A Hopf-bifurcation theorem for the vorticity formulation of the Navier-Stokes equations in $\mathbb{R}^3$

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January 31, 2008

Abstract

We prove a Hopf-bifurcation theorem for the vorticity formulation of the Navier-Stokes equations in $\mathbb{R}^3$ in case of spatially localized external forcing. The difficulties are due to essential spectrum up to the imaginary axis for all values of the bifurcation parameter which a priori no longer allows to reduce the problem to a finite dimensional one.

1 Introduction

The flow around some obstacle is the paradigm for the successive occurrence of bifurcations leading to more and more complicated dynamics. For increasing Reynolds number the laminar flow undergoes a number of bifurcations and finally becomes turbulent. Although a number of results are known for the steady flow, very little is rigorously known about the bifurcations cf. [Fin65, Fin73, Gal94]. One reason for this is essential spectrum up to the imaginary axis for all Reynolds numbers. Hence, classical methods like the center manifold theorem or the Lyapunov-Schmidt method a priori fail to reduce the bifurcation problem to a finite dimensional one.

Based on the invertibility of the Oseen operator from L^p to L^q, with $p < q$ suitably chosen, in [Saz94] a Hopf-bifurcation result has been established. In this paper we prove a similar result for the vorticity formulation of the Navier-Stokes equations in $\mathbb{R}^3$ subject to some localized external forcing. Our work is motivated by [BKSS04] where the spatial structure of bifurcating time-periodic solutions in reaction-diffusion convection problems with similar properties has been analyzed. There, it turned out that the nontrivial time-periodic part decays with some exponential rate in space. Decay in x corresponds to smoothness in the Fourier wave number k. However, the Fourier space symbol of the projection operator onto the divergence-free vector fields is not smooth. Therefore, exponential decay cannot be expected for the velocity field. Here, we obtain L^p decay for the vorticity field. This yields an L^q decay for the velocity which complements the result in [Saz94]. See [vB07] for a different approach.

1.1 The vorticity formulation

We consider the Navier-Stokes equations

$$\partial_t U + (U \cdot \nabla)U = \Delta U - \nabla p + f_\alpha, \qquad \nabla \cdot U = 0, \tag{1}$$

with spatial variable $x \in \mathbb{R}^3$, time variable $t \in \mathbb{R}$, velocity field $U(x,t) \in \mathbb{R}^3$, pressure field $p(x,t) \in \mathbb{R}$, and external time-independent forcing $f_\alpha(x) \in \mathbb{R}^3$. We assume that the external forcing

f_α depends smoothly on some parameter α and that it is chosen in such a way that there exists a stationary solution $(U_\alpha, p_\alpha) = (U_\alpha, p_\alpha)(x)$. Furthermore, we assume that $U_\alpha(x) = U_c + u_\alpha(x)$ with $U_c = (c, 0, 0)^T$, $\lim_{|x|\to\infty} u_\alpha(x) = 0$, and $u_\alpha(\cdot)$ has certain decay and smoothness properties specified below.
The deviation (u, q) from the stationary solution (U_α, p_α) satisfies

$$\partial_t u = \Delta u - \nabla q - c\partial_{x_1} u - \nabla \cdot (u_\alpha u^T) - \nabla \cdot (u u_\alpha^T) - \nabla \cdot (u u^T), \qquad \nabla \cdot u = 0, \tag{2}$$

where we used $\nabla \cdot U = 0$ to rewrite the nonlinear terms, and where

$$\nabla \cdot G = \begin{pmatrix} \partial_{x_1} g_{11} + \partial_{x_2} g_{12} + \partial_{x_3} g_{13} \\ \partial_{x_1} g_{21} + \partial_{x_2} g_{22} + \partial_{x_3} g_{23} \\ \partial_{x_1} g_{31} + \partial_{x_2} g_{32} + \partial_{x_3} g_{33} \end{pmatrix} \quad \text{for general matrices} \quad G = (g_{ij})_{i,j=1,2,3}. \tag{3}$$

Notation. From now on we denote with u the velocity field of the fluid and with ω the associated vorticity defined by $\omega = \nabla \times u$. Similarly, we denote with ω_j the vorticity associated with the velocity u_j, and vice versa.
In order to derive the vorticity formulation for the Navier-Stokes equations we use

$$\nabla \times \nabla \cdot (u u^T) = \nabla \cdot (\omega u^T - u \omega^T)$$

which implies $\nabla \times \nabla \cdot (u_\alpha u^T + u u_\alpha^T) = \nabla \cdot (\omega_\alpha u^T + \omega u_\alpha^T - u_\alpha \omega^T - u \omega_\alpha^T)$. Therefore, we find

$$\partial_t \omega = B\omega + 2\nabla \cdot Q(\omega_\alpha, \omega) + \nabla \cdot Q(\omega, \omega), \tag{4}$$

where

$$B\omega = \Delta\omega - c\partial_{x_1}\omega, \qquad 2Q(\omega_1, \omega_2) = \omega_2 u_1^T + \omega_1 u_2^T - u_2 \omega_1^T - u_1 \omega_2^T.$$

The space of divergence-free vector fields is invariant under the evolution of (4), i.e., additionally we assume that $\nabla \cdot \omega = 0$. Note that (4) still contains the velocity u which can be reconstructed from the vorticity ω by solving the equations $\nabla \cdot u = 0$ and $\nabla \times u = \omega$.
Since we work in the whole space $\mathbb{R}^3$ it turns out to be advantageous to work in Fourier space.

