Local existence and uniqueness of solutions of the weak electrolyte model describing electro-convection in nematic liquid crystals

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Abstract

We show the local existence and uniqueness of solutions of the most advanced model for the description of electro-convection in nematic liquid crystals, namely the weak electrolyte model (WEM), which is a mixture of quasilinear parabolic equations and balance laws. We do this by bringing the WEM in a form where a standard iteration scheme can be applied.

Keywords: quasilinear hyperbolic–parabolic system, iteration scheme

1 Introduction

Electro-convection in nematic liquid crystals is a paradigm for pattern formation in non–isotropic media. Experimentally, a thin layer of such a material is contained in between two spatially extended electrode plates. When an alternating current is applied to the electrodes an electro-hydrodynamic instability occurs if the voltage is above a certain threshold. The trivial spatially homogeneous solution becomes unstable and bifurcates into non-trivial pattern [Cha77, PB98].

There are essentially two models for the mathematical description of electro-convection in nematic liquid crystals. These are the standard model ([ZK85] and the references therein) and the weak electrolyte model (WEM). The latter has been introduced by Kramer and Treiber in [Tre96, TK98] to overcome a number of insufficiencies of the standard model. In particular, the WEM has a number of pattern forming instabilities which agree well with experimental results [Tre96], see also [SU07].

The local existence and uniqueness of solutions of the WEM is a nontrivial task since the governing equations are a relatively complicated mixture of quasilinear parabolic equations and balance laws. Therefore, in [SU07] a regularized semilinear parabolic WEM was considered. Here, we solve the problem for the original WEM by combining optimal regularity theory for quasilinear parabolic systems and Kato's method for quasilinear hyperbolic systems. As a consequence, the justification results for the approximation of a regularized WEM by Ginzburg–Landau equations from [SU07] also hold for the original WEM.

The following presentation and non-dimensionalization of the WEM follows [DO04, SU07]. We consider a layer of nematic liquid crystals in between two infinitely extended horizontal plates of height π, i.e. in the following $(x, y, z) \in \Omega = \mathbb{R}^2 \times (0, \pi)$. In the WEM the average molecular axis of the nematic liquid crystals is described locally by a director field n of unit vectors. The Leslie-Erickson equations for n and the generalized Navier-Stokes equations for the fluid velocity v and the pressure p in the presence of an electric field E are given by

$$(\partial_t + v \cdot \nabla)n = \omega \times n + \delta^{\perp}(\lambda A n - h)\,, \tag{1}$$

$$P_2(\partial_t + v \cdot \nabla)v = -\nabla p - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E\,, \tag{2}$$

$$\nabla \cdot v = 0\,, \tag{3}$$

for $(x, y, z) \in \Omega$. These equations turn out to be a quasilinear parabolic system. The meaning of the quantities is as follows. The vorticity is

$$\omega = \frac{1}{2}(\nabla \times v), \tag{4}$$

and the molecular field h is given by

$$h = 2\left(\frac{\partial f}{\partial n} - \nabla \cdot \frac{\partial f}{\partial \nabla n}\right) - \varepsilon_a \pi^2 (n \cdot E)E \tag{5}$$

where

$$2f = (\nabla \cdot n)^2 + K_2[n \times (\nabla \times n)]^2 + K_3[n \cdot (\nabla \times n)]^2\,, \tag{6}$$

is the elastic energy density describing splay, twist (K_2), and bend (K_3) deformations, and where

$$\left(\frac{\partial f}{\partial \nabla n}\right)_{ij} := \frac{\partial f}{\partial n_{i,j}} \text{ with } n_{i,j} = \partial_{x_j} n_i.$$

We refer to [DO04] for a physical interpretation of the constants P_2, λ, K_2, K_3, and ε_a. The electric field $E = E(x, y, z, t) \in \mathbb{R}^3$ is considered to be quasistationary, i.e. rot $E = 0$. It is then split into an external forcing and some potential part, i.e.

$$E = E_p(t)(0, 0, 1)^T - \nabla \phi\,, \tag{7}$$

where usually in the experiments

$$E_p(t) = E_0 \cos \omega_0 t \tag{8}$$

with an $E_0 > 0$ and $\omega_0 > 0$.

