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Amplitude equations – an invitation to multi-scale analysis

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Abstract. Despite the steady increase in computing power many nonlinear partial differential equations still cannot be solved by brute-force numerical methods in acceptable time. However, often the method of multiple scales resulting in so-called amplitude equations can be used to first reduce the complexity of the problem. The amplitude equations give valuable mathematical insight and, moreover, can be treated numerically by orders of magnitude faster and hence be used to approximate physically or technologically interesting solutions with high accuracy. We explain this method using some simple examples, starting with ODEs and progressing to some pattern-forming systems and nonlinear wave equations.

Contents

1 Introduction

Despite the ever increasing computing power and improved algorithms, for many models it is useful or even necessary to first reduce the problem, or, in other words, approximate a complicated system by a reduced system. Here, reduced system in particular means that it lives on longer spatio-temporal scales. Then

a) the reduced model (together with the process of reduction) allows a much better and comprehensive analytical understanding of the problem, sometimes even some closed-form analytical solution;

b) the reduced model *often* falls into some well-known class of equations, for which a great deal is already known;

c) the reduced model *usually* reduces computational costs by (several) orders of magnitude compared to the original system.

It is the purpose of this lecture to explain a) – c) using some simple examples ranging from ordinary differential equations (ODE) to nonlinear partial differential equations (PDEs), describing, e.g., some nonlinear oscillations, pattern-forming systems, nonlinear optics, and some simple fluid mechanics. Thus, concerning computational methods this is rather a lecture about possible analytical steps *before* one starts actual numerics. Nevertheless, as already said in c), these steps often allow to speed up numerical simulations by orders of magnitude. Another message is: besides its intrinsic value, analysis also helps a lot to set up the "right" numerics for a given problem.

The problems we have in mind concern so-called multi-scale problems, where there is a separation between different temporal and spatial scales, e.g., a fast time-scale and a slow time-scale. *Multiscale analysis* is also called *amplitude* or *modulation formalism*, and in a broad sense both are sometimes also refered to as *averaging* or *homogenization*, although in a narrower sense there are substantial differences between all four methods. Moreover, all these methods sometimes are subsumed under the name of *asymptotic expansion*.

Additionally to a) – c) above, there is a point

d) validity of the reduction of the original to the reduced system: does the formalism produce good approximations on the relevant time-scale?

As we shall see in the very first example below (Example 1.2), naive asymptotic expansions may well fail, and answering d) is often not easy. In this lecture, the

validation will only be in the sense that (numerical) solutions of the reduced problem will be compared with (numerical) solutions of the original system. This obviously very much contradicts the spirit of the reduction, and there are methods to define and prove validity without having to solve the original system. These, however, are mathematically somewhat involved and therefore will not be discussed in this lecture, although we will give some hints to the literature.

We also want to stress that even if d) holds that does not mean that *all* solutions of the original system can be approximated via the reduced system. In the reduction process we usually restrict to specific classes of solutions, and the best one can in general hope for is that the original system has solutions in this class. This may fail, and even if it holds there may still be other solutions of the original system not at all described by the reduced system.

In the remainder of this introduction we start with multiscale analysis using some ODE examples. In Sec. 2 we consider some toy problems for pattern formation which can be reduced to Ginzburg–Landau equations, while Sec. 3 treats some nonlinear wave equations from Nonlinear Optics which can be reduced to the Nonlinear Schrödinger equation. In Sec. 4 we consider flow in porous media as one of the simplest possible systems from fluid dynamics.

As prerequisite for this lecture we only assume some understanding of ODE, as taught in most undergraduate science curricula. Some basic understanding of PDEs and numerical methods for ODEs and PDEs is helpful but not strictly necessary.

There is a vast and somewhat scattered literature on the amplitude formalism. The methods of center manifold reduction, averaging, and bifurcation for nonlinear ODEs are treated, for instance, in [GH83, Wig88, Ver96]. For the amplitude formalism for PDEs see for instance [Man92, CH93, Deb05], and [PS08] for problems including stochastical effects. A highly recommended textbook containing multi-scale analysis and much more from an applied point of view is [Kee88]. All these books go way beyond the scope of this lecture. As already said, here the purpose is to give some introductory examples to nonspecialists (from mathematics or computational or applied science in general) which we hope will help the reader to get some understanding of multiple scales and, for instance, will motivate him or her to investigate whether there might be multiple scales at work in a given problem, instead of immediately setting up some brute-force "discretizing everthing in sight" numerical method.

1.1 ODE examples

First we recall some basic formulas for linear (scalar) ODEs with constant coeffients. Consider

$$\partial_t^2 u + a_1 \partial_t u + a_0 u = g(t), \quad u = u(t) \in \mathbb{R}, \tag{1}$$

with initial conditions $u(0) = u_0, \partial_t u(0) = u_1$, where $a_1, a_0 \in \mathbb{R}$ and $g \in C(\mathbb{R}, \mathbb{R})$ is an inhomogeneity. Here and in the following, the symbols $\partial_\star$ denotes the derivatives of a function with respect to the variable $\star$. If u depends only on one variable, i.e., $u = u(t)$, then we may as well write $\partial_t u(t) = u'(t)$. However, later we shall consider

PDEs where $u = u(t, x)$. Then $\partial_t u$ denotes the partial derivative with respect to t, and to unify notation we use the symbol $\partial_\star$ throughout.

To solve (1) we first consider the homogeneous system

$$\partial_t^2 u + a_1 \partial_t u + a_0 u = 0. \tag{2}$$

The ansatz $u(t) = \mathrm{e}^{\lambda t}$ yields the *characteristic equation*

$$P(\lambda) := \lambda^2 + a_1 \lambda + a_0 = 0. \tag{3}$$

In the case that we have two distinct roots $\lambda_1 \neq \lambda_2$ the general solution of (2) is given by

$$u_h(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} \text{ where } c_{1,2} \in \mathbb{C} \text{ are arbitrary constants.} \tag{4}$$

Here and henceforth we make extensive use of complex calculus. However, since (1) is a real equation, i.e., $a_1, a_2, u_0, u_1 \in \mathbb{R}$ and we look for $u \in C^2(\mathbb{R}, \mathbb{R})$, we have a number of symmetries: the roots $\lambda_{1,2}$ are either both real, or they are a complex conjugate pair $\lambda_1 = \lambda_{1r} + \mathrm{i}\lambda_{1i}$ and $\lambda_2 = \overline{\lambda}_1 = \lambda_{1r} - \mathrm{i}\lambda_{1i}$, where $\mathrm{i} = \sqrt{-1}$. If $\lambda_{1,2} \in \mathbb{R}$, then c_1, c_2 in (4) are also real, while for $\mathrm{Im}\lambda_1 \neq 0$ we have $c_2 = \overline{c}_1$. Equivalently, u_h may then also be written as

$$u_h(t) = \tilde{c}_1 \mathrm{e}^{\lambda_{1r} t} \cos(\lambda_{1i} t) + \tilde{c}_2 \mathrm{e}^{\lambda_{1r} t} \sin(\lambda_{1i} t) \text{ where } \tilde{c}_{1,2} \in \mathbb{R}. \tag{5}$$

The general solution of (1) is given as $u(t) = u_h(t) + u_s(t)$ where u_s is an arbitrary special solution of $\partial_t^2 u + a_1 \partial_t u + a_0 u = g(t)$, and where u_h is then chosen such that u fulfills the initial conditions.

To find u_s one may use the variation of constant formula. However, very often g is of the form $g(t) = \mathrm{e}^{\mu t}$ for some $\mu \in \mathbb{C}$, or a sum of such terms, where we shall exploit linearity of (2) and again complex notation. In this case there are explicit formulas for u_s. For instance, $\cos(t) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})$ and if u_{s1} is a special solution for $g_1(t) = \frac{1}{2}\mathrm{e}^{\mathrm{i}t}$ and u_{s2} a special solution for $g_2(t) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}t}$, then $u_s = u_{s1} + u_{s2}$ is a special solution for $g(t) = g_1(t) + g_2(t)$. The formulas for $g(t) = \mathrm{e}^{\mu t}$ now read:

(i) If μ is not a resonance value, i.e., if μ is not a root of (3), then $u_s(t) = \mathrm{e}^{\mu t}/P(\mu)$. In particular, if $\mathrm{Re}\mu = 0$, then the solution stays bounded for all times.

(ii) If $P(\mu) = 0$, i.e., $\mu = \lambda_1$ or $\mu = \lambda_2$, then we have so-called secular growth, i.e., there exists a special solution of (1) of the form $u_s(t) = \alpha t \mathrm{e}^{\mu t}$.

Similar formulas exist for constant coefficient linear ODEs of arbitrary order, see any textbook dealing with ODEs or applied mathematics.

Example 1.1 Consider an oscillator with eigenfrequency $\omega_0 = 1$ driven with frequency ω, i.e.,

$$\partial_t^2 u + u = 2\cos(\omega t) = \mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}, \quad u(0) = \partial_t u(0) = 0. \tag{6}$$

$P(\lambda) = \lambda^2 + 1 = 0$ yields $\lambda = \pm\mathrm{i}$ (eigenfrequency 1), hence

$$u_h(t) = c_1 \mathrm{e}^{\mathrm{i}t} + c_2 \mathrm{e}^{-\mathrm{i}t} = \tilde{c}_1 \cos(t) + \tilde{c}_2 \sin(t).$$

Thus, if $\omega \neq 1$, then $P(\mu) \neq 0$, where $\mu = \mathrm{i}\omega$, hence

$$u_s(t) = \frac{1}{P(\mathrm{i}\omega)}\mathrm{e}^{\mathrm{i}\omega t} + \frac{1}{P(-\mathrm{i}\omega)}\mathrm{e}^{-\mathrm{i}\omega t} = \frac{2}{1-\omega^2}\cos(\omega t),$$

and the solution of the initial value problem (6) is given by $u(t) = \frac{2}{1-\omega^2}(\cos(\omega t) - \cos(t))$. Thus, for ω close to (but not equal to) 1 the solution becomes large but remains (quasi)periodic (and hence bounded for all t), see Fig.1 for $\omega = 0, \omega = \sqrt{1/2}$ and $\omega = \sqrt{0.9}$. However, if $\omega = 1$, then $u_s(t) = t\sin t$, which is also the solution of (6), and which grows without bounds. ⌋

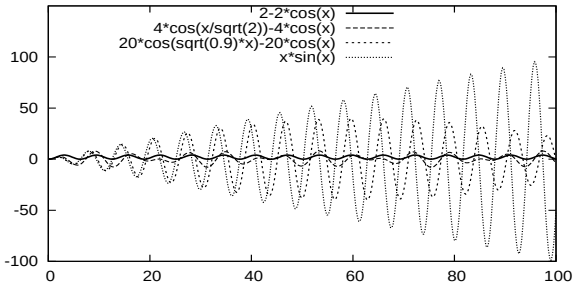

Figure 1: Resonance catastrophe for (6) as $\omega \nearrow 1$.