Notation. The Fourier transform$\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are given by

$$\begin{aligned} \mathcal{F}(f)(\xi) &= \widehat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x) \exp(-ix \cdot \xi) dx, \\ \mathcal{F}^{-1}(\widehat{f})(x) &= f(x) = \int_{\mathbb{R}^3} \widehat{f}(\xi) \exp(ix \cdot \xi) d\xi. \end{aligned}$$

For $s \geq 0$ and $q \geq 1$ let $W^{s,q}$ be the standard Sobolev space equipped with the norm $\|\omega\|_{W^{s,q}} = \left(\sum_{|\alpha|\leq s} \|D^\alpha \omega\|_{L^q}^q\right)^{\frac{1}{q}}$. In general, we do not distinguish between scalar and vector-valued functions or real- and complex-valued functions. We introduce $L_s^p(\mathbb{R}^3)$, $p \geq 1$, as the spatially weighted Lebesgue spaces equipped with the norm $\|f\|_{L_s^p} = \|f\rho^s\|_{L^p}$, where $\rho(x) = \sqrt{1 + |x|^2}$. For $p \in [1, 2]$, the Fourier transform is a continuous mapping from L_s^p into $W^{s,q}$ if $1/p + 1/q = 1$. For $p = 2$, the Fourier transform is an isomorphism between these spaces. Many different constants are denoted with the same symbol C.

Applying the Fourier transform to (4) yields

$$\partial_t \widehat{\omega} = \widehat{B}\widehat{\omega} + 2i\xi \cdot \widehat{Q}(\widehat{\omega}_\alpha, \widehat{\omega}) + i\xi \cdot \widehat{Q}(\widehat{\omega}, \widehat{\omega}) \tag{5}$$

where

$$(\widehat{B}\widehat{\omega})(\xi) = (-|\xi|^2 - ic\xi_1)\widehat{\omega}(\xi), \qquad 2\widehat{Q}(\widehat{\omega}_1, \widehat{\omega}_2) = \widehat{\omega}_2 * \widehat{u}_1^T + \widehat{\omega}_1 * \widehat{u}_2^T - \widehat{u}_2 * \widehat{\omega}_1^T - \widehat{u}_1 * \widehat{\omega}_2^T,$$

where $*$ denotes the convolution, i.e., $(\widehat{u} * \widehat{v})(\xi) = \int_{\mathbb{R}^3} \widehat{u}(\xi - \eta)\widehat{v}(\eta) d\eta$, and where, like in (3),

$$i\xi \cdot G = i \begin{pmatrix} \xi_1 g_{11} + \xi_2 g_{12} + \xi_3 g_{13} \\ \xi_1 g_{21} + \xi_2 g_{22} + \xi_3 g_{23} \\ \xi_1 g_{31} + \xi_2 g_{32} + \xi_3 g_{33} \end{pmatrix} \quad \text{for general matrices} \quad G = (g_{ij})_{i,j=1,2,3}. \tag{6}$$

1.2 Assumptions on the linearized problem

Due to Lemma 2.3 below, for $\widehat{\omega}_\alpha \in L^p_s$ with $p > 3/2$ and $s \geq 3(p-1)/p$ the operator

$$\widehat{L}\cdot = \widehat{B} \cdot + 2i\xi \cdot \widehat{Q}(\widehat{\omega}_\alpha, \cdot) \tag{7}$$

is well defined in the space L^p_s, with domain of definition given by L^p_{s+2}. Moreover, by Lemma 2.8, for $p \in (3,4),\ s > 3(p-1)/p$, the operator $2i\xi \cdot \widehat{Q}(\widehat{\omega}_\alpha, \cdot)$ is a relatively compact perturbation of $\widehat{B}$, and hence the essential spectrum of $\widehat{L}$ equals the essential spectrum

$$\mathrm{essspec}(\widehat{B}) = \{\lambda \in \mathbb{C} : \lambda = -|\xi|^2 - ic\xi_1,\ \xi \in \mathbb{R}^3\}$$

of $\widehat{B}$, i.e., the spectra of $\hat{L}$ and $\hat{B}$ only differ by isolated eigenvalues of finite multiplicity, cf. [Hen81, p.136].

Thus, for the family $U_\alpha(x) = U_c + u_\alpha(x)$, $\alpha \in [\alpha_c - \delta_0, \alpha_c + \delta_0]$, of stationary solutions we we assume that $\widehat{\omega}_\alpha \in L^p_s$, $p \in (3,4),\ s > 3(p-1)/p$, and that:

(A1) $\lambda = 0$ is not an eigenvalue of $\widehat{L}$ for any value of $\alpha \in [\alpha_c - \delta_0, \alpha_c + \delta_0]$.

(A2) For $\alpha = \alpha_c$ the operator L has two eigenvalues $\lambda_0^\pm(\alpha)$ which satisfy

$$\lambda_0^\pm(\alpha_c) = \pm i\Omega_c \neq 0,\ \Omega_c > 0, \quad \text{and} \quad \frac{d}{d\alpha}\mathrm{Re}(\lambda_0^\pm(\alpha))\bigg|_{\alpha=\alpha_c} > 0.$$

(A3) All other eigenvalues of $\widehat{L}$ are strictly bounded away from the imaginary axis in the left half plane for all $\alpha \in [\alpha_c - \delta_0, \alpha_c + \delta_0]$.

