A remark about the justification of the nonlinear Schrödinger equation in quadratic spatially periodic media

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Abstract

We prove the validity of a technical assumption necessary in a proof of the validity of the nonlinear Schrödinger equation as envelope equation in quadratic spatially periodic media.

The dynamics of the envelopes of spatially and temporarily oscillating wave packets advancing in dispersive spatially periodic media can be approximated by solutions of a Nonlinear Schrödinger equation. In [1] the semilinear wave equation

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) - \chi_3(x)u^{\vartheta}(x,t) \tag{1}$$

has been considered as a model problem for this, where $x \in \mathbb{R}, t \in \mathbb{R}, u(x,t) \in \mathbb{R}, \vartheta = 2$ or $\vartheta = 3$, and $\chi_j(x) = \chi_j(x+2\pi)$ for $j = 1,2,3$.

Under a number of technical assumptions in [1] a proof for the validity of the Nonlinear Schrödinger equation as an amplitude equation has been given. In the present paper we explain that the technical assumption (7) in [1] for the much more advanced quadratic case $\vartheta = 2$ is always satisfied if $\chi_1 \in C^2_{\mathrm{per}}$.

The linearized problem

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) \tag{2}$$

is solved by the Bloch waves

$$u(x,t) = f_n(\ell,x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\pm\mathrm{i}\omega_n(\ell)t}$$

where $n \in \mathbb{N}, \ell \in (-1/2, 1/2]$, with $\omega_n(\ell) \in \mathbb{R}$ satisfying $\omega_{n+1}(\ell) \geq \omega_n(\ell)$, and $f_n(x,\ell)$ satisfying $f_n(\ell,x) = f_n(\ell,x+2\pi)$ and $f_n(\ell,x) = f_n(\ell+1,x)\mathrm{e}^{\mathrm{i}x}$.

Slow modulations in time and space of such a Bloch mode (indexed with n_0) are described by the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x+c_{\mathrm{g}}t), \varepsilon^2 t) f_{n_0}(\ell_0,x)\mathrm{e}^{\mathrm{i}\ell_0 x}\mathrm{e}^{\mathrm{i}\omega_{n_0}(\ell_0)t} + \mathrm{cc} + \mathrm{h.o.t.}, \tag{3}$$

where cc means complex conjugate, h.o.t. means terms of order ε^2 and higher, $0 < \varepsilon \ll 1$ is a small parameter, $c_{\mathrm{g}} = \partial_\ell \omega_{n_0}(\ell_0)$ is the negative group velocity, and where

A is the slowly varying envelope. Plugging the ansatz into (1) one finds that A has to satisfy a NLS equation

$$\partial_T A = \mathrm{i}\nu_1 \partial_X^2 A + \mathrm{i}\nu_2 A|A|^2 \tag{4}$$

with coefficients $\nu_1 \in \mathbb{R}$ and $\nu_2 \in \mathbb{R}$. This describes via the complex valued amplitude $A(X,T) \in \mathbb{C}$ slow modulations in time $T = \varepsilon^2 t$, and space $X = \varepsilon(x + c_{\mathrm{g}} t)$, of the underlying wave $f_{n_0}(\ell_0, x)\mathrm{e}^{\mathrm{i}\ell_0 x}\mathrm{e}^{\mathrm{i}\omega_{n_0}(\ell_0)t}$.

Validity means that, given a solution A of (4) for $T \in [0, T_0]$, for all small $\varepsilon > 0$ the difference between the formal approximation and exact solutions of (1) stays small for all t in the long time interval $[0, T_0/\varepsilon^2]$. In [1], in order to prove this, besides a number of non-resonance conditions, in case $\vartheta = 2$ we also needed a technical assumption on the quadratic interaction of the Bloch modes $f_n(\ell)$, namely: there exists an $\alpha > 1/2$ and a $C > 0$ such that for all $j, j_1, j_2 \in \mathbb{N}$ and $\ell_1, \ell_2, \ell_3 \in (-1/2, 1/2]$ we have

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f_j(\ell_1, x)\chi_3(x)\overline{f_{j_1}(\ell_2, x) f_{j_2}(\ell_3, x)} \frac{1}{\chi_1(x)} \,\mathrm{d}x \right| \leq \left(\frac{C}{1 + |j - j_1 - j_2|} \right)^{\alpha}, \tag{5}$$

In [1] assumption (5) has been verified with $\alpha = 2 - \delta$ for a $\delta > 0$ arbitrary in case that χ_1 is independent of x. In the present paper we prove (5) in case $\chi_1 \in C^2_{\mathrm{per}}$ not a constant by applying a change of coordinates making χ_1 a constant.