In applications, ODEs often involve some small parameter. We consider two simple examples to motivate, introduce and illustrate the method of multiple scales which will be transfered to PDEs below.

Example 1.2 Consider the weakly damped oscillator

$$\partial_t^2 u + 2\varepsilon\partial_t u + u = 0, \quad u(t) \in \mathbb{R}, \quad u(0) = a \in \mathbb{R}, \quad \partial_t u(0) = 0, \quad 0 < \varepsilon \ll 1. \tag{7}$$

Using the above calculus, the explicit exact solution is

$$u(t) = \mathrm{e}^{-\varepsilon t}(a\cos(\omega t) + \frac{\varepsilon a}{\omega}\sin(\omega t)), \text{ where } \omega = \sqrt{1-\varepsilon^2}.$$

However, we might also try an expansion in ε, i.e., $u(t) = u_0(t) + \varepsilon u_1(t) + \mathcal{O}(\varepsilon^2)$. Plugging this ansatz into (7) and sorting with respect to powers in ε yields

$$\begin{aligned}
&\mathcal{O}(\varepsilon^0): \quad u_0''(0) + u_0 = 0, \quad u_0(0) = a,\ u_0'(0) = 0 \quad \Rightarrow u_0(t) = a\cos t,\\
&\mathcal{O}(\varepsilon^1): \quad u_1'' + u_1 = 2a\sin t, \quad u_1(0) = 0,\ u_1'(0) = 0 \quad \Rightarrow u_1(t) = -at\cos t + a\sin t,
\end{aligned}$$

and hence $u_{\mathrm{app}_1}(t) = a\cos t - \varepsilon t a\cos t + \varepsilon a\sin t + \mathcal{O}(\varepsilon^2)$. Comparing with u shows that the expansion only makes sense for $t = \mathcal{O}(\varepsilon^{-1})$, and becomes completely useless after that, and this shows that formal expansions may well fail on natural time scales.

With some physical (or mathematical) insight, we may however directly see from (7) that $2\varepsilon\partial_t u$ corresponds to a weak or slow damping, and hence suspect that there are two time-scales involved in (7). Thus we may try a multi-scale ansatz of the form

$$u(t) = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} + \varepsilon u_1(t), \tag{8}$$

with $\omega \in \mathbb{R}$ an a priori unknown (fast) frequency, and where $A = A(\tau) \in \mathbb{C}$ is a slowly varying (complex valued) amplitude. The symbol cc stands for "complex conjugate", i.e., $A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega t} + \overline{A}(\varepsilon t)\mathrm{e}^{-\mathrm{i}\omega t}$. Then, e.g, $\partial_t u = (\mathrm{i}\omega + \varepsilon\partial_\tau)A\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} + \varepsilon\partial_t u_1$, and plugging into (7) we obtain

$$\begin{aligned} \mathcal{O}(\varepsilon^0): &\quad -\omega^2 + 1 = 0, \quad A(0) = a/2 \qquad \Rightarrow \omega = 1, \\ \mathcal{O}(\varepsilon^1): &\quad u_1'' + u_1 = -2\mathrm{i}(\partial_\tau A + A)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \end{aligned} \tag{9}$$

together with appropriate initial conditions for u_1. Now, since A varies on the long time scale, $\partial_\tau A + A$ should be considered to be constant in (9). Thus, to avoid secular growth of u_1 we obtain the so-called *solvability condition* $\partial_\tau A + A = 0$, from which we obtain $A(\tau) = \mathrm{e}^{-\tau}A(0)$. In principle we could now solve for u_1, which however is often omitted: all we want to know is that *there exists* a bounded solution u_1, *provided* that $\partial_\tau A + A = 0$. We thus obtain

$$u_{\mathrm{app}_2}(t) = A(\tau)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc} + \mathcal{O}(\varepsilon) = a\mathrm{e}^{-\varepsilon t}\cos(t) + \mathcal{O}(\varepsilon),$$

which at least is a much better approximation of the true solution than u_{app_1}, see Fig. 2. In fact, solving for u_1 and subsequently for higher-order terms we can make the approximation arbitrary good, uniformly for arbitrary large times.

The equation $\partial_\tau A = -A$ is called the *amplitude equation*, and here can be solved explicitly, like the original system. However, already in simple nonlinear ODEs in general neither the original equation nor the amplitude equation can be solved explicitly. We also like to stress that although the amplitude equation is usually a bit "simpler", this is not the essential characteristic. The main points are that the amplitude equation often falls into some universality class and that it describes the system on long scales. Thus, if one has to use numerical methods, then the numerical costs are greatly reduced. For instance, in the present example we would then have reduced the numerical costs by a factor $1/\varepsilon$, e.g., by factor 10 if $\varepsilon = 0.1$. (Much) more drastic cost reductions occur for PDEs, see Secs. 2 – 4. ⌋

Remark 1.3 For the mathematically inclined reader we remark that the name "solvability condition" in (9) is due to the Fredholm alternative theorem, see, e.g. [Kee88], of which we only state the following matrix version: For $L \in \mathbb{R}^{n\times n}$ the equation $Lu = g$ has a solution $u \in \mathbb{R}^n$ if and only if we have the solvability condition $\langle g, v\rangle = 0$ for every v in the null space $\ker(L^*) := \{v \in \mathbb{R}^n : L^*v = 0\}$ of the adjoint of L.

The name "alternative" comes from the following reformulation: *either* $Lu = g$ has a unique solution, *or* there there exists a $v \in \ker(L^*)$ with $\langle g, v\rangle \neq 0$. In the latter

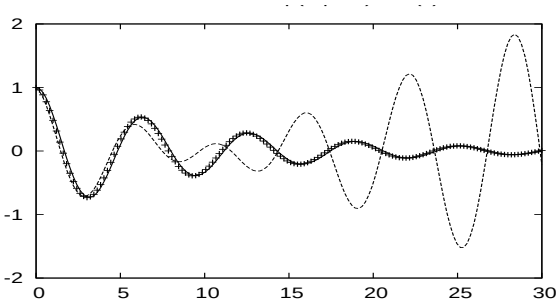

Figure 2: Exact solution and the two approximations for (7); $\varepsilon = 0.1$, $a = 1$.

case, there may be no solution of $Lu = g$ (if, e.g., $L^* = L$ and $g \in \ker(L)$) or infinitely many solutions (if, e.g., $L^* = L, g = 0$ and $\ker(L) \neq \emptyset$).

This generalizes immediately to bounded linear operators L in Hilbert spaces, and with some more effort also to unbounded (Fredholm) operators.

Now, for simplicity, consider (9) as an equation in the Hilbert space H of 2π–periodic functions equipped, e.g., with the scalar product $\langle u, v\rangle = \int_0^{2\pi} u(t)\overline{v}(t)\,\mathrm{d}t$. The left hand side $Lu_1 := u_1'' + u_1$ in (9) then is a linear operator in H, and from integration by parts we have $L = L^*$, i.e., L is selfadjoint. Since $\ker(L) = \mathrm{span}\{\mathrm{e}^{\pm \mathrm{i}t}\}$, the solvability condition $\langle g, v\rangle = 0$ from the Fredholm alternative becomes $\left\langle -2\mathrm{i}(\partial_\tau A + A)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}, \mathrm{e}^{\pm \mathrm{i}t}\right\rangle = 0$ which yields $\partial_\tau A + A = 0$. ⌋

Example 1.4 The van der Pol equation is given by

$$\partial_t^2 u + \varepsilon(u^2 - \alpha)\partial_t u + u = 0, \quad u(t) \in \mathbb{R}, \tag{10}$$

where $\alpha > 0$ and $0 \leq \varepsilon \ll 1$ are some parameters, and as initial conditions we take $u(0) = a$ and $u'(0) = 0$. This describes some oscillator with small amplitude-dependent damping. It is known (and might be expected from the form of the equation), that for every fixed $\alpha > 0$ and small $\varepsilon > 0$ there is a unique periodic solution, a so-called *limit-cycle*, which however cannot be given in closed form. For $\varepsilon = 0$ we have solutions $u(t) = A\mathrm{e}^{\mathrm{i}t} + \mathrm{cc}$ with $A \in \mathbb{C}$ arbitrary, and thus for $\varepsilon > 0$ we try a two-scale ansatz of the form

$$u(t) = A(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{cc} + \varepsilon u_1(t). \tag{11}$$