1.3 The Hopf-bifurcation theorem

Even though $\widehat{L}$ has essential spectrum up to the imaginary axis, a Lyapunov-Schmidt reduction to a finite-dimensional bifurcation problem is possible due to the following reasons. First, the invertibility of the Oseen operator $\widehat{B}$ in $\mathbb{R}^3$ from L^∞ into some L^p-space, cf. Lemma 2.7. Second, the assumption (A1) which allows to transfer this invertibility to $\widehat{L}$, cf. Lemma 2.9, and, third, the fact that for suitable

p and s the nonlinearity $\widehat{Q}$ is a bilinear mapping from $L^p_s \times L^p_s$ into L^∞, cf. Corollary 2.3. To state our Hopf-bifurcation theorem for the vorticity formulation (5) we introduce the space

$$\widehat{\mathcal{X}}^p_s := \{\widehat{\omega} = (\widehat{\omega}_n)_{n\in\mathbb{Z}} : \|\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s} < \infty\}, \quad \|\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s} = \sum_{n\in\mathbb{Z}} \|\widehat{\omega}_n\|_{L^p_s}.$$

Under the generic assumption that the cubic coefficient γ in the reduced system defined subsequently in (10) does not vanish, we have:

Theorem 1.1 *Assume (A1)–(A3) with $p \in (3,4)$ and $s > 3(p-1)/p$. Then there exists an $\varepsilon_0 > 0$ such that for all $\alpha = \alpha_c + \varepsilon^2$ with $\varepsilon \in (0, \varepsilon_0)$ there exists a time-periodic solution*

$$\widehat{\omega}^{\mathrm{per}}(\xi, t) = \sum_{n\in\mathbb{Z}} \widehat{\omega}^{\mathrm{per}}_n(\xi) \exp(in\Omega t)$$

to (5)*, with $(\widehat{\omega}^{\mathrm{per}}_n)_{n\in\mathbb{Z}} \in \widehat{\mathcal{X}}^p_s$, $\|\widehat{\omega}_{\mathrm{per}}(\cdot,t)\|_{L^p_s} = \mathcal{O}(\varepsilon)$, and $\Omega - \Omega_c = \mathcal{O}(\varepsilon^2)$.*

Remark 1.2 For the velocity field we obtain, using the Biot-Savart law, cf. Lemma 2.2 below, $(\widehat{u}^{\mathrm{per}}_n) \in \widehat{\mathcal{X}}^{\tilde{p}}_s$ with $\tilde{p} \in [1, 12/7)$. Since $\widehat{g} \in L^p_s$ with $p \in [1,2]$ implies $g \in W^{s,q}$ where $1/p + 1/q = 1$ it follows that $u \in \mathcal{X}^{s,\tilde{q}}$, $1/\tilde{p} + 1/\tilde{q} = 1$, where

$$\mathcal{X}^{s,q} := \{\omega = (\omega_n)_{n\in\mathbb{Z}} : \|\omega\|^{s,q}_{\mathcal{X}} < \infty\}, \quad \|\omega\|_{\mathcal{X}^{s,q}} = \sum_{n\in\mathbb{Z}} \|\omega_n\|_{W^{s,q}}.$$

In particular, by standard results on Fourier series, for

$$u^{\mathrm{per}}(x,t) = \sum_{n\in\mathbb{Z}} u^{\mathrm{per}}_n(x) \exp(in\Omega t)$$

we have $u^{\mathrm{per}} \in C([0,2\pi), W^{s,\tilde{q}}(\mathbb{R}^3))$. From $\tilde{p} \in [1, 12/7)$ we have $\tilde{q} \in (12/5, \infty]$. In this sense, our result complements the result of [Saz94]. Finally, by Sobolev embeddings in space we also have $u^{\mathrm{per}} \in C([0,2\pi), C^0_b(\mathbb{R}^3, \mathbb{R}))$.

2 Preliminary estimates

2.1 Sobolev's embedding theorem in $\mathbf{L}^p_s$ spaces

Sobolev's embedding in L^p_s spaces is as follows.

Lemma 2.1 *For $p \geq r$ and $s > d\frac{p-r}{pr}$ we have the continuous embedding $L^p_s(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$.*

Proof. With $\rho(\xi) = \sqrt{1 + |\xi|^2}$ and Hölder's inequality for $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ we have

$$\|f\|_{L^r} = \|f\rho^s\rho^{-s}\|_{L^r} \leq \|f\rho^s\|_{L^p}\|\rho^{-s}\|_{L^q} = \|f\|_{L^p_s}\|\rho^{-s}\|_{L^q}.$$

We estimate

$$\|\rho^{-s}\|^q_{L^q} = \int_{\mathbb{R}^3} \frac{d\xi}{(1+|\xi|^2)^{\frac{sq}{2}}} = \int_{|\xi|\leq 1} \frac{d\xi}{(1+|\xi|^2)^{\frac{sq}{2}}} + \int_{|\xi|>1} \frac{d\xi}{(1+|\xi|^2)^{\frac{sq}{2}}}.$$

Obviously, the first integral is bounded. For the second integral we find

$$\int_{|\xi|>1} \frac{d\xi}{(1+|\xi|^2)^{\frac{sq}{2}}} \leq C\int_1^\infty \frac{r^{d-1}dr}{(1+r^2)^{\frac{sq}{2}}} \leq C\int_1^\infty \frac{dr}{r^{sq-d+1}}$$

which is bounded for $sq - d + 1 > 1$, i.e., if $sq > d$. ■

2.2 Reconstruction of the velocity from the vorticity

In the following lemma we estimate $\widehat{u}$ in terms of the vorticity $\widehat{\omega}$ in Fourier space, see also, e.g., [GW02] for estimates in x-space using the Biot-Savart law

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y)}{|x-y|^3}dy.$$

Lemma 2.2 *For $\widehat{\omega} \in L^q(\mathbb{R}^3)^3$, $q \in [1,\infty]$, and $j = 1,2,3$, we have*

$$\|i\xi_j\widehat{u}\|_{L^q} \leq C\|\widehat{\omega}\|_{L^q}. \tag{1}$$

Moreover, for every $r \in [1,3)$ and $\tilde{p}, q \in [1,\infty]$ with $1/q = 1/\tilde{p} + 1/r$ there exists a $C > 0$ such that the following holds. If $\widehat{\omega} \in L^{\tilde{p}}(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ then $\widehat{u} \in L^q(\mathbb{R}^3)^3$, and

$$\|\widehat{u}\|_{L^q} \leq C(\|\widehat{\omega}\|_{L^{\tilde{p}}} + \|\widehat{\omega}\|_{L^q}).$$

