\ell^N)_{\ell\in\mathbb{Z}^\star}) &= G_{0,j-1}(V_0^N(\xi), (\Psi_{\ell,j}(V_0^N, W_\ell^N))_{\ell\in\mathbb{Z}^\star}) \\ &\quad - G_{0,j-1}(V_0^N(\xi), (\Psi_{\ell,j}(V_0^N, 0))_{\ell\in\mathbb{Z}^\star}). \end{aligned}$$

Our aim is to make $G_{0,j}$ small to achieve the form (1.8). Using $G_{0,j}$ and $F_{0,j}$ the terms for $m \neq 0$ can be given explicitly as

$$
\begin{aligned}
F_{m,j}(V_0^N) &= KG_{m,j-1}(V_0^N, (\Psi_{\ell,j}(V_0^N, 0))_{\ell\in\mathbb{Z}^\star}) \\
&\quad + K(A_m)^{-1} DF_{0,j-1}(V_0^N)\Big[P^N A_0 V_0^N + F_{0,j}(V_0^N)\Big], \\
G_{m,j}(V_0^N, (W_\ell^N)_{\ell\in\mathbb{Z}^\star}) &= KG_{m,j-1}(V_0^N, (\Psi_{\ell,j}(V_0^N, W_\ell^N))_{\ell\in\mathbb{Z}^\star}) \\
&\quad - KG_{m,j-1}(V_0^N, (\Psi_{\ell,j}(V_0^N, 0))_{\ell\in\mathbb{Z}^\star}) \\
&\quad + K(P^N A_m)^{-1} DF_{0,j-1}(V_0^N)\Big[G_{0,j}(V_0^N, (W_\ell^N)_{\ell\in\mathbb{Z}^\star})\Big].
\end{aligned}
$$

Step 2: Iterative estimates

To show (1.7,1.8) in the Galerkin approximation we now prove estimates for the smallness of $\Psi_m(V_0^N, W_m^N) - \Psi_m(V_0^N, 0)$ and of $G_{0,j}$. The main part are inductive estimates of $F_{m,j}$ and $G_{m,j}$ for increasing j simultaneously for all $m \in \mathbb{Z}$.

For a given Z and $\mathcal{D} \subset Z_{\mathbb{C}}$ we use the notation

$$
\|F\|_{\mathcal{D}} := \sup_{V\in\mathcal{D}} \|F(V)\|_Z, \quad \|DF\|_{\mathcal{D}} := \sup_{V\in\mathcal{D}} \|DF(V)\|_{L(Z,Z)}.
$$

We also write

$$
\|(F_m)_{m\in\mathbb{Z}}\|_{\mathcal{D}} := \|F\|_{\mathcal{D}} \quad \text{with} \quad F = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}mz} F_m,
$$

and similar for $m \in \mathbb{Z}^\star$. We show estimates on the transformed nonlinearities as long as normal form transformations are defined on a domain $\mathcal{D}_j \subset Z_{\mathbb{C}}$ which is the complex extension of a large ball in $Z = \mathcal{H}^N$, i.e.,

$$
\mathcal{D}_j = \mathcal{N}_{\delta - j\eta, Z}(B_Z(R)).
$$

Under the assumptions of estimates on $\mathcal{D}_{j-1}$:

$$
\begin{aligned}
&\|V_m^N\|_{\mathcal{D}_{j-1}} < C_W, \quad \|(G_{\ell,j-1})_{\ell\in\mathbb{Z}}\|_{\mathcal{D}_{j-1}} < C_G, \\
&\|(F_{\ell,j-1})_{\ell\in\mathbb{Z}}\|_{\mathcal{D}_{j-1}} < C_F = \bar{C}_F(1+\varepsilon^{1/2})^j,
\end{aligned}
\tag{3.4}
$$

we show on $\mathcal{D}_j \subset \mathcal{D}_{j-1}$:

$$
\begin{aligned}
\|W_m^N\|_{\mathcal{D}_j} &< C_W, & \|(G_{\ell,j})_{\ell\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} &< C_G, \\
\|(F_{\ell,j})_{\ell\in\mathbb{Z}}\|_{\mathcal{D}_j} &< C_F(1+\varepsilon^{1/2}), & \|G_{0,j}\|_{\mathcal{D}_j} &\le \frac{1}{2}\|G_{0,j-1}\|_{\mathcal{D}_{j-1}}.
\end{aligned}
\tag{3.5}
$$

Then $(W_m^N)_{m\in\mathbb{Z}}$, $F_{\ell,j}$ for $\ell \in \mathbb{Z}$ and $G_{\ell,j}$ for $\ell \in \mathbb{Z}^\star$ stay bounded under $j^* = \mathcal{O}(\varepsilon^{-1/2})$ many transformations (see (3.6) below), while $G_{0,j}$ will go to zero exponentially in j.