Using $u^2 = A^2\mathrm{e}^{2\mathrm{i}\omega t} + 2|A|^2 + \overline{A}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \mathcal{O}(\varepsilon^2)$ this yields

$$\begin{aligned} &\mathcal{O}(\varepsilon^0): \quad -\omega^2 + 1 = 0, \quad \Rightarrow \omega = 1, \\ &\mathcal{O}(\varepsilon^1): \quad u_1'' + u_1 = \mathrm{i}(-2\partial_\tau A + \alpha A - A|A|^2)\mathrm{e}^{\mathrm{i}t} - \mathrm{i}A^3\mathrm{e}^{3\mathrm{i}t} + \mathrm{cc}, \end{aligned} \tag{12}$$

and thus the solvability condition

$$\partial_\tau A = \frac{1}{2}A(\alpha - |A|^2), \tag{13}$$

which is often called *Landau* equation, and which has the following phase symmetry: setting $A(\tau) = \rho(\tau)\mathrm{e}^{\mathrm{i}\phi(\tau)}$ we obtain $A' = (\rho' + \mathrm{i}\phi'\rho)\mathrm{e}^{\mathrm{i}\phi} = \frac{1}{2}\rho(\alpha - \rho^2)\mathrm{e}^{\mathrm{i}\phi}$, and for $\rho \neq 0$ this is equivalent to $\rho' = \frac{1}{2}\rho(\alpha - \rho^2), \phi' = 0$. From this, or directly from (13) we can see that $|A|$ converges to $\sqrt{\alpha}$, which predicts that u approaches the circle with radius $2\sqrt{\alpha}$ up to $\mathcal{O}(\varepsilon)$ terms. Incidentically, although nonlinear, (13) can again be explicitly solved. For $r = \rho^2$ we find $r' = r(\alpha - r)$, with solution (substitute $v = 1/r$ to obtain $v' = -\alpha v + 1$) $r(t) = \alpha r_0/(r_0 + (\alpha - r_0)\mathrm{e}^{-\tau})$, and hence

$$\rho(\tau) = \rho_0 \left(\frac{\alpha}{(\alpha - \rho_0^2)\mathrm{e}^{-\alpha\tau} + \rho_0^2} \right)^{1/2}, \ \rho(0) = \rho_0 = a/2, \quad \phi(\tau) = \phi_0 = 0. \tag{14}$$

Figure 3 compares some numerical solutions to (10) with approximations via (11) and illustrates the distortion of the limit cycles of (10) from the circles described by (11) as ε becomes larger. ⌋

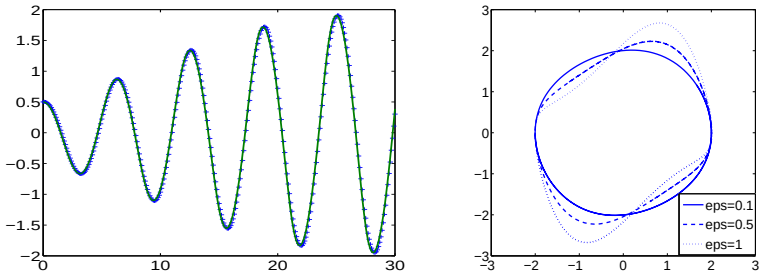

Figure 3: Left: numerical solution of (10) and approximation via (11), $\alpha = 1$, $\varepsilon = 0.2$. Right: Distortion of circle $\rho = 2\sqrt{\alpha}$ by higher-order terms.

Additionally to slow time scales in the examples above, in applications often also small amplitudes play a role, but we skip this here. Already for ODE, amplitude equations are an extremely important tool, in particular for their analytical understanding, for instance to study bifurcations. They can be rigorously justified in a number of cases, usually associated with the so-called center manifold theorem. Apart from numerical comparisons, here we do *not* justify the approximations, i.e., we do *not* prove estimates for the error $\|u(t) - (A(\varepsilon t)\mathrm{e}^{\mathrm{i}t} + \mathrm{cc})\|$ between the true (unknown) solution and the approximation. For this, see the literature cited above. Instead, in the next section we consider a simple PDE situation where the computational advantages of amplitude equations become even more striking.

Exercise 1.5 Consider the ordinary differential equation $\ddot{y} = -(1+\varepsilon)y$ for $y(t) \in \mathbb{R}$, with $y(0) = 1$, $\dot{y}(0) = 0$ and small $\varepsilon > 0$. Discuss the ansatz $y(t) = y_0(t) + \mathcal{O}(\varepsilon)$ to approximate solutions. ⌋

Exercise 1.6 Derive the Landau equation for the weakly damped oscillator

$$\partial_t^2 u + \varepsilon(\partial_t u)^3 + u = 0,$$

and discuss the obtained prediction for its behaviour as $t \to \infty$. ⌋

2 Pattern forming systems

2.1 The Swift–Hohenberg equation

The Swift–Hohenberg (SH) equation [SH77]

$$\partial_t u = -(1 + \partial_x^2)^2 u + \alpha u - u^3, \quad t \geq 0, \quad x \in \mathbb{R}, \quad u = u(t,x) \in \mathbb{R}, \tag{15}$$

is a phenomenological model for the onset of thermal convection in Bénard's problem, which concerns heat conduction in and the motion of a layer of fluid confined between two parallel plates and heated from below, see [Man92, Chap. 8]. Here $\alpha \in \mathbb{R}$ is called the stress parameter and is related to the temperature difference between the bottom and the top of the fluid. We split (15) into a linear part

$$\partial_t u = Au := -(1 + \partial_x^2)^2 u + \alpha u = -(1-\alpha)u - 2\partial_x^2 u - \partial_x^4 u,$$

and the nonlinear part $-u^3$. The linear part is best understood by a Fourier transform. The ansatz $u(x,t) = \exp(\lambda(k)t + \mathrm{i}kx)$, where $k \in \mathbb{R}$ is called the wavenumber, yields

$$\lambda(k) = -(1-k^2)^2 + \alpha, \tag{16}$$

such that for $\alpha < 0$ all modes are exponentially damped. However, for $\alpha > 0$ we have a band of unstable modes around $k = \pm 1$, i.e., modes which grow exponentially in time, see Fig. 4; $k_c = 1$ is then called the critical wavenumber. However, we expect this growth to be saturated by the nonlinearity $-u^3$.

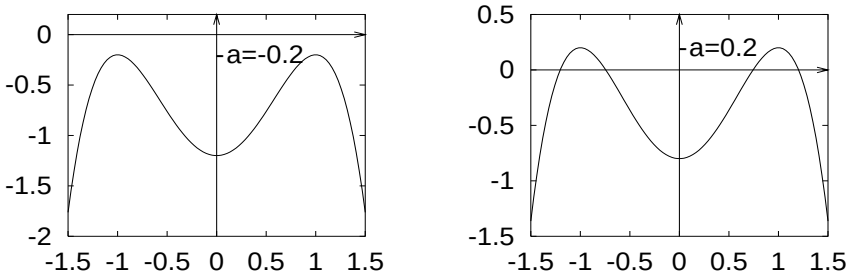

Figure 4: Eigenvalue curve $\lambda(k)$ for (15)

The SH equation is (one of) the simplest PDE examples where multiple scale analysis, which is here also called Ginzburg–Landau formalism, can be used to describe the slowly varying amplitude of the unstable modes. Let $\alpha = \varepsilon^2 > 0$. Since the bands

of unstable wavenumbers k have width $\mathcal{O}(\varepsilon)$ and the instability is $\mathcal{O}(\varepsilon^2)$ ($\lambda(1) = \varepsilon^2$) we expect that the solution can be described by the ansatz

$$u(x,t) = \varepsilon\psi_A(x,t) := \varepsilon A(X,T)\mathrm{e}^{\mathrm{i}x} + \mathrm{cc},\ X = \varepsilon x,\ T = \varepsilon^2 t. \tag{17}$$

Plugging this into (15) yields

$$\begin{aligned}
\partial_t u =& \varepsilon^3(A_T e_1 + \mathrm{cc}) \stackrel{!}{=} -\partial_x^4 u - 2\partial_x^2 u - u + \varepsilon^2 u - u^3 \\
=& \varepsilon\left[(-(\mathrm{i}+\varepsilon\partial_X)^4 - 2(\mathrm{i}+\varepsilon\partial_X)^2 - (1-\varepsilon^2))A\right] e_1 + \mathrm{cc} \\
& - \varepsilon^3(Ae_1 + \overline{A}e_{-1})^3 \\
=& \varepsilon\left[-(1 - 4\mathrm{i}\varepsilon\partial_X - 6\varepsilon^2\partial_X^2 + 4\mathrm{i}\varepsilon^3\partial_X^3 + \varepsilon^4\partial_X^4)A\right] e_1 + \mathrm{cc} \\
& + \varepsilon\left[-2(-1 + 2\mathrm{i}\varepsilon\partial_X + \varepsilon^2\partial_X^2)A - (1-\varepsilon^2)A\right] e_1 + \mathrm{cc} \\
& - \varepsilon^3\left(A^3 e_3 + 3|A|^2 A e_1 + 3|A|^2\overline{A}e_{-1} + 3\overline{A}^3 e_{-3}\right),
\end{aligned}$$

where $e_k = \mathrm{e}^{\mathrm{i}kx}$. Comparing coefficients in front of $\varepsilon^j e_k$ gives

$$\begin{array}{lll}
\varepsilon e_1: & 0 = -A + 2A - A, & \text{i.e. } 0 = 0 \\
\varepsilon^2 e_1: & 0 = 4\mathrm{i}\partial_X A - 4\mathrm{i}\partial_X A, & \text{i.e. } 0 = 0, \\
\varepsilon^3 e_1: & A_T = (6-2)\partial_X^2 A + A - 3|A|^2 A, & \text{equation for } A, \\
\varepsilon^3 e_3: & 0 = A^3, \quad \text{a residual that shall later} & \text{be removed}, \\
\varepsilon^4 e_1: & 0 = -4\partial_X^3 A, & \text{more residual}, \\
\varepsilon^5 e_1: & \vdots & \vdots
\end{array} \tag{18}$$

This means that the so-called residual is minimized if $A(X,T)$ fulfills

$$\partial_T A = 4\partial_X^2 A + A - 3|A|A, \tag{19}$$

where the residual

$$\mathrm{Res}(u) = -\partial_t u - (1+\partial_x^2)^2 u + \alpha u - u^3$$

contains the terms which do not cancel after inserting an ansatz into the equation. If $\mathrm{Res}(u) = 0$, then u is an exact solution.