Proof. The velocity u is defined in terms of the vorticity ω by solving the equations

$$\nabla\times u = \omega \qquad \text{and} \qquad \nabla\cdot u = 0$$

for ω satisfying $\nabla\cdot\omega = 0$. This leads in Fourier space to

$$\begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_3 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix}\begin{pmatrix}\widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3\end{pmatrix} = \begin{pmatrix}\widehat{\omega}_1 \\ \widehat{\omega}_2 \\ \widehat{\omega}_3 \\ 0\end{pmatrix},$$

which is solved by $\widehat{u} = \widehat{M}\widehat{\omega}$ where

$$\widehat{M}(\xi) = -\frac{1}{|\xi|^2}\begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix}.$$

With Hölder's inequality we obtain

$$\|\widehat{u}\|_{L^q} \leq C\left(\|\chi_{\{|\xi|\leq 1\}}\widehat{M}\|_{L^r}\|\widehat{\omega}\|_{L^p} + \|\chi_{\{|\xi|>1\}}\widehat{M}\|_{L^\infty}\|\widehat{\omega}\|_{L^q}\right)$$

with $1/q = 1/p + 1/r$. Hence it remains to estimate terms of the form

$$K_j^\infty(\xi) = \chi_{\{|\xi|>1\}}\frac{i\xi_j}{|\xi|^2} \quad \text{and} \quad K_j(\xi) = \chi_{\{|\xi|\leq 1\}}\frac{i\xi_j}{|\xi|^2}$$

in the spaces $L^\infty(\mathbb{R}^3)$ and $L^r(\mathbb{R}^3)$, respectively. The estimate for K_j^∞ is obvious. For K_j we have

$$\|K_j(\xi)\|_{L^r}^r = \int_{|\xi|\leq 1}\left|\frac{\xi_j}{|\xi|^2}\right|^r d\xi \leq C\int_0^1 \frac{\rho^r}{\rho^{2r}}\rho^2 d\rho = \int_0^1 \frac{d\rho}{\rho^{r-2}},$$

which is bounded for $r < 3$. Estimate (1) follows from $\|i\xi_j\widehat{u}\|_{L^q} \leq \|i\xi_j\widehat{M}(\xi)\|_{L^\infty}\|\widehat{\omega}\|_{L^q} \leq C\|\widehat{\omega}\|_{L^q}$. ■

2.3 Estimates for the bilinear term $\widehat{\mathbf{Q}}(\widehat{\omega}_\mathbf{1}, \widehat{\omega}_\mathbf{2})$

Lemma 2.3 *For $p \in (3/2, \infty]$ and $s > 3(p-1)/p$ there exists a $C > 0$ such that for all $\widehat{\omega}_1, \widehat{\omega}_2 \in L^p_s$ we have*

$$\|\widehat{\omega}_1 * \widehat{u}_2\|_{L^p_s} \leq C\|\widehat{\omega}_1\|_{L^p_s}\|\widehat{\omega}_2\|_{L^p_s}.$$

Proof. Using Young's inequality, Lemma 2.2 with $1 = 1/\tilde{p} + 1/r$, where $r \in [1, 3)$ which yields $\tilde{p} \in (3/2, \infty]$, we have

$$\begin{aligned}\|\widehat{\omega}_1 * \widehat{u}_2\|_{L^p_s} &\leq C\left(\|\widehat{\omega}_1\|_{L^p}\|\widehat{u}_2\|_{L^1} + \|\xi^s\widehat{\omega}_1\|_{L^p}\|\widehat{u}_2\|_{L^1} + \|\widehat{\omega}_1\|_{L^1}\|\xi^s\widehat{u}_2\|_{L^p}\right) \\ &\leq C\big(\|\widehat{\omega}_1\|_{L^p}(\|\widehat{\omega}_2\|_{L^1} + \|\widehat{\omega}_2\|_{L^{\tilde{p}}}) + \|\xi^s\widehat{\omega}_1\|_{L^p}(\|\widehat{\omega}_2\|_{L^1} + \|\widehat{\omega}_2\|_{L^{\tilde{p}}}) + \|\widehat{\omega}_1\|_{L^1}\|\xi^s\widehat{u}_2\|_{L^p}\big).\end{aligned}$$

Now using $\|\xi^s\widehat{u}_2\|_{L^p} \leq C\|\xi^{s-1}\widehat{\omega}_2\|_{L^p}$ as in the proof of (1), and Sobolev's embedding $L^p_s \subset L^1 \cap L^{\tilde{p}}$ for $s > 3(p-1)/p$ and $p > \tilde{p}$, yields the result. ■

Lemma 2.4 *For $p \in (3, 4)$ and $s > 1$ there exists a $C > 0$ such that for all $\widehat{\omega}_1, \widehat{\omega}_2 \in L^p_s$ we have*

$$\|\widehat{\omega}_1 * \widehat{u}_2\|_{L^\infty} \leq C\|\widehat{\omega}_1\|_{L^p_s}\|\widehat{\omega}_2\|_{L^p_s}.$$

Proof. By Young's inequality with $1 = 1/p + 1/q$ and Lemma 2.2 with $1/q = 1/\tilde{q} + 1/r^*$, $r^* \in [1, 3)$, we have

$$\|\widehat{\omega}_1 * \widehat{u}_2\|_{L^\infty} \leq \|\widehat{\omega}_1\|_{L^p}\|\widehat{u}_2\|_{L^q} \leq \|\widehat{\omega}_1\|_{L^p}(\|\widehat{\omega}_2\|_{L^q} + \|\widehat{\omega}_2\|_{L^{\tilde{q}}}).$$