The idea is based on [2] where the spectral problem has been discussed. Introducing

$$\tilde{u}(y,t) = \chi_1^{-1/4}(x) u(x,t) \quad \text{where} \quad y = \int_0^x \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi, \tag{6}$$

equation (1) with $\vartheta = 2$ transforms into

$$\partial_t^2 \tilde{u}(y,t) = \partial_y^2 \tilde{u}(y,t) - \tilde{\chi}_2(y)\tilde{u}(y,t) - \tilde{\chi}_3(y)\tilde{u}^2(y,t), \tag{7}$$

where

$$\tilde{\chi}_2(y) = \chi_2(x) - \chi_1^{3/4}(x)(\chi_1^{1/4}(x))'', \qquad \tilde{\chi}_3(y) = \chi_3(x)\chi_1^{1/4}(x),$$

with $\tilde{\chi}_j(y) = \tilde{\chi}_j(y + \tilde{L})$, $\tilde{L} = \int_0^{2\pi} \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi$, and consequently $\ell \in (-1/(2\tilde{L}), 1/(2\tilde{L}))$ in the Bloch representation.

Thus (1) can be transformed via (6) into (7) with constant coefficient in front of the second spatial derivative and (5) is also satisfied in case χ_1 not being a constant. Since $\tilde{\chi}_3 \in C_b^2$ is needed in [1, Lemma A.1] we require at least $\chi_1 \in C^2_{\mathrm{per}}$. Moreover, in the variation of constant formula used to obtain local existence and uniqueness [1, Sections 4.2 and 5.2], given $u(\cdot, t) \in H^s$ we need

$$\left(\partial_x^2 + \chi_2 - \chi_1^{3/4}(\cdot)(\chi_1^{1/4}(\cdot))''\right)\left(\chi_3(\cdot)\chi_1^{1/4}(\cdot)u^2(\cdot,t)\right) \overset{!}{\in} H^{s-2}.$$

Here and in the following, we use the abbreviation H^s for $H^s(\mathbb{R}, \mathbb{R})$ or $H^s(\mathbb{R}, \mathbb{C})$.

Therefore, if we want an approximation result in high order Sobolev spaces, then we need additional regularity of χ_1 and χ_3. In detail, for $s \in \mathbb{R}$ we define $\lceil s \rceil$ as the smallest integer greater or equal to s. The improved result then is as follows:

Theorem 1 *Let* $s \in (1/2, 5/2)$, $s_A \geq 4$, *and assume that* $\chi_2 \in C_{\mathrm{per}}^{\max\{0,\lceil s-2\rceil\}}$ *and* $\chi_{1,3} \in C_{\mathrm{per}}^{\max\{2,\lceil s\rceil\}}$ *in* (1) *are chosen in such a way that the nonresonance conditions*

$$\begin{aligned}\inf_{n\in\mathbb{Z}\setminus\{0\},|j|\leq 4,(n,j)\notin\{-(n_0,1),(n_0,1)\}} |\omega_n(j\ell_0) - j\omega_{n_0}(\ell_0)| &> 0,\\ \inf_{r,n\in\mathbb{Z}\setminus\{0\},\ell,m\in\left(-\frac{1}{2\pi},\frac{1}{2\pi}\right],|\ell-m-\ell_0|<\delta} |-\omega_r(\ell) - \omega_{n_0}(\ell-m) + \omega_n(m)| &> 0,\end{aligned}$$

hold. Then for all C_1 *and* $T_0 > 0$ *there exist* $\varepsilon_0 > 0$ *and* $C_2 > 0$ *such that for all solutions* $A \in C([0, T_0], H^{s_A})$ *of* (1) *with*