Equation (19) is an example for a so-called *complex Ginzburg–Landau (cGL) equation*, which in most general form can be written as

$$u_t = (1+\mathrm{i}\nu)u_{xx} + Ru - (1+\mathrm{i}\mu)|u|^2 u, \quad \nu,\mu,R,x \in \mathbb{R}. \tag{20}$$

The cGL can be derived in a great variety of problems, ranging from fluid dynamics and various other physical systems to reaction diffusion systems from chemistry and mathematical biology. It is also important as a model to study various phenomena ranging from stability and instability to turbulence and chaos in the context of PDEs,

see, e.g., [CH93, LO96, Mie02, AK02]. The particular cGL (19) is actually called real Ginzburg–Landau equation since it has real coefficients.

Mathematically, the next step would be to show the validity of the approximation of solutions u of (15) by solutions of (19) via (17), i.e., to estimate the error of the approximation on a suitable time-scale. Suitable here means of at least order $1/\varepsilon^2$ in t since otherwise there are no interesting dynamics in (19). It turns out that such error estimates can be proved, see, e.g., [Sch94, MS96], but here we content ourselves with one numerical simulation, see Fig. 5. For the numerical solution of (15) and (19) we recommend spectral methods, see, e.g., [Uec09].

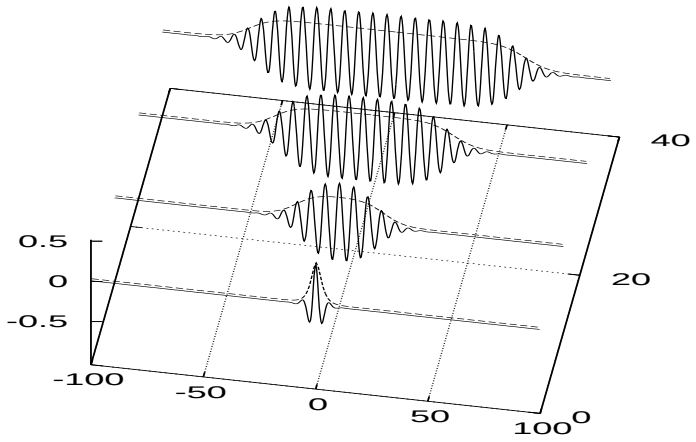

Figure 5: Comparison of the true (numerical) solution of the SH equation with ε=0.5 and initial condition $u_0(x)$=$A_0(\varepsilon x)\cos(x)$, $A_0(X) = 1/\cosh(X)$, with the (numerical) solution A (dashed line), which is real, of the GL with IC $A_0(X)$. Clearly, $\psi_A(x,t) = \varepsilon A(\varepsilon x, \varepsilon^2 t)\cos(x)$ gives an approximation up to higher-order terms for all times considered. Numerically, solving (19) instead of (15) reduces costs by factors of at least ε(for space)$\times\varepsilon^2$(for time)$= \varepsilon^3$ (in total). For $\varepsilon = 1/2$ this is a factor 1/8, but for, e.g., $\varepsilon = 1/10$ this is already a factor 1/1000. This factor is actually rather conservative since for reasonable (implicit) numerical methods for parabolic equations the complexity is of order at least $\mathcal{O}(n \log n)$, where $n \sim 1/\varepsilon$ is the number of spatial discretization points.

Remark 2.1 In the introduction we pointed out that in the reduction we restrict to some specific class of solutions, i.e., here described by the ansatz (17), and that a given system may well have many other solutions, not described by the ansatz. In fact, for the Swift–Hohenberg equation and similar dissipative systems the situation

is somewhat better: One can prove [Eck93, MS96] that *all small solutions*, i.e. of amplitude ε, can be described by (19), in a suitable sense. ⌋

Remark 2.2 (19) can again be understood as a solvability condition as follows: suppose that we make the ansatz

$$u(x,t) = \varepsilon\psi_A(x,t) + \varepsilon^3 u_3(x,T), \tag{21}$$

where we stipulate that similar to ψ_A the higher-order terms depend on t only via $T = \varepsilon^2 t$, and where we used that the lowest order terms generated by cubic nonlinearity of the Swift–Hohenberg equation are of order ε^3. Then the equation for u_3 reads

$$Lu_3 := (1+\partial_x^2)^2 u_3 = (-\partial_T A + 4\partial_X^2 A + A - 3|A|^2 A)\mathrm{e}^{\mathrm{i}x} - A^3\mathrm{e}^{3\mathrm{i}x} + \mathrm{cc} + \mathcal{O}(\varepsilon). \tag{22}$$

Now L can be treated as a selfadjoint linear operator in the Hilbert space $L^2(\mathbb{R})$, and at least formally we have $\mathrm{e}^{\mathrm{i}x} \in \ker L$. Thus, by the Fredholm alternative, we need $-\partial_T A + 4\partial_X^2 A + A - 3|A|^2 A{=}0$ to solve (22) for u_3. Here, although equivalent, this point of view is somewhat more involved than simply trying to minimize the residual as outlined above. However, for *systems* of PDEs the formalism of solvalibility conditions is usually needed to derive amplitude equations, see Sec. 4. ⌋

Remark 2.3 After choosing $\partial_T A = 4\partial_X^2 A + A - 3|A|^2 A$, the lowest order residual in (18) is A^3 at $\varepsilon^3 e_3$. Here we briefly outline how this and in principle also all other higher-order terms can be removed. To remove $\varepsilon^3 A^3 e_3$ we refine our ansatz to $u(x,t) = \varepsilon A(X,T)e_1 + \varepsilon^3 A(X,T)e_3 + \mathrm{cc}$. This gives

$$\begin{aligned}
&\varepsilon^3 A_T e_1 + \varepsilon^5 \partial_T A_3 e_3 + \mathrm{cc}\\
&\quad = \varepsilon\left[(-(\mathrm{i}+\varepsilon\partial_X)^4 - 2(\mathrm{i}+\varepsilon\partial_X)^2 - (1-\varepsilon^2))A\right]e_1 + \mathrm{cc}\\
&\qquad + \varepsilon^3\left[(-(3\mathrm{i}+\varepsilon\partial_X)^4 - 2(3\mathrm{i}+\varepsilon\partial_X)^2 - (1-\varepsilon^2))A_3\right]e_3 + \mathrm{cc}\\
&\qquad - \varepsilon^3(Ae_1 + \overline{A}e_{-1} + \varepsilon^2 A_3 e_3 + \varepsilon^2\overline{A}_3 e_{-3})^3\\
&\quad = \varepsilon^3(4\partial_X^2 A + A - 3|A|^2 A)e_1 + \varepsilon^3(-81 + 2\cdot 9 - 1)A_3 e_3 - \varepsilon^3 A^3 e_3 + \mathcal{O}(\varepsilon^4) + \mathrm{cc},
\end{aligned}$$

and hence the residual is $\mathcal{O}(\varepsilon^4)$ if we choose $A_3 = -\frac{1}{64}A^3$. Similarly, more corrections can be added in order to have an arbitrarily small residual. ⌋

2.2 Quadratic nonlinearity

The above derivation heavily relies on the fact that the nonlinearity in the Swift–Hohenberg equation is cubic. As a consequence, the ansatz (17) directly yields the cGL at $\mathcal{O}(\varepsilon^3 e_1)$ since the cubic interaction of modes e_1, e_{-1} couples back to e_1, e_{-1}.

If the nonlinearity is quadratic, or, more generally, if the nonlinearity contains quadratic terms, then we need to modify our ansatz since the quadratic interaction of e_1, e_{-1} only couples to e_{-2}, e_0, e_2.

As an example we consider the Kuramoto-Sivashinsky type of equation

$$\partial_t u = L(\partial_x)u + f(u, u_x), \quad t \geq 0, \quad x \in \mathbb{R}, \quad u = u(t,x) \in \mathbb{R}, \tag{23}$$

where again

$$Lu = [-(1+\partial_x^2)^2 + \alpha_0\varepsilon^2]u,$$

with $\alpha_0 \in \mathbb{R}$, $0 < \varepsilon^2 \ll 1$, and $f(u, u_x) = f_1 u^2 + f_2 u u_x$ with $f_1, f_2 \in \mathbb{R}$. We make the ansatz

$$u(x,t) = \varepsilon\psi(x,t) := \varepsilon A_1(X,T)e_j + \frac{\varepsilon^2}{2}A_0(X,T) + \varepsilon^2 A_2(X,T)e_2 + \text{cc}, \tag{24}$$

$X = \varepsilon x$, $T = \varepsilon^2 t$, $e_j = \mathrm{e}^{\mathrm{i}jx}$ and derive equations for A_0, for A_2, and finally for A_1 such that $\text{Res}(\varepsilon\psi) := -\partial_t(\varepsilon\psi) + L(\partial_x)u + f(u, u_x)$ becomes small. Indeed, inserting (24) into (23) and equating coefficients in front of $\varepsilon^j e_j$ we obtain the closed system of equations

$$\begin{aligned}
\varepsilon^2 e_0 : &\qquad 0 = -A_0 + 2f_1|A|^2 \\
\varepsilon^2 e_2 : &\qquad 0 = -9A_2 + (f_1 + \mathrm{i}f_2)A^2 \\
\varepsilon^3 e_1 : &\quad A_T = 4\partial_X^2 A + \alpha_0 A + (2f_1 + \mathrm{i}f_2)(A_0 A + A_2\overline{A}).
\end{aligned}$$

Eliminating A_0 and A_2 we obtain

$$A_T = 4\partial_X^2 A + \alpha_0 A + c_3|A|^2 A \text{ with } c_3 = (2f_1 + \mathrm{i}f_2)\,(2f_1 + (f_1 + \mathrm{i}f_2)/9)\,.$$

3 Nonlinear optics

The transport of information through glass fibers by light is a key technology. Information is encoded digitally by ones and zeroes, i.e., by sending a light pulse through the optical fiber or not. Physically such a light pulse is a complicated structure. It consists of an underlying electromagnetic carrier wave moving with phase velocity c_p and of a pulse-like envelope moving with group velocity c_g, see Fig. 6.