Then

$$\|\widehat{\omega}_1 * \widehat{u}_2\|_{L^\infty} \leq \|\widehat{\omega}_1\|_{L^p}\|\widehat{\omega}_2\|_{L^p_s}$$

by Sobolev's embedding if $L^p_s \subset L^q$ and $L^p_s \subset L^{\tilde{q}}$. This holds for $p \geq \tilde{q}$ and $s > 3\frac{p-\tilde{q}}{p\tilde{q}}$, respectively $p \geq q$ and $s > 3\frac{p-q}{pq}$. With $0 < \delta < 1$, $\tilde{\delta} > 0$ sufficiently small and $s > 1$, these conditions are fulfilled by choosing $p = 3 + \delta$, $q = (3+\delta)/(2+\delta)$, $r^* = 3 - \mathcal{O}(\tilde{\delta})$ and hence $\tilde{q} = 3(3+\delta)/(3 + 2\delta) + \mathcal{O}(\tilde{\delta})$. ■

Remark 2.5 Lemma 2.3 will be used for the noncritical modes associated with $n \neq 0$ in the Liapunov-Schmidt reduction, while Lemma 2.4 will be used for $n = 0$. The upper bound $p < 4$ in Lemma 2.4 is not optimal but it is also obtained from Lemma 2.7 below and, therefore, we omit a more detailed discussion.

Corollary 2.6 *For $p \in (3/2, \infty]$ and $s > 3(p-1)/p$ there exists a $C > 0$ such that for all $\widehat{\omega}_1, \widehat{\omega}_2 \in L^p_s$ we have*

$$\|\widehat{Q}(\widehat{\omega}_1, \widehat{\omega}_2)\|_{L^p_s} \leq C\|\widehat{\omega}_1\|_{L^p_s}\|\widehat{\omega}_2\|_{L^p_s}.$$

Moreover, for $p \in (3, 4)$ and $s > 0$ there exists a $C > 0$ such that for all $\widehat{\omega}_1, \widehat{\omega}_2 \in L^p_s$ we have

$$\|\widehat{Q}(\widehat{\omega}_1, \widehat{\omega}_2)\|_{L^\infty} \leq C\|\widehat{\omega}_1\|_{L^p_s}\|\widehat{\omega}_2\|_{L^p_s}.$$

Proof. This is a direct consequence of Lemmas 2.3 and 2.4. ■

2.4 Estimates for the Oseen operator $\widehat{B}$

The linear operator $\widehat{B}$ which has essential spectrum up to the imaginary axis can be inverted in the following sense.

Lemma 2.7 *Let $s \geq 0$. For $p \geq 1$ we have $\widehat{B}^{-1} i\xi_1 \in L(L^p_s, L^p_s)$. For $1 \leq p < 4$ and $j = 2,3$ we have $\widehat{B}^{-1} i\xi_j \in (L^p_s \cap L^\infty, L^p_s)$.*

Proof. We have

$$\widehat{\omega}(\xi) = \widehat{B}(\xi)^{-1} i\xi_j \widehat{f(\xi)} = -\frac{i\xi_j}{|\xi|^2 + ic\xi_1} \widehat{f(\xi)}.$$

The result for $j = 1$ follows from the uniform boundedness of $\frac{i\xi_1}{|\xi|^2+ic\xi_1}$. For $j = 2,3$, we find

$$\|\widehat{\omega}\|_{L^p} \;\; \leq \;\; C\|\widehat{f}\|_{L^\infty} \int_{|\xi|\leq 1} \left| \frac{i\xi_j}{|\xi|^2 + ic\xi_1} \right|^p d\xi + C\|f\|_{L^p} \left\| \frac{i\xi_j}{|\xi|^2 + ic\xi_1} \chi_{|\xi|\geq 1} \right\|_{L^\infty}.$$

Obviously, $\left\| \frac{i\xi_j}{|\xi|^2+ic\xi_1} \chi_{|\xi|\geq 1} \right\|_{L^\infty}$ is bounded for all $p \in [1,\infty)$. Next we have

$$\begin{aligned}
\int_{|\xi|\leq 1} \left| \frac{i\xi_j}{|\xi|^2 + ic\xi_1} \right|^p d\xi \quad & \leq \quad C \int_0^1 \int_0^1 \int_0^1 \left| \frac{i\xi_j}{|\xi|^2 + ic\xi_1} \right|^p d\xi_1 d\xi_2 d\xi_3 \\
& \leq \quad C \int_0^1 \int_0^1 \int_0^1 \frac{|\xi_j|^p}{|\xi|^{2p} + |c\xi_1|^p} d\xi_1 d\xi_2 d\xi_3 \\
& \underbrace{\leq}_{|\xi^*|^2 = \xi_2^2 + \xi_3^2} \quad C \int_0^1 \int_0^1 \int_0^1 \frac{|\xi_j|^p}{|\xi^*|^{2p}} \frac{1}{1 + \frac{|c\xi_1|^p}{|\xi^*|^{2p}}} d\xi_1 d\xi_2 d\xi_3 \\
& \underbrace{\leq}_{y = \frac{|c\xi_1|}{|\xi^*|^2}} \quad C \int_0^1 \int_0^1 \frac{|\xi_j|^p}{|\xi^*|^{2p-2}} d\xi_2 d\xi_3 \int_0^\infty \frac{1}{1+y^p} dy \\
& \leq \quad C \int_{|\xi^*|\leq\sqrt{2}} \frac{|\xi_j|^p}{|\xi^*|^{2p-2}} d\xi^* \\
& \leq \quad C \int_0^{\sqrt{2}} \int_0^{2\pi} \frac{r^{p+1}}{r^{2p-2}} d\phi dr \;\leq\; C \int_0^{\sqrt{2}} \frac{1}{r^{p-3}} dr
\end{aligned}$$

which is bounded for $p < 4$. The estimates for $s > 0$ are exactly the same. ■

2.5 Compactness properties

Lemma 2.8 *For $p \in (3,4)$ and $s > 3(p-1)/p$ the operators L and B differ by a relatively compact perturbation in L^p_s.*

Proof. By Corollary 2.6, the difference maps L^p_s into $L^p_{s-1} \cap L^\infty$. By the theorem of Riesz [Alt99, Theorem 2.15], this space is compactly embedded in L^p_{s-2} the domain of definition of the sectorial operator B . ■

2.6 Estimates for the operator $\widehat{L}$

Combining the estimates for the operator $\widehat{B}$ from Lemma 2.7 with the assumptions (A1)–(A3) allows us to prove a similar result for the operator $\widehat{L}$.