Using the Cauchy estimate in Lemma 2.4 and Lemma 2.2(ii) we first obtain

$$\begin{aligned}\|F_{0,j}\|_{\mathcal{D}_j} &\leq \|F_{0,j-1}\|_{\mathcal{D}_j} + \|D_2 G_{0,j-1}\|_{\mathcal{D}_j}\|(P^N A_m)^{-1}\|_{L(Z,Z)} C_F \\ &\leq C_F + \frac{C_G}{\eta} C_A \varepsilon C_F = C_F(1+\varepsilon\frac{C_G}{\eta}C_A),\end{aligned}$$

and, for $m \neq 0$,

$$\begin{aligned}\|(F_{m,j})_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} &\leq K\|\big((D_2 G_{m,j-1})(P^N A_m)^{-1}F_{m,j-1}\big)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} \\ &\quad +K\Big\|\big(|(P^N A_m)^{-1}(DF_{m,j-1})(N|V_0|+F_{0,j})\big)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} \\ &\leq K\frac{C_G}{\eta}C_A\varepsilon C_F + KC_A\varepsilon\frac{C_F}{\eta}(NC_W + \|F_{0,j}\|_{\mathcal{D}_j}).\end{aligned}$$

The terms depending on $W_\ell^N, \ell \in \mathbb{Z}^\star$, are estimated as

$$\|G_{0,j}\|_{\mathcal{D}_j} \leq \|D_2 G_{0,j-1}\|_{\mathcal{D}_j}\|\frac{1}{K}W_m^N\|_{\mathcal{D}_j} \leq \frac{C_G}{\eta}\frac{1}{K}\|W_m^N\|_{\mathcal{D}_j},$$

and, for $m \neq 0$,

$$\begin{aligned}\|(G_{m,j})_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} &\leq K\|D_2(G_{m,j-1})_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j}\|\frac{1}{K}(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} \\ &\quad +K\|((P^N A_m)^{-1}DF_{m,j-1})_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j}\|G_{0,j}\|_{\mathcal{D}_j} \\ &\leq \frac{C_G}{\eta}\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} + KC_A\varepsilon\frac{C_F}{\eta}\|G_{0,j}\|_{\mathcal{D}_j}.\end{aligned}$$

To bound $G_{m,j}$, we first estimate W_m^N applying Lemma 2.5, i.e.,

$$\begin{aligned}&\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} \leq C_A\varepsilon^{1/2}(\|(F_{m,j})_{m\in\mathbb{Z}}\|_{\mathcal{D}_j} + \|(G_{m,j})_{m\in\mathbb{Z}}\|_{\mathcal{D}_j}) \\ &\qquad \leq C_A\varepsilon^{1/2}\Big(\|(F_{m,j})_{m\in\mathbb{Z}}\|_{\mathcal{D}_j} + \frac{C_G}{\eta}\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} + C_A\varepsilon\frac{C_F C_G}{\eta^2}\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j}\Big).\end{aligned}$$

Therefore, if

$$C_A\varepsilon^{1/2}\Big(\frac{C_G}{\eta} + C_A\varepsilon\frac{C_F C_G}{\eta^2}\Big) \leq \frac{1}{2},$$

then

$$\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_j} \leq 2C_A\varepsilon^{1/2}\|(F_{m,j})_{m\in\mathbb{Z}}\|_{\mathcal{D}_j}.$$

Thus we altogether need the following relations to ensure (3.5)

$$\begin{aligned}\frac{C_G}{\eta}C_A\varepsilon C_F &\leq C_F\varepsilon^{1/2}, \\ \varepsilon K\frac{C_A}{\eta}(C_G + NC_W + C_F(1+\varepsilon^{1/2})) &< 1, \\ C_A\varepsilon^{1/2}\Big(\frac{C_G}{\eta} + C_A\varepsilon\frac{C_F C_G}{\eta^2}\Big) &\leq \frac{1}{2}, \\ 2C_A\varepsilon^{1/2}C_F(1+\varepsilon^{1/2}) &< C_W.\end{aligned}$$