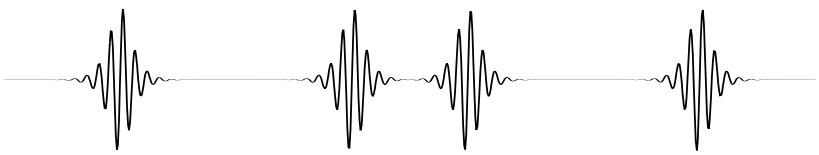

Figure 6: 0's and 1's are encoded physically by sending a light pulse or not; thus, for instance, the above electromagnetic wave encodes the sequence 101101.

The analysis of the evolution of such a light pulse is a nontrivial task. The system shows dispersion and (weak) dissipation, i.e., harmonic waves with different wavenumbers travel at different speeds and energy is lost in a wavenumber-dependent way. Moreover, there is a nonlinear response by the optical fiber. Thus, at a first glance it looks like a typical example for the application of numerical methods. However, a direct simulation of Maxwell's equations which describe these electromagnetic waves

is beyond any present possibilities. This can be seen as follows: The wavelength of the carrier wave is around 10^{-7} m. Resolving this structure in a fiber of 10 km $=10^4$ m gives in uniform one-dimensional spatial discretization 10^{11} points, not to speak about the transverse directions and the temporal discretization. Therefore, before making any numerical investigations, the system has to be analyzed and simpler, numerically more suitable, models have to be derived. In particular we shall see that a great deal can be learned about optical pulses (and related systems) using only paper and pen, by deriving a Nonlinear Schrödinger (NLS) equation as the amplitude equation for wavepackets in nonlinear dispersive media.

3.1 Physical background

Light pulses are electromagnetic waves and described by Maxwell's equations, namely

$$\begin{aligned} \nabla \cdot \vec{B} = 0 \quad &, \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0, \\ \nabla \cdot \vec{D} = \rho \quad &, \quad \nabla \times \vec{H} - \partial_t \vec{D} = \vec{J}, \end{aligned}$$

with $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ and $\vec{H} = \vec{B}/\mu_0 - \vec{M}$. Here $\vec{E} = \vec{E}(\vec{x}, t)$ is the electric field, $\vec{x} = (x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$ is the time, ε_0 the permittivity of vacuum, $\vec{P}$ the material polarization, $\vec{B}$ the magnetic flux, μ_0 the magnetic permeability of vacuum, $\vec{M}$ the material magnetization, ρ the charge density and $\vec{J}$ the electric current. These equations have to be closed with constitutive laws $\vec{P} = \vec{P}(\vec{E}, \vec{H})$ and $\vec{M} = \vec{M}(\vec{E}, \vec{H})$ describing the behavior of the medium. Depending on this choice there are linear and nonlinear, instantaneous and history-dependent, dispersive and dissipative models.

In typical optical fibers there is no magnetization $\vec{M}$, no charge density ρ, and no electric current $\vec{J}$, and therefore, using $\nabla \times \nabla \vec{E} = \Delta \vec{E} - \nabla(\nabla \cdot \vec{E})$, Maxwell's equations for light in nonlinear optical material are given by

$$\triangle \vec{E} - \nabla(\nabla \cdot \vec{E}) - \partial_t^2 \vec{E} = \partial_t^2 \vec{P}, \tag{25}$$

where we scaled the speed of light in vacuum and the dielectric constant to 1.

The constitutive law for the polarization $\vec{P} = \vec{P}_{\mathrm{l}} + \vec{P}_{\mathrm{nl}}$ splits into a linear and a nonlinear part, which in general both depend on the history of the electric field. In centrosymmetric isotropic bulk material, the constitutive law for the linear response $\vec{P}_{\mathrm{l}}$ is given by an instantaneous part $\vec{P}_{\mathrm{l}}^{\mathrm{i}}(\vec{x}, \vec{E}(\vec{x}, t))$ and a history-dependent term

$$\vec{P}_{\mathrm{l}}^{\mathrm{h}}(\vec{x}, t) = (\chi_1 *_t \vec{E})(\vec{x}, t) = \int_{-\infty}^{\infty} \chi_1(t - \tau) \vec{E}(\vec{x}, \tau) \, \mathrm{d}\tau, \tag{26}$$

where χ_1 in (26) is a scalar function, independent of $\vec{x}$, with $\chi_1(t) = 0$ for $t < 0$ due to causality, and similar for the nonlinear polarization. In the case of optical fibers χ_1 does also depend on the transverse directions y, z, and in the case of photonic crystals also on the longitudinal direction x.

In the simplest case $\vec{E}$ is linearly polarized and only depends on x, i.e.,

$$\vec{E}(\vec{x}, t) = u(x, t)\hat{k} \quad \text{with} \quad \|\hat{k}\|_{\mathbb{R}^3} = 1, \quad (1, 0, 0) \cdot \hat{k} = 0. \tag{27}$$

Then, (25) simplifies to

$$\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - \partial_t^2 p_{\mathrm{l}}(x,t) - \partial_t^2 p_{\mathrm{nl}}(x,t), \tag{28}$$

with $u(x,t), p_{\mathrm{l}}(x,t), p_{\mathrm{nl}}(x,t) \in \mathbb{R}$ such that $\vec{P}_{\mathrm{l}}(t,\vec{x}) = p_{\mathrm{l}}(x,t)\hat{k}$, $\vec{P}_{\mathrm{nl}}(t,\vec{x}) = p_{\mathrm{nl}}(x,t)\hat{k}$. The symmetry $(y,z) \mapsto -(y,z)$, which is present in most optical materials, prevents the occurrence of even terms in p with respect to u. Thus, in general p_{nl} starts with cubic terms.

Due to the fact that we are mainly interested in the underlying mathematical structures, throughout the rest of the paper we choose

$$\partial_t^2 p(x,t) = u(x,t) - u^3(x,t)$$

as constitutive law, thus the toy problem for this paper is

$$\partial_t^2 u = \partial_x^2 u - u + u^3. \tag{29}$$

This choice is rather unphysical; however, it delivers a system with all properties in which we are interested, namely dispersive and nonlinear behavior. We refer to [SU03] for a mathematical discussion of a physically more realistic choice which includes dissipation and history dependence additionally to dispersion and nonlinearity.

3.2 Derivation of the NLS equation

3.2.1 Linearization, modes, and dispersion

The description of light pulses, i.e., here of localized solutions of (29), is based on the derivation of a Nonlinear Schrödinger (NLS) equation by formal perturbation analysis. A priori there are no separate scales in (29). However, even if this may appear somewhat artificial, we can simply introduce a small perturbation parameter

$$0 < \varepsilon \ll 1$$

which will relate the amplitude with the spatial and temporal scales. We start with the linear problem

$$\partial_t^2 u = \partial_x^2 u - u \tag{30}$$

and seek solutions of the form $u(x,t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ with wavenumber $k \in \mathbb{R}$ and (temporal) frequency ω. Plugging this ansatz into (30) yields the so called dispersion relation

$$\omega^2 = k^2 + 1 \Leftrightarrow \omega = \pm\sqrt{1+k^2}. \tag{31}$$

From this the phase speed c_p is calculated as

$$kx - \omega t = \mathrm{const} \overset{wlog}{=} 0 \Leftrightarrow x = x(t) = \frac{\omega}{k} t =: c_p(k)t.$$

(30) is called dispersive since the phase speed c_p is not constant, i.e., the speed of harmonic waves depends on their "color" k. However, for the transport of information (or energy) the group speed c_g is the relevant quantity, which we explain now. Consider the sum of two harmonics

$$u(x,t) = \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + A_2 \mathrm{e}^{\mathrm{i}[(k_0+\varepsilon)x - \omega(k_0+\varepsilon)t]} \tag{32}$$

with small wavenumber difference ε (and arbitrary $A_2 \in \mathbb{C}$. Since (30) is linear, (32) is an exact solution of (32), but the problem is that this does not tell us much. The solution is to Taylor expand $\omega(k_0+\varepsilon)$, i.e., to write

$$\begin{aligned} u(x,t) &= \mathrm{e}^{i(k_0 x - \omega_0 t)} + A_2 \mathrm{e}^{i((k_0+\varepsilon)x - \omega(k_0+\varepsilon)t)} + \mathrm{cc} \\ &= \mathrm{e}^{i(k_0 x - \omega_0 t)} \underbrace{\left(1 + A_2 \mathrm{e}^{i(\varepsilon(k_0 x - \omega'(k_0)t) - \frac{1}{2}\omega''(k_0)\varepsilon^2 t + \mathrm{h.o.t})}\right)}_{=:A(X,T)} + \mathrm{cc}, \end{aligned}$$

where $X = \varepsilon(k_0 x - \omega'(k_0)t)$ and $T = \varepsilon^2 t$. This shows that in lowest order (32) is a long wave modulation of the basic harmonic $\mathrm{e}^{\mathrm{i}k_0 x}$, which is constant in the frame comoving with group speed $\omega'(k_0)$. In music this is called a "Schwebung"; the listener perceives a pulsation of the tone of basic frequency ω_0, see also Fig. 7. In second order we obtain the linear Schrödinger equation

$$\partial_T A = \frac{\mathrm{i}}{2}\omega''(k_0)\partial_X^2 A, \tag{33}$$

which describes the evolution of (32) on long spatio-temporal scales.

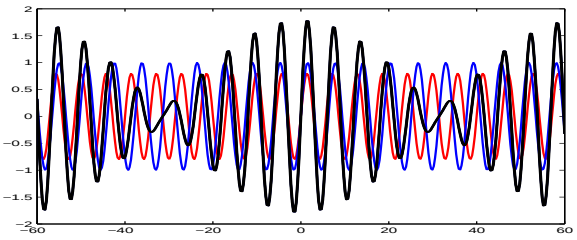

Figure 7: A "Schwebung" as a pseudo wavepacket.