Lemma 2.9 *Let $s \geq 0$ and assume (A1)–(A3). For $p \geq 1$ we have $\widehat{L}^{-1} i\xi_1 \in L(L^p_s, L^p_s)$. For $1 < p < 4$ and $j = 2,3$ we have $\widehat{L}^{-1} i\xi_j \in L(L^p_s \cap L^\infty, L^p_s)$.*

Proof. We have $\hat{L} = \widehat{B} + \widehat{G}$ with $\widehat{G}\cdot = 2i\xi \cdot \widehat{Q}(\widehat{\omega}_\alpha, \cdot)$. Then $(\widehat{B} + \widehat{G})w = i\xi_j f$ is equivalent to

$$\widehat{B}(I + \widehat{B}^{-1}\widehat{G})w = i\xi_j f \quad \text{resp.} \quad w = (I + \widehat{B}^{-1}\widehat{G})^{-1}\widehat{B}^{-1} i\xi_j f.$$

The existence of $(I + \widehat{B}^{-1}\widehat{G})^{-1}$ is established as follows. By Lemma 2.8, the operator $\widehat{B}^{-1}\widehat{G} : L^p_s \to L^p_s$ is compact. Hence, $I + \widehat{B}^{-1}\widehat{G}$ is Fredholm with index 0. If $(I + \widehat{B}^{-1}\widehat{G})w = 0$ had a nontrivial solution, then $\widehat{L}w = \widehat{B}(I + \widehat{B}^{-1}\widehat{G})w = 0$ would also have a nontrivial solution, which would contradict (A1). Therefore, the Fredholm property implies the existence of $(I+\widehat{B}^{-1}\widehat{G})^{-1} : L^p_s \to L^p_s$. The estimates for $\widehat{L}$ now follow from

$$\|w\|_{L^p_s} \leq \|(I + \widehat{B}^{-1}\widehat{G})^{-1}\|_{L^p_s \to L^p_s} \|\widehat{B}^{-1} i\xi_j f\|_{L^p_s}$$

and Lemma 2.7. ■

Remark 2.10 The nonlinearity $i\xi \cdot \widehat{Q}(\widehat{\omega}, \widehat{\omega})$ contains all combinations of all components of ξ and $\widehat{\omega}$. Therefore, below we shall need $1 < p < 4$ when estimating $\widehat{L}^{-1} i\xi \cdot \widehat{Q}(\widehat{\omega}, \widehat{\omega})$ and the estimate for $\widehat{L}^{-1} i\xi_1$ is only for the sake of completeness. Similarly, it is easy to see that in fact $\widehat{L}^{-1} i\xi \cdot \in L(L^p_s \cap L^\infty_s, L^p_{s+1})$. However, the gain in weight ξ is not helpful since the difficulties arise near $\xi = 0$.

3 Proof of the Hopf-Bifurcation theorem

For small $|\alpha - \alpha_c|$ and $|\Omega - \Omega_c|$ we look for $2\pi/\Omega$-time periodic solutions of (5), i.e., we look for solutions $\widehat{\omega}$ of

$$\partial_t \widehat{\omega} = \widehat{L}\widehat{\omega} + i\xi \cdot \widehat{Q}(\widehat{\omega}, \widehat{\omega}) \tag{1}$$

which satisfy $\widehat{\omega}(\xi, t) = \widehat{\omega}(\xi, t + 2\pi/\Omega)$. This system has the trivial solution $\widehat{\omega} = 0$. By assumption (A2), the linear operator $(\widehat{L} \pm i\Omega I)_{n\in\mathbb{Z}}$ is not invertible for $\alpha = \alpha_c$. Therefore, the implicit function theorem no longer applies and the necessary condition for the bifurcation of time-periodic solutions is satisfied. In order to establish a Hopf-bifurcation, we use a Lyapunov-Schmidt reduction to reduce the bifurcation problem to a finite-dimensional one. Thus, we make the ansatz

$$\widehat{\omega}(\xi, t) = \sum_{n\in\mathbb{Z}} \widehat{\omega}_n(\xi) \exp(in\Omega t),$$

with

$$(\widehat{\omega}_n) \in \widehat{\mathcal{X}}^p_s := \{(\widehat{\omega}_n)_{n\in\mathbb{Z}} : \|\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s} < \infty\}, \quad \|\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s} = \sum_{n\in\mathbb{Z}} \|\widehat{\omega}_n\|_{L^p_s}.$$

We introduce projections P_n onto the n-th Fourier mode, i.e.,

$$(P_n \widehat{\omega})(\xi) = \frac{\Omega}{2\pi} \int_0^{\frac{2\pi}{\Omega}} \exp(in\Omega t)\widehat{\omega}(\xi, t) dt,$$

and split (1) into the infinitely many equations for the Fourier modes $\widehat{\omega}_n$, namely

$$in\Omega\widehat{\omega}_n = \widehat{L}\widehat{\omega}_n + i\xi \cdot N_n(\widehat{\omega}), \quad n \in \mathbb{Z}, \tag{2}$$

with

$$N_n(\widehat{\omega}) = \sum_{m\in\mathbb{Z}} \widehat{Q}(\widehat{\omega}_{n-m}, \widehat{\omega}_m).$$