These inequalities are fulfilled for $0 < \varepsilon < \varepsilon_0$ if

$$\begin{aligned}
&\eta(\varepsilon) = M\varepsilon^{1/2}, \qquad N(\varepsilon) = \varepsilon^{-1/2} \quad \text{with} \\
&C_F = \bar{C}_F \mathrm{e}^{\delta/(2M)}, \\
&M \geq \max\Big(C_G C_A, K C_A(C_W + \varepsilon_0^{1/2}(C_G + C_F(1+\varepsilon_0^{1/2}))), 2C_A(C_G + \varepsilon_0^{1/2} C_G C_A C_F)\Big), \\
&\varepsilon_0 \leq \left(\frac{C_W}{2C_A\bar{C}_F}\right)^2.
\end{aligned}$$

Thus we can perform

$$j^* = \left[\frac{\delta}{2M\varepsilon^{1/2}}\right] \tag{3.6}$$

normal form transformations with $\mathcal{N}_{\delta/2,Z}(B_Z(R)) \subset \mathcal{D}_{j^*}$ and all resulting nonlinearities are uniformly bounded on bounded sets in $\mathcal{X}$ respectively $\mathcal{Y}$, i.e.,

$$\begin{aligned}
\|G_{0,j^*}\|_{\mathcal{D}_{j^*}} &\leq 2^{-j^*} C_G = C\mathrm{e}^{-c_2\varepsilon^{-1/2}}, \\
\|(G_{m,j})_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_{j^*}} &\leq C_G, \\
\|(W_m^N)_{m\in\mathbb{Z}^\star}\|_{\mathcal{D}_{j^*}} &\leq C_W, \\
\|(F_{m,j})_{m\in\mathbb{Z}}\|_{\mathcal{D}_{j^*}} &\leq \bar{C}_F(1+\varepsilon^{1/2})^{\delta/(2M\varepsilon^{1/2})} \leq C_F.
\end{aligned}$$

Furthermore all transformations are well-defined in $B_{\mathcal{X}}(R)$ and $B_{\mathcal{Y}}(R)$. Indeed, we have for the first and second transformation

$$\begin{aligned}
V_m^N &= \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, W_m^N)) \\
&= \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, 0)) + \Big(\Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, W_m^N)) - \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, 0)\Big) \\
&= \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, 0)) + \frac{1}{K}\Big[\Psi_{m,2}(V_0^N, W_m^N) - \Psi_{m,2}(V_0^N, 0)\Big] \\
&= \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, 0)) + \frac{1}{K^2} W_m^N.
\end{aligned}$$

After j transformations we then have

$$V_m^N = \Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, \Psi_{m,3}(V_0^N, \ldots, \Psi_{m,j}(V_0^N, 0)\ldots))) + \frac{1}{K^j} W_m^N.$$

For all iterations we have, due to(3.2), (3.4), and Lemma 2.2, for $K \geq 2$ and uniformly in $m \in \mathbb{Z}^\star$,

$$\begin{aligned}
\|\Psi_{m,j}(V_0^N, 0)\|_{\mathcal{D}_j} &\leq C_A\varepsilon C_F, \\
\|\Psi_{m,j-1}(V_0^N, \Psi_{m,j}(V_0^N, 0))\|_{\mathcal{D}_j} &\leq C_A\varepsilon C_F + \frac{1}{2} C_A\varepsilon C_F.
\end{aligned}$$

Iteratively we obtain

$$\|\Psi_{m,1}(V_0^N, \Psi_{m,2}(V_0^N, \Psi_{m,3}(V_0^N, \ldots, \Psi_{m,j}(V_0^N, 0)\ldots)))\|_{\mathcal{D}_j} < 2C_A\varepsilon C_F.$$

Then we define the transformations T_m as

$$\varepsilon T_m(V_0,\varepsilon) := \Psi_{m,1}(V_0^{N(\varepsilon)}, \Psi_{m,2}(V_0^{N(\varepsilon)}, \Psi_{m,3}(V_0^{N(\varepsilon)}, \ldots, \Psi_{m,j^*(\varepsilon)}(V_0^{N(\varepsilon)}, 0)\ldots))).$$

For each T_m the norm is bounded by $2C_AC_F$ on the extended domain, hence it is analytic in V_0. For the remainder

$$R_T^N((V_m^N)_{m\in\mathbb{Z}}) = \sum_{m\in\mathbb{Z}^\star} \frac{1}{K^{j^*}} W_m^N \mathrm{e}^{\mathrm{i}mz}$$

we obtain, by (3.5),

$$\|R_T^N\|_{\mathcal{D}_{j^*}} \leq C_W K^{-M\varepsilon^{-1/2}} = C_1 \mathrm{e}^{-c_1\varepsilon^{-1/2}}, \tag{3.7}$$

which defines $C_1, c_1 > 0$.