3.2.2 The weakly nonlinear problem

Following the above heuristics we now seek $\mathcal{O}(\varepsilon)$-amplitude solutions of the *nonlinear* problem (29), which are slow spatial and temporal modulations of an underlying wave train $\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$. Thus we make an ansatz

$$u_A(x,t) = \varepsilon(A(X,T)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + \mathrm{cc}) + \mathcal{O}(\varepsilon^2), \tag{34}$$

where $X = \varepsilon(x - c_g t)$, $T = \varepsilon^2 t$, and hence $A(X, T)$ is a complex-valued amplitude on a long spatial scale in a frame comoving with the group speed c_g to be determined, and on a very long time scale. Substituting (34) into (29) and sorting the coefficients of $e^{i(k_0 x - \omega_0 t)}$ with respect to powers of ε, at order $\mathcal{O}(\varepsilon)$ we recover the dispersion relation, i.e.,

$$\mathcal{O}(\varepsilon^1): \quad -\omega_0^2 A = -(k_0^2 + 1)A, \quad \Rightarrow \omega_0^2 = k_0^2 + 1,$$

while at $\mathcal{O}(\varepsilon^2)$ we obtain the equation for the so-called group speed c_g, namely

$$\mathcal{O}(\varepsilon^2): \quad 2ic_g\omega_0 A_X = 2ik_0 A_X \;\Rightarrow\; c_g = k_0/\omega_0 = \omega'(k_0).$$

The frequency ω depends nonlinearly on the wavenumber ω. As a consequence, the group speed $c_g(k) = \omega'(k)$ is not constant but depends nontrivially on k. Thus, wavepackets with different wavenumbers, i.e. colors, travel at different speed, and precisely this effect is called dispersion.

At $\mathcal{O}(\varepsilon^3 e^{i(k_0 x - \omega_0 t)})$ we find that A should satisfy the NLS equation

$$2i\omega_0 \partial_T A + (1 - c_g^2)\partial_X^2 A + 3|A|^2 A = 0,$$

which after regrouping is often written as

$$\partial_T A = i(c_2 \partial_X^2 A + c_3 |A|^2 A), \qquad c_2 = \frac{1 - c_g^2}{2\omega_0}, \; c_3 = \frac{3}{2\omega_0}. \tag{35}$$

Note that $c_2 = \frac{i}{2}\omega''(k_0)$ in agreement with the linear calculations above.

As usual, there will be more terms in the residual, for instance $\varepsilon^3 A^3 e^{3i(k_0 x - \omega_0 t)}$, but it again turns out that these can be made arbitrarily small be refining the approximation similar to Remark 2.3. We skip the details, and likewise only refer to the literature for the mathematical justification of the approximation of solutions of (29) via (35), e.g. [KSM92, SU07b].

Remark 3.1 (35) is an equation with complex coefficients and for a complex field. This could be rewritten a real 2D system, but on the face of it (35) is in no obvious way "simpler" than the original system (29). Again, conceptually the main point is that (35) lives on long scales.

Additionally, (35) is universal: similar to the cGL (20) for nonlinear dissipative systems in Sec. 2, the NLS is the fundamental amplitude equation for wavepackets in nonlinear dispersive systems: additionally to (29) (or the basic Maxwell equations of which (29) is a toy model), it can be derived for wide a variety of problems, for instance: water waves, plasma waves, elastic waves, lasers, molecular dynamics, see [Gib90, CH93, SS99].

Moreover, the NLS has a lot of special structure, which is well understood partly due to the ubiquity of the NLS. We are now going to exploit some very basic results about the NLS. ⌋

The NLS is a so-called integrable system, and in particular there are quite a number of explicit solutions known. For nonlinear optics, the most important special solutions are the so-called solitons. Equation (35) has a four-dimensional family of solutions of the form

$$A(X,T) = \tilde{A}(X - vT - X_0)\mathrm{e}^{\mathrm{i}(\tilde{v}X - \gamma_0 T + \phi_0)}, \quad \tilde{v} = (\omega_0 v)/(1 - c_\mathrm{g}^2),$$

$v, \gamma_0, \phi_0, X_0 \in \mathbb{R}$, in which the real-valued function $\tilde{A}$ satisfies the second-order ordinary differential equation

$$\partial_X^2 \tilde{A} = b_1 \tilde{A} - b_2 \tilde{A}^3, \tag{36}$$

where

$$b_1 = \tilde{v}^2 - \frac{2\gamma_0\omega_0}{1 - c_\mathrm{g}^2}, \quad b_2 = \frac{3}{1 - c_\mathrm{g}^2}.$$

Since $c_\mathrm{g} < 1$, we always have $b_2 > 0$, and for $b_1 > 0$ there exist two explicit homoclinic solutions of (36), namely

$$\tilde{A}_\mathrm{pulse}(X) = \pm\sqrt{\frac{2b_1}{b_2}}\,\mathrm{sech}\,(\sqrt{b_1}X). \tag{37}$$

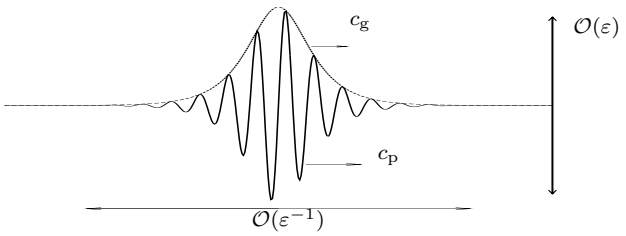

Figure 8: A modulating pulse for (29) described by the NLS equation.

Example 3.2 Recalling the purpose of this lecture we give a numerical example illustrating the NLS formalism to calculate the propagation of a light pulse through a medium described by (29). For simplicity we consider the propagation of a single pulse of NLS form, with $\varepsilon = 0.1, k_0 = 1, \gamma_0 = 1, \tilde{v} = 0, \phi_0 = 0$ and $X_0 = 5$, and hence $A_0(X) = 2\varepsilon\sqrt{2b_1/b_2}\mathrm{sech}\,(\sqrt{b_1}(X-5))$ and compare it to the prediction by the NLS. Thus, as initial conditions for (29) we take

$$u_0(x) = 2\varepsilon\sqrt{\frac{2b_1}{b_2}}\cos(x)\mathrm{sech}\,(\sqrt{b_1}\varepsilon(x-50))), \tag{38}$$

$$u_1(x) = 2\varepsilon^2 c_\mathrm{g}\sqrt{\frac{2b_1^2}{b_2}}\cos(x)\tanh(\varepsilon(x-50)) + 2\varepsilon\sqrt{\frac{2b_1}{b_2}}\omega_0\sin(x)\mathrm{sech}\,(\sqrt{b_1}\varepsilon(x-50))). \tag{39}$$

The NLS predicts the solution

$$u(x,t) \approx 2\varepsilon\sqrt{2b_1/b_2}\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}(x-t-\varepsilon t)}\mathrm{sech}\left(\sqrt{b_1}\varepsilon(x-50-ct)\right)\right], \tag{40}$$

which fits rather well with the numerical solution, see Fig. 9.

However, in general, given some initial condition $A(0,X)$ the NLS has to be solved numerically. But even then, similar to Fig. 5, the speed-up in numerics is of the order ε(for space)$\times\varepsilon^2$(for time)$=\varepsilon^3$(in total). See [CBCSU08] for some numerical illustrations including some higher-order approximation of the dynamics of (29) by (extensions of) the NLS equation. ⌋

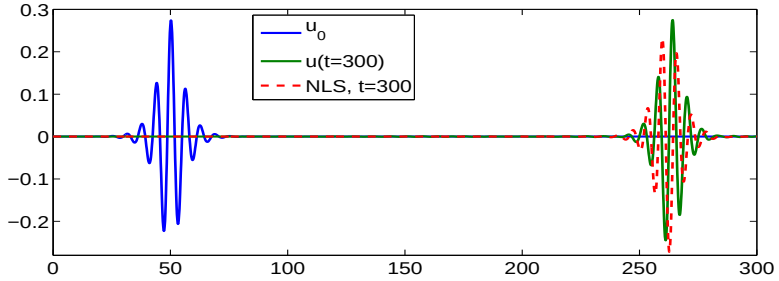

Figure 9: Comparison of the NLS prediction with the numerical solution of 29, see text.

The reduction in computational costs becomes even more dramatic in real life problems. A real fiber is a three-dimensional object, and hence three spatial dimensions have to be discretized. Typically, the transverse dimensions are rather small, but for instance the small number of 20 discretization points in each transverse direction yields an additional factor of 400 for Maxwell simulations, while the NLS discretization remains unchanged; see, e.g. [Agr01] for the derivation of the NLS from a realistic 3D fiber model.

Exercise 3.3 Some so-called χ^2 materials have a quadratic law for their polarizations. As a toy problem, derive the amplitude equation for the propagation of wavepackets in the nonlinear wave equation with a quadratic nonlinearity, i.e.,

$$\partial_t^2 u = \partial_x^2 u - u + u^2, \quad u = u(x,t) \in \mathbb{R}, \quad u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x). \tag{41}$$

Hint: Make an ansatz $u(x,t) = \varepsilon A_1(X,T)e_1 + \frac{\varepsilon^2}{2}A_0(X,T) + \varepsilon^2 A_2(X,T)e_2 + \mathrm{cc}$. ⌋

4 Convection in porous media

In this final section we turn to a vector valued problem in two space dimensions, where in particular we can explain the role of transverse directions and the Fredholm alternative in the derivation of amplitude equations in more detail.

A classical hydrodynamical stability problem is the so-called Rayleigh–Bénard problem which concerns a layer of fluid heated from below, for instance, a fluid between two horizontal plates. In Sec. 2.1 we considered the Swift–Hohenberg equation as a toy problem for this. First-principle models couple the Navier–Stokes equations for the fluid motion with an equation for the temperature in the fluid, where in the so-called Boussinesq approximation the only place where the temperature affects the motion of the fluid is in the buoyancy. There is a trivial solution, a purely conducting state with an affine temperature profile and no motion of the fluid. This state is stable if the temperature difference between the lower and the upper plate is sufficiently small but if the temperature difference becomes large then it loses stability and convection rolls appear. For even larger temperature differences the motion becomes more complicated and eventually turbulent. This can be studied in your kitchen.