To reduce (2) to a finite dimensional bifurcation problem we invert the linear operators $in\Omega I - \widehat{L}$ in the biggest possible subspaces. For $n = \pm 1$, let $P_{n,c}$ be the $\widehat{L}$–invariant orthogonal projection onto the subspace spanned by the eigenvector associated with the eigenvalue $in\Omega$, let $P_{n,s} = 1 - P_{n,c}$, and consider

$$\begin{aligned}
in\Omega\widehat{\omega}_n &= \widehat{L}\widehat{\omega}_n + i\xi \cdot N_n(\widehat{\omega}), \qquad (n = \pm 2, \pm 3 \ldots), && (3)\\
in\Omega\widehat{\omega}_{n,s} &= \widehat{L}\widehat{\omega}_{n,s} + P_{n,s} i\xi \cdot N_n(\widehat{\omega}), \qquad (n = \pm 1), && (4)\\
0 &= \widehat{L}\widehat{\omega}_0 + i\xi \cdot N_0(\widehat{\omega}), && (5)\\
in\Omega\widehat{\omega}_{n,c} &= \widehat{L}\widehat{\omega}_{n,c} + P_{n,c} i\xi \cdot N_n(\widehat{\omega}), \qquad (n = \pm 1). && (6)
\end{aligned}$$

Due to the spectral assumptions on $\widehat{L}$, we have in L^p_s the invertibility of $in\Omega I - \widehat{L}$ for $n = \pm 2, \pm 3, \ldots$, the invertibility of $(in\Omega I - \widehat{L})P_{n,s}$ for $n = \pm 1$, and, moreover, the existence of $\widehat{L}^{-1} i\xi\cdot$ as a bounded operator from $L^p_s \cap L^\infty$ to L^p_s if $p \in (1, 4)$, cf. Lemma 2.9. By Corollary 2.6, the nonlinear terms N_n map L^p_s into L^p_s if $p > 3/2$ and $s > 3(p-1)/p$, and into L^∞ if $p \in (3, 4)$ and $s > 1$. Thus we rewrite (3)–(5) as

$$\begin{aligned}
\widehat{\omega}_n &= (in\Omega I - \widehat{L})^{-1} i\xi \cdot N_n(\widehat{\omega}), \qquad (n = \pm 2, \pm 3 \ldots), && (7)\\
\widehat{\omega}_{n,s} &= (in\Omega I - \widehat{L})^{-1} P_{n,s} i\xi \cdot iN_n(\widehat{\omega}), \qquad (n = \pm 1), && (8)\\
\widehat{\omega}_0 &= \widehat{L}^{-1} i\xi \cdot N_0(\widehat{\omega}), && (9)
\end{aligned}$$

and expect that (7)–(9) can be solved for $\omega_n \in L^p_s$, $n \neq \pm 1$, $\omega_{n,s} \in L^p_s$, $n = \pm 1$, and $\omega_0 \in L^p_s$ in terms of $\omega_{1,c} = P_{1,c}\omega_1 \in L^p_s$ and $\omega_{-1,c} = P_{-1,c}\omega_{-1} \in L^p_s$, if $p \in (3, 4)$ and $s > 3(p-1)/p$. In detail, we use the following lemmas.

Lemma 3.1 *Let $\widehat{M} = (\widehat{M}_l)_{l\in\mathbb{Z}}$ with $\widehat{M}_l : L^p_s \to L^p_s$. Defining the action of $\widehat{M}$ on $\widehat{\omega} = (\widehat{\omega}_l)_{l\in\mathbb{Z}}$ by $(\widehat{M}\widehat{\omega})_l = \widehat{M}_l\widehat{\omega}_l$ we find*

$$\|\widehat{M}\widehat{\omega}\|_{\widehat{\mathcal{X}}^k_s} \leq \sup_{l\in\mathbb{Z}} \|\widehat{M}_l\|_{L^p_s \mapsto L^p_s} \|\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s}.$$

Proof. $\|\widehat{M}\widehat{\omega}\|_{\widehat{\mathcal{X}}^p_s} = \sum_{l\in\mathbb{Z}} \|\widehat{M}_l\widehat{\omega}_l\|_{L^p_s} \leq \sup_{l\in\mathbb{Z}} \|\widehat{M}_l\|_{L^p_s \mapsto L^p_s} \sum_{l\in\mathbb{Z}} \|\widehat{\omega}_l\|_{L^p_s}$. ■

Lemma 3.2 *Let $p > 3/2$ and $s > 3(p-1)/p$. Then there exists a $C > 0$ such that for $\widehat{\omega} \in \widehat{\mathcal{X}}^p_s$ we have*

$$\|(N_n(\widehat{\omega}, \widehat{\omega}))_{n\in\mathbb{Z}}\|_{\widehat{\mathcal{X}}^p_s} \leq C\|\widehat{\omega}\|^2_{\widehat{\mathcal{X}}^p_s}.$$

Moreover, for $p \in (3, 4)$ and $s > 1$ we have $\|N_0(\widehat{\omega}, \widehat{\omega})\|_{L^\infty} \leq C\|\widehat{\omega}\|^2_{\widehat{\mathcal{X}}^p_s}$.