$$\sup_{T\in[0,T_0]} \|A(\cdot, T)\|_{H^{s_A}} \leq C_1$$

the following holds. For all $\varepsilon \in (0, \varepsilon_0)$ *there are solutions* $u \in C([0, T_0/\varepsilon^2], H^s)$ *of* (1) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \left\|u(\cdot,t) - \left(\varepsilon A\left(\varepsilon(\cdot + c_{\mathrm{g}}t), \varepsilon^2 t\right) f_{n_0}(\cdot,\ell_0)\mathrm{e}^{\mathrm{i}\ell_0\cdot}\mathrm{e}^{\mathrm{i}\tilde{\omega}_{n_0}(\ell_0)t} + \mathrm{cc}\right)\right\|_{H^s} \leq C_2\varepsilon^{3/2}.$$

Proof. Theorem 1 has been established in [1] under the additional condition (5). Hence, it remains to prove the validity of (5).

Denote the inverse of $y = \int_0^x \chi_1^{-1/2}(\xi)\,\mathrm{d}\xi$ by $x = h(y)$. The solutions

$$u(x,t) = f_n(\ell, x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\pm\mathrm{i}\omega_n(\ell)t}$$

of (2) transform under (6) into

$$\tilde{u}(y,t) = \chi_1^{-1/4}(h(y)) f_n(\ell, h(y))\mathrm{e}^{\mathrm{i}\ell h(y)}\mathrm{e}^{\pm\mathrm{i}\omega_n(\ell)t}.$$

Since $\mathrm{e}^{\mathrm{i}\ell h(y)}$ can be split into $\mathrm{e}^{\mathrm{i}\ell 2\pi y/\tilde{L}}\mathrm{e}^{\mathrm{i}\ell(h(y)-2\pi y/\tilde{L})}$, where the second factor is $\tilde{L}$-periodic w.r.t. y due to $h(\tilde{L}) = 2\pi$ this can be written as

$$\tilde{u}(y,t) = \tilde{f}_n(\tilde{\ell}, y)\mathrm{e}^{\mathrm{i}\tilde{\ell}y}\mathrm{e}^{\pm\mathrm{i}\omega_n(\tilde{\ell})t}$$

where

$$\tilde{f}_n(\tilde{\ell}, y) = \chi_1^{-1/4}(h(y)) f_n(\ell, h(y))\mathrm{e}^{\mathrm{i}\ell(h(y)-2\pi y/\tilde{L})}$$

and $\tilde{\ell} = 2\pi\ell y/\tilde{L}$.

Since $\mathrm{d}y = \chi_1^{-1/2}(x)\,\mathrm{d}x$ the condition (5) transforms into

$$\left|\int_0^{\tilde{L}} \tilde{f}_j(\ell_1, y)\check{\chi}_3(y)\overline{\tilde{f}_{j_1}(\ell_2, y)\tilde{f}_{j_2}(\ell_3, y)}\,\mathrm{d}y\right| \leq C\left(\frac{C}{1 + |j - j_1 - j_2|}\right)^{\alpha} \tag{8}$$

where $\check{\chi}_3(y) = \tilde{\chi}_3(y)\mathrm{e}^{\mathrm{i}(\ell_1-\ell_2-\ell_3)(h(y)+2\pi y/\tilde{L})}$. Since the $\tilde{f}_j(\tilde{\ell}, y)\mathrm{e}^{\mathrm{i}\tilde{\ell}y}$ are solutions of

$$\partial_y^2 u - \tilde{\chi}_2\tilde{u} = -\omega^2\tilde{u},$$

since $\tilde{\ell} = 2\pi\ell/\tilde{L}$ is linearly related with ℓ, and since $\check{\chi}_3$ satisfies the assumptions of [1, Lemma A.1], the validity of (8) has already been established in [1, Lemma A.1]. Therefore, we are done. □

In most photonic crystals the χ_j only take two values, i.e., the χ_j are only in L^∞ and the validity of (5) is still an open question.

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