Convection also plays a big role in geophysics. The movement of the tectonic plates on earth is induced by convection in the mantle of the earth, i.e., in between the core and the surface of the earth. Convection also plays a role in the description of hot springs and geysers, and of so-called black smokers on the ocean floor. The rock between the air or the sea at the top and of the magma chambers at the bottom is highly fractured and thus modeled as a so-called porous medium. Compared to classical hydrodynamical stability problems the associated system of partial differential equations for convection in porous media is easier since in this case the velocity field of the fluid is determined by a constitutive law, namely Darcy's law, and has not to be computed as a solution of the Navier–Stokes equations.

As a model problem we are interested in the velocity field $u = (u_1, u_2)$ and the temperature field T of a fluid in a strip $\mathbb{R} \times [0,1]$ of porous media, heated from below. If we denote the coordinates in the strip with $(x, y) \in \mathbb{R} \times [0,1]$, we have to solve

$$\nabla \cdot u = 0, \tag{42}$$

$$u = -\nabla p + RTe_2, \tag{43}$$

$$\partial_t T + u \cdot \nabla T = \Delta T, \tag{44}$$

with the boundary conditions $T = 1,\ u_2 = 0$ at $y = 0$ and $T = 0,\ u_2 = 0$ at $y = 1$. Here, $\nabla = (\partial_x, \partial_y)^T$, $\Delta = \partial_x^2 + \partial_y^2$, $e_2 = (0,1)^T$, p denotes a pressure field, and the so-called Rayleigh number R is a dimensionless parameter, proportional for instance to the (physical) distance of the plates and the (physical) temperature difference.

For a detailed derivation of (42)-(44) see for instance [Fow97, Section 14]. Conservation of mass for an incompressible fluid is described by (42), while (43) is the balance of forces based on the Boussinesq approximation and Darcy's law. The heat equation (44) is derived from an energy balance.

The purely conducting state of (42)-(44) is given by

$$u = 0, \quad T = 1 - y, \quad p = -\frac{R}{2}(1 - y^2). \tag{45}$$

Since (42)-(44) is supposed to be a model for convection we expect that for large R, e.g., for large temperature difference δT between the upper and lower plate, convection

sets in, resulting in some pattern of convection rolls. In the following we explain that this is indeed the case, and that it can conveniently be described using a Ginzburg–Landau equation as the amplitude equations for the convection rolls.

4.1 Linearized stability

The first step is to find the dispersion relation for the linearized system; in a certain sense, this will turn out to be very similar to that of the Swift–Hohenberg equation, cf. (16). We eliminate the pressure p by introducing the stream function ψ such that

$$u_1 = \partial_y \psi \quad \text{and} \quad u_2 = -\partial_x \psi$$

and introduce the deviation θ from the linear temperature profile by $T = 1 - y + \theta$. This yields

$$\Delta\psi = -R\partial_x\theta, \quad \partial_t\theta + \partial_x\psi + (\partial_y\psi\partial_x\theta - \partial_x\psi\partial_y\theta) = \Delta\theta. \tag{46}$$

The linearized system is

$$\Delta\psi = -R\partial_x\theta, \qquad \partial_t\theta + \partial_x\psi = \Delta\theta,$$

with the boundary conditions $\theta = \psi = 0$ at $y = 0, 1$. Due to the boundary conditions we make the ansatz

$$\psi = f\sin(n\pi y)e^{\lambda t + ikx}, \qquad \theta = g\sin(n\pi y)e^{\lambda t + ikx}$$

with $n \in \mathbb{N}$, $k \in \mathbb{R}$, and complex-valued coefficients f and g. This gives the system of linear equations

$$-(\pi^2 n^2 + k^2)f = -ikRg, \qquad -(\pi^2 n^2 + k^2)g = ikf + \lambda g. \tag{47}$$

We find

$$g = \frac{n^2\pi^2 + k^2}{ikR} f \text{ and } \lambda = \frac{Rk^2}{n^2\pi^2 + k^2} - (n^2\pi^2 + k^2),$$

i.e., we have a family of curves $k \mapsto \lambda_n(k) \in \mathbb{R}$ of eigenvalues with $n \in \mathbb{N}$ and $k \in \mathbb{R}$. It is easy to see that $\lambda_{n+1}(k) \le \lambda_n(k) \in \mathbb{R}$ for each fixed $k \in \mathbb{R}$. Moreover, $\lambda_n(k) \to -\infty$ for $k \to \infty$ or $n \to \infty$, or both.

Hence $\theta = \psi = 0$ is stable if $\lambda_1(k) < 0$ for all $k \in \mathbb{R}$. Instability occurs when the curve λ_1 touches the axis $\lambda = 0$ at a wavenumber $k = k_c \in \mathbb{R}$ for a parameter value $R = R_c$. This leads to the conditions

$$\lambda_1 = \frac{Rk^2}{\pi^2 + k^2} - (\pi^2 + k^2) = 0$$

and

$$\partial_{k^2}\lambda_1 = \frac{R}{\pi^2 + k^2} - \frac{Rk^2}{(\pi^2 + k^2)^2} - 1 = \frac{\pi^2 R}{(\pi^2 + k^2)^2} - 1 = 0.$$

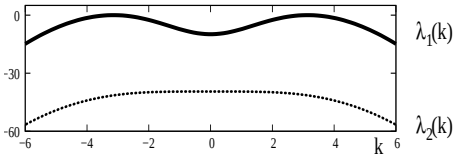

Figure 10: The curve of eigenvalues $k \mapsto \lambda_n(k)$ for $n = 1, 2$.

From this we find $\pi R^{1/2} = \pi^2 + k^2$ and $\lambda = R - 2\pi R^{1/2}$, and this shows that $\lambda = 0$ for $R = R_c = 4\pi^2 \approx 39.48$ at the critical wavenumber $k = k_c = \pi$, see Fig. 10.

Thus, for $R > R_c$, say $R = R_c + \varepsilon^2$ with $0 < \varepsilon^2 \ll 1$, the linearized problem has modes

$$\begin{pmatrix} 2\pi i \\ 1 \end{pmatrix} e^{ikx} \sin \pi y + \text{cc}$$

with $k \approx \pi$, which grow exponentially in time with rate $R - R_c = \varepsilon^2$. If the model makes sense physically, then we expect some nonlinear saturation at some small amplitude ε and thus expect stationary convection roll solutions of the form

$$\begin{pmatrix} 0 \\ 0 \\ 1-y \end{pmatrix} + \varepsilon \begin{pmatrix} u_1 \\ u_2 \\ \theta \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 1-y \end{pmatrix} + \varepsilon \begin{pmatrix} 4\pi^2 \cos \pi y \sin kx \\ -4\pi k \sin \pi y \cos kx \\ \sin \pi y \end{pmatrix} + \mathcal{O}(\varepsilon^2).$$

We now derive an amplitude equation for these rolls.

4.2 Weakly nonlinear analysis

The idea is that the dynamics of (42)–(44) is dominated by the unstable modes since all other modes are linearly exponentially damped and hence "slaved" to the critical modes. Thus, in the near critical regime we set

$$R = R + s\varepsilon^2, \tag{48}$$

where $s \in \mathbb{R}$ and $0 < \varepsilon \ll 1$ is a small parameter. The use of s and ε^2 instead of, say $-1 \ll \varepsilon \ll 1$ is for convenience. We make the ansatz

$$\begin{aligned} \begin{pmatrix} \psi \\ \theta \end{pmatrix}(x,y,t) &= \varepsilon \psi_A(x,y,t) \\ &:= \varepsilon A(\xi,\tau) \begin{pmatrix} 2\pi i \\ 1 \end{pmatrix} e^{i\pi x} \sin \pi y + \text{cc} + \varepsilon^2 \begin{pmatrix} \psi_2 \\ \theta_2 \end{pmatrix}(x,y,t) + \varepsilon^3 \begin{pmatrix} \psi_3 \\ \theta_3 \end{pmatrix}(x,y,t) \end{aligned} \tag{49}$$

where $\xi = \varepsilon x$ and$\tau = \varepsilon^2 t$ are the long spatial and very long temporal scale. This describes small amplitude long spatial and temporal modulations of the convection pattern

$$\begin{pmatrix} 2\pi i \\ 1 \end{pmatrix} e^{i\pi x} \sin \pi y + \text{cc},$$

see Fig. 11.

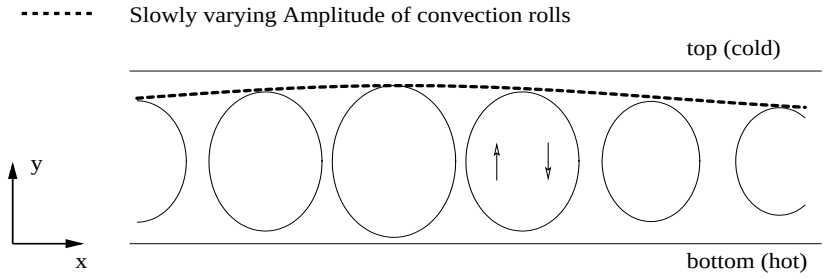

Figure 11: Long wave modulation of convection rolls.