Step 3: Regularity estimates

So far all estimates have been on the Galerkin approximation V^N. In the final step we estimate the error of this approximation using the high regularity in y of bounded solutions from Lemma 2.3. First we consider the additional error term in (1.7), when setting

$$V(\xi,y,z) = V_0(\xi,y) + \varepsilon \sum_{m\in\mathbb{Z}^\star} T_m(P^{N(\varepsilon)}V_0,\varepsilon)(y)\mathrm{e}^{\mathrm{i}mz} + R_T(V(\xi),\varepsilon)(y,z).$$

The remainder term is bounded by

$$\begin{aligned}
\|R_T(V(\xi),\varepsilon)\|_{\mathcal{X}} &\leq \|(I-P^N)V(\xi)\|_{\mathcal{X}} + \Big\|P^NV(\xi) - \Big(V_0(\xi) + \varepsilon \sum_{m\in\mathbb{Z}^\star} T_m(P^{N(\varepsilon)}V_0,\varepsilon)(\cdot)\mathrm{e}^{\mathrm{i}m\cdot}\Big)\Big\|_{\mathcal{X}} \\
&\leq \|V\|_{\mathcal{Y}}\mathrm{e}^{-\sigma N} + \|R_T^N\|_{\mathcal{D}_{j^*}} \\
&\leq \|V\|_{\mathcal{Y}}\mathrm{e}^{-\sigma N} + C_1\mathrm{e}^{-c_1\varepsilon^{-1/2}},
\end{aligned}$$

using (3.7) and Lemma 2.2(iii). Our choice $N(\varepsilon) = [\varepsilon^{-1/2}]$ gives the desired exponential estimate in (1.7). This choice is the optimal coupling of N and ε for our estimates as both exponential estimates balance.

To derive (1.7), we extend the equation (3.3) for the Galerkin approximation V_0^N after j^* normal form transformations to the full space X respectively G_σ^s. As V_0^N remains unchanged under the transformations we have with $N = N(\varepsilon)$

$$\begin{aligned}
\partial_\xi V_0(\xi) &= A_0V_0(\xi) + \Pi_0 F(P^NV(\xi)) + \Pi_0\big[F(V(\xi)) - F(P^NV(\xi))\big) \\
&= A_0V_0(\xi) + F_{0,j^*}(P^NV_0(\xi)) + G_{0,j^*}(V_0^N, (W_m^N)_{m\in\mathbb{Z}^\star}) + \Pi_0\big[F(V(\xi)) - F(P^NV(\xi))\big].
\end{aligned}$$

Then with

$$\overline{F}(V_0(\xi),\varepsilon) := F_{0,j^*}(P^{N(\varepsilon)}V_0(\xi))$$

we obtain

$$R_E(V(\xi),\varepsilon) = G_{0,j^*}(V_0^N,(W_m^N)_{m\in\mathbb{Z}^\star}) + \Pi_0\big[F(V(\xi)) - F(P^N V(\xi))\big]$$

with the estimate

$$\begin{aligned} \|R_E(V(\xi),\varepsilon)\|_X &\leq \|G_{0,j^*}\|_{\mathcal{D}_{j^*}} + \|\Pi_0 DF\|_{B_X(R)}\|V - P^N V\|_{\mathcal{X}} \\ &\leq C\mathrm{e}^{-c_2\varepsilon^{-1/2}} + C\|V\|_{\mathcal{Y}}\mathrm{e}^{-\sigma N}. \end{aligned}$$

This is the final required exponential estimate in (1.8). ∎

Remark 3.1 In lowest order, $\overline{F}$ in (1.8) is given by the first transformation

$$\overline{F}(V_0,\varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} P^N F\Big(P^N V_0 - \sum_{m\in\mathbb{Z}^\star}(P^N A_m)^{-1}F_m(P^N V_0)\mathrm{e}^{\mathrm{i}mz}\Big)\mathrm{d}z + h.o.t.$$

where $N = N(\varepsilon) = [\varepsilon^{-1/2}]$ was chosen in the proof. Due to the structure of A_m and F_m, we have for $\varepsilon \to 0$

$$\overline{F}(V_0,0) = \frac{1}{2\pi}\int_0^{2\pi} P^N F(P^N V_0)\mathrm{d}z.$$

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