Proof. By Corollary 2.6, we have

$$\begin{aligned}
\|(N_n(\widehat{\omega}, \widehat{\omega}))_{n\in\mathbb{Z}}\|_{\widehat{\mathcal{X}}^p_s} &= \sum_{l\in\mathbb{Z}} \|(\widehat{Q}(\widehat{\omega}, \widehat{\omega}))_l\|_{L^p_s} = \sum_{l\in\mathbb{Z}} \|\sum_{j\in\mathbb{Z}} \widehat{Q}(\widehat{\omega}_{l-j}, \widehat{\omega}_j)\|_{L^p_s}\\
&\leq C \sum_{l\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \|\widehat{\omega}_{l-j}\|_{L^p_s} \|\widehat{\omega}_j\|_{L^p_s} \leq C \sum_{l\in\mathbb{Z}} \|\widehat{\omega}_l\|_{L^p_s} \sum_{j\in\mathbb{Z}} \|\widehat{\omega}_j\|_{L^p_s} = C\|\widehat{\omega}\|^2_{\widehat{\mathcal{X}}^p_s},
\end{aligned}$$

and the L^∞_s-estimate is also a trivial consequence of Corollary 2.6. ■

Lemma 3.3 *There exists a $C > 0$ such that*

$$\begin{aligned} \|(in\Omega I - \widehat{L})^{-1} i\xi \cdot \|_{L^p_s \mapsto L^p_s} &\leq C, \qquad n \in \mathbb{Z}\backslash\{-1,0,1\}, \\ \|(in\Omega I - \widehat{L})^{-1} \widehat{P}_{n,s} i\xi \cdot \|_{L^p_s \mapsto L^p_s} &\leq C, \qquad n = \pm 1. \end{aligned}$$

Proof. $\widehat{L} = \widehat{B} + 2i\xi \cdot \widehat{Q}(\widehat{\omega}_c, \widehat{\omega})$ is sectorial in L^p_s since $\widehat{B}$ is a sectorial operator in L^p_s and $2i\xi \cdot \widehat{Q}(\widehat{\omega}_c, \widehat{\omega})$ is $\widehat{B}$ relatively bounded (in fact relatively compact due to Lemma 2.8). Thus, for the invertibility of $in\Omega I - \widehat{L}$ it is sufficient that the spectrum is strictly bounded away from zero, which holds due to (A3). The estimates follow from Lemma 2.9. ■

To proceed, we abbreviate (7)–(9) as $F = F(\widehat{\omega}_c, \widehat{\omega}_s) = 0$ where

$$\widehat{\omega}_c = (\ldots, 0, \widehat{\omega}_{-1c}, 0, \widehat{\omega}_{1c}, 0, \ldots) \qquad \text{and} \qquad \widehat{\omega}_s = (\ldots, \widehat{\omega}_{-2}, \widehat{\omega}_{-1s}, \widehat{\omega}_0, \widehat{\omega}_{1s}, \widehat{\omega}_2, \ldots).$$

By Lemmas 3.1 to 3.3, $F : \widehat{\mathcal{X}}^p_s \times \widehat{\mathcal{X}}^p_s \to \widehat{\mathcal{X}}^p_s$ is well defined and smooth for $p \in (3,4)$ and $s > 3(p-1)/p$. In order to resolve $F(\omega_c, \omega_s) = 0$ with respect to $\widehat{\omega}_s$ we have to prove $F(0,0) = 0$ and the invertibility of $D_{\omega_s} F(0,0) : \widehat{\mathcal{X}}^p_s \to \widehat{\mathcal{X}}^p_s$. The first condition trivially holds, and we have $D_{\widehat{\omega}_s} F(0,0) = I$. Thus, there exists a unique smooth function $\widehat{\omega}_s = \widehat{\omega}_s(\widehat{\omega}_c)$ with $\widehat{\omega}_s : \widehat{\mathcal{X}}^p_s \mapsto \widehat{\mathcal{X}}^p_s$ satisfying $\|\widehat{\omega}_s(\widehat{\omega}_c)\|_{\widehat{\mathcal{X}}^p_s} \leq C\|\widehat{\omega}_c\|^2_{\widehat{\mathcal{X}}^p_s}$.

Thus, the bifurcation problem can be reduced to a problem for $\omega_{1,c}$ and $\omega_{-1,c}$ alone which has exactly the same properties as the one in case of a classical Hopf-bifurcation. Thus, we only sketch the concluding arguments. Setting $\omega_n = A_n \varphi_n$, $n = \pm 1$, where $\widehat{\varphi}_n \in L^p(s)$ are the eigenfunctions associated with the eigenvalues $\pm i\Omega_c$ and $A_n \in \mathbb{C}$ with $A_{-1} = \overline{A_1}$, we find the reduced problem

$$\begin{aligned} g_1(\alpha - \alpha_c, \Omega - \Omega_c, A_1, A_{-1}) &= 0, \\ g_{-1}(\alpha - \alpha_c, \Omega - \Omega_c, A_1, A_{-1}) &= 0. \end{aligned}$$

Since we have an autonomous problem, the reduced problem has to be invariant under $A_1 \mapsto A_1 \exp(i\phi)$ and $A_{-1} \mapsto A_{-1} \exp(-i\phi)$. Therefore, g_1 and g_{-1} are of the form

$$\begin{aligned} A_1 \tilde{g}_1(\alpha - \alpha_c, \Omega - \Omega_c, |A_1|^2) &= 0, \\ A_{-1} \tilde{g}_{-1}(\alpha - \alpha_c, \Omega - \Omega_c, |A_1|^2) &= 0. \end{aligned}$$

Introducing polar coordinates $A_1 = r\exp(i\phi)$ yields

$$\begin{aligned} (\alpha - \alpha_c) + \gamma r^2 + \mathcal{O}(|\alpha - \alpha_c|^2 + |\Omega - \Omega_c|^2 + r^4) &= 0, \\ \Omega - \Omega_0 + \mathcal{O}(|r|^2 + |\alpha - \alpha_c|^2 + |\Omega - \Omega_c|^2) &= 0, \end{aligned} \tag{10}$$

which is the well-known reduced system for a Hopf-bifurcation. For given $\alpha - \alpha_c$ the second equation can be solved with respect to $\Omega - \Omega_c$ and then the first equation with respect to r. Therefore, we are done. ■

Acknowledgment. The paper is partially supported by the Deutsche Forschungsgemeinschaft DFG under the grant Schn520/4-1/2.

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