Again the goal is to make the residual

$$\text{Res}(\varepsilon\psi_A) := \begin{pmatrix} \Delta\psi + R\partial_x\theta \\ \partial_t\theta + \partial_x\psi + (\partial_y\psi\partial_x\theta - \partial_x\psi\partial_y\theta) - \Delta\theta \end{pmatrix} \tag{50}$$

small in ε, in an appropriate sense. Thus we plug (49) into (46) and sort with respect to ε. Here we use the following notation: applying

$$L = \begin{pmatrix} \Delta & R\partial_x \\ -\partial_x & \Delta \end{pmatrix} \quad \text{to} \quad v = A(\varepsilon x)e^{ikx} \sin(n\pi y) \begin{pmatrix} a \\ b \end{pmatrix},$$

we obtain

$$Lv = \hat{L}(k,n)v + \mathcal{O}(\varepsilon), \quad \text{with} \quad \hat{L}(k,n) = \begin{pmatrix} -k^2 - n^2\pi^2 & ikR \\ -ik & -k^2 - n^2\pi^2 \end{pmatrix}, \tag{51}$$

and where the $\mathcal{O}(\varepsilon)$ terms contain the ∂_ξ derivatives of A.

The $\mathcal{O}(\varepsilon)$ terms in (50) vanish by construction of ψ_A. At $\mathcal{O}(\varepsilon^2)$ we obtain, by

calculus, and since $R_c = 4\pi^2$,

$$\begin{aligned} L\begin{pmatrix}\psi_2\\ \theta_2\end{pmatrix} &= \begin{pmatrix}0\\ \psi_{1y}\theta_{1x} - \psi_{1x}\theta_{1y}\end{pmatrix} \\ &= -\begin{pmatrix}(-4\pi^2 + R_c)\\ (-2\pi i + 2\pi i)\end{pmatrix}\partial_X A + |A|^2\begin{pmatrix}0\\ 4\pi^3\end{pmatrix}\sin 2\pi y \\ &\quad + A^2 \mathrm{e}^{2\mathrm{i}x}\cos(\pi y)\sin(\pi y)\begin{pmatrix}0\\ -2\pi + 2\pi\end{pmatrix} + \mathrm{cc} \\ &= |A|^2\begin{pmatrix}0\\ 4\pi^3\end{pmatrix}\sin(2\pi y). \end{aligned} \tag{52}$$

Hence

$$\begin{pmatrix}\psi_2\\ \theta_2\end{pmatrix} = |A|^2\hat{L}(0,2)^{-1}\begin{pmatrix}0\\ 4\pi^2\end{pmatrix} = |A|^2\begin{pmatrix}0\\ -\pi\sin 2\pi y\end{pmatrix}.$$

At $\mathcal{O}(\varepsilon^3)e^{\mathrm{i}\pi x}\sin(\pi y)$ we obtain

$$L(\pi,1)\begin{pmatrix}\psi_3\\ \theta_3\end{pmatrix} = \begin{pmatrix}-i\pi s A - 2\pi i\partial_X^2 A\\ \partial_T A + 4\pi^4|A|^2 A - \partial_X^2 A\end{pmatrix}. \tag{53}$$

Since $L(\pi,1)(2\pi i,1) = 0$ we need a solvability condition for (53). By the Fredholm alternative we obtain

$$\left\langle \psi^*(\pi,n), \begin{pmatrix}-i\pi s A - 2\pi i\partial_X^2 A\\ \partial_T A + 4\pi^4|A|^2 A - \partial_X^2 A\end{pmatrix}\right\rangle = 0, \tag{54}$$

where $\psi^*(\pi,n)$ is the null-eigenvector of the adjoint

$$L^*(\pi,1) = \begin{pmatrix}-2\pi^2 & i\pi\\ -4i\pi^3 & -2\pi^2\end{pmatrix}, \text{ i.e. } \psi^* = \begin{pmatrix}i\\ 2\pi\end{pmatrix}.$$

The solvability condition (54) thus yields

$$\partial_T A = \partial_X^2 A + \frac{s}{2}A - 4\pi^4|A|^2 A. \tag{55}$$

In (53) we have additional terms on the right hand side, i.e., additionally $\mathcal{O}(\varepsilon^3)$ in $\mathrm{Res}(\varepsilon\psi_A)$, but these are uncritical since they do not lie in the kernel of L. Also note that we do not actually solve for (ψ_3,θ_3) but only use (53) to derive the solvability condition.

An immediate observation from the GL equation (55) is again that for $s = 1$ it has stable spatially constant steady solutions

$$Ae^{i\phi}, \quad \phi \in [0, 2\pi], \quad |A| \equiv 1/(2\sqrt{2}\pi^2).$$

For $\phi = 0$ this (formally) yields the steady convection rolls

$$\begin{pmatrix} 0 \\ 0 \\ 1-y \end{pmatrix} + \varepsilon \begin{pmatrix} u_1 \\ u_2 \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1-y \end{pmatrix} + \frac{\varepsilon}{2\sqrt{2}\pi^2} \begin{pmatrix} 4\pi^2 \pi \cos \pi y \sin kx \\ -4\pi k \sin \pi y \cos kx \\ \sin \pi y \end{pmatrix} + \mathcal{O}(\varepsilon^2). \qquad (56)$$

Remark 4.1 a) Details of the analogous calculations for the full Navier–Stokes problem can be found in [Man92].
b) The formal calculations above do not guarantee that (56) is an $\mathcal{O}(\varepsilon)$ approximation of steady convection rolls for (42) – (44), nor that such steady rolls exist at all. However, this does hold, as can, for instance, be shown by Lyapunov–Schmidt reduction, see again [Fow97, Section 14].
c) Thus, the next step should be the mathematical justification of (55) by proving error estimates between a solution of (42) – (44) and approximations via (49) and (55). Again we refer to the literature, for instance [Sch94].
d) A numerical validation of, e.g., (56) is left as an exercise to the (ambitious) reader.
e) As already said, many more (and much more complicated) problems than the simple examples considered in this lecture can be analyzed using the amplitude formalism. For a classical enzyclopedic review we again refer to [CH93]. Additionally to the literature already cited we refer to [Uec03, BSTU06, SU07a, Uec07, DU09] for a selection of recent analysis and applications of the amplitude formalism. ⌋

Solutions to some exercises

Solution to Exercise 1.5. We obtain the ordinary differential equation

$$\ddot{y}_0 = -y_0$$

as first approximation. A comparison of the two solutions $y(t) = \cos(\sqrt{1+\varepsilon}\, t)$ and $y_0(t) = \cos(t)$ immediately shows that for $t = \mathcal{O}(\varepsilon^{-1})$ the difference $y(t) - y_0(t)$ is of order $\mathcal{O}(1)$ and hence y_0 provides no longer a good approximation of y for $t \geq \mathcal{O}(1/\varepsilon)$.

Solution to Exercise 1.6. Since $(\partial_t u)^3$ in $\partial_t^2 u + \varepsilon(\partial_t u)^3 + u = 0$ has the same sign as $\partial_t u$, this again describes an oscillator with small nonlinear damping. The ansatz $u(t) = \frac{1}{2}A(\varepsilon t)e^{ix} + \text{cc} + \varepsilon u_1(t)$ yields, at $\mathcal{O}(\varepsilon e^{ix})$, $i\partial_\tau A + \frac{3}{8}i|A|^2 A = 0$. This yields $A(\tau) = \dfrac{2A_0}{\sqrt{3A_0\tau + 4}}$, i.e., $A(\tau) \sim 1/\sqrt{\tau}$ as $\tau \to \infty$. For u we obtain

$$u(t) \sim \frac{1}{\sqrt{\varepsilon t}} \cos(t + \phi_0) + \mathcal{O}(\varepsilon). \qquad (.1)$$

This prediction is somewhat unsatisfying since we expect that $(u, u') \to 0$ as $t \to \infty$. For this consider the energy $E(t) = \frac{1}{2}(u'(t)^2 + u(t)^2)$. Then $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\varepsilon u'(t)^2$ and using the ODE this implies $E(t) \to 0$ as $t \to \infty$. In fact, solving for u_1 we find that the $\mathcal{O}(\varepsilon)$ terms in (.1) again decay. However, next there will be $\mathcal{O}(\varepsilon^2)$ terms, and so on. Although with a bit more theory (center manifolds) this can be treated systematically, this example also shows that so far the behaviour of some system as $t \to \infty$ usually cannot be studied using only amplitude equations.

Solution to Exercise 3.3. The linear dispersion relation for

$$\partial_t^2 u = \partial_x^2 u - u + u^2 \tag{.2}$$

is as in (29). Thus, let $e_j = \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$ with $\omega_0^2 = \sqrt{1 + k_0^2}$. Since the quadratic interaction of e_1 yields modes at e_{-2}, e_0, e_2 we need to take these into account in our ansatz. Thus, let

$$u(x,t) = \varepsilon A_1(X,T)e_1 + \frac{\varepsilon^2}{2}A_0(X,T) + \varepsilon^2 A_2(X,T)e_2 + \mathrm{cc}, \tag{.3}$$

where as before $X = \varepsilon(x - c_{\mathrm{g}}t)$, $T = \varepsilon^2 t$, $c_{\mathrm{g}} = \omega_0/k_0$. Plugging into (41) we obtain new $\mathcal{O}(\varepsilon^2)$ terms, i.e.,

$$\mathcal{O}(\varepsilon^2 e_0) : 0 = -A_0 + 2|A_1|^2 \quad \Rightarrow \quad A_0 = 2|A_1|^2,$$

$$\mathcal{O}(\varepsilon^2 e_2) : -(2\omega_0)^2 A_2 = -((2k_0)^2+1)A_2 + A^2 \;\Rightarrow\; A_2 = (4k_0^2+1-4\omega_0^2)^{-1}A_1^2 = -\frac{1}{3}A_1^2.$$

Plugging this into the $\mathcal{O}(\varepsilon^3 e_1)$ equation $c_{\mathrm{g}}^2\partial_X^2 A_1 - 2\mathrm{i}\omega_0\partial_T A_1 = \partial_X^2 A + 2A_1A_0 + 2A_2\overline{A}_1$ we obtain the NLS for A_1 in the form

$$2\mathrm{i}\omega_0\partial_T A_1 + (1 - c_{\mathrm{g}}^2)\partial_X^2 A + \frac{10}{3}|A|^2 A = 0. \tag{.4}$$

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