The tensors A, and T^{visc} are, respectively, the shear flow tensor

$$A_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i), \tag{9}$$

and the viscous stress tensor

$$\begin{aligned} -T_{ij}^{visc} &= \sum_{k=1}^{3} \left(\alpha_1 n_i n_j n_k \Big(\sum_{l=1}^{3} n_l A_{kl} \Big) + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj} \right) \\ &\quad + \alpha_2 n_j m_i + \alpha_3 n_i m_j + \alpha_4 A_{ij} \end{aligned} \tag{10}$$

where

$$m = \delta^{\perp}(\lambda A n - h) \tag{11}$$

and with constant coefficients $\alpha_1, \ldots, \alpha_6$. The tensor Π with

$$\Pi_{ij} = \sum_{k=1}^{3} \frac{\partial f}{\partial n_{k,j}} n_{k,i} \tag{12}$$

is called the nonlinear Ericksen stress tensor. The projection tensor

$$\delta_{ij}^{\perp} = \delta_{ij} - n_i n_j \tag{13}$$

in (1) guarantees that $|n| = 1$ as long as the solution exists. This follows from writing $\partial_t n = (-v \cdot \nabla) n + \omega \times n + \delta^{\perp} f$ with $f = \lambda A n - h$. The transport terms $(-v \cdot \nabla) n + \omega \times n$ conserve lengths, and for $\partial_t n = \delta^{\perp} f$ we obtain, for any f,

$$\frac{1}{2}\partial_t |n|^2 = n \cdot \partial_t n = n \cdot \delta^{\perp} f = n \cdot f - n \cdot (n \cdot f) n = 0$$

using $n \cdot n = n_1^2 + n_2^2 + n_3^2 = 1$.

The second part of the WEM comes from the quasi-static Maxwell equations. In the WEM [Tre96, TK98] there are two species of oppositely charged mobile ions. Under the assumption of a linear recombination and zero diffusivity, the WEM consists of (1)-(3) and two balance equations for the charge density ρ and the deviation σ of the local conductivity from 1, namely

$$P_1(\partial_t + v \cdot \nabla)\rho = -\nabla \cdot (\mu E \sigma)\,, \tag{14}$$

$$(\partial_t + v \cdot \nabla)\sigma = -\alpha^2 \pi^2 \nabla \cdot (\mu E \rho) - \frac{r}{2}\left(2\sigma + \sigma^2 - P_1 \pi^2 \alpha \rho^2\right). \tag{15}$$

Finally the system is closed by Poisson's law

$$\rho = \nabla \cdot (\varepsilon E) \ . \tag{16}$$

The dielectric tensor ε and the conductivity tensor μ are given by

$$\varepsilon_{ij} = \delta_{ij} + \varepsilon_a n_i n_j \quad \text{and} \quad \mu_{ij} = \delta_{ij} + \sigma_a n_i n_j.$$

Similar to P_2, the parameter P_1 is a Prandtl–type time scale ratio. For a physical interpretation of the constants P_1, σ_a, α, and r we again refer to [DO04]. The WEM will be considered with the boundary conditions

$$\partial_z n_2 = n_3 = \partial_z v_1 = \partial_z v_2 = v_3 = \phi = 0 \tag{17}$$

at $z = 0, \pi$.

Using Poisson's law E, resp. ϕ, can be expressed in terms of (n, ρ) and so (1)-(3) and (14)-(15) can be rewritten as a system of dynamical equations for $V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma)$. Thus, (1)-(3), (14), (15) is abbreviated as

$$\partial_t V = M(t)V + \tilde{N}(t, V) \tag{18}$$

where $M(t)V$ stands for the linear and $\tilde{N}(t, V)$ for the nonlinear terms with respect to V.

The WEM equations are invariant under arbitrary translations in x and y and under the reflections

$$\mathcal{S}_1 : (x, n_2, n_3, v_1) \quad \to \quad -(x, n_2, n_3, v_1), \tag{19}$$
$$\mathcal{S}_2 : (y, n_2, v_2) \quad \to \quad -(y, n_2, v_2), \tag{20}$$
$$\mathcal{S}_3 : (z, n_3, v_3, \phi) \quad \to \quad -(z, n_3, v_3, \phi). \tag{21}$$

The local existence and uniqueness of solutions is nontrivial due to the relatively complicated mixture of quasilinear parabolic equations and balance laws. To our knowledge no local existence and uniqueness result is documented so far in the literature. Thus, here we make a first step and prove the local existence and uniqueness for initial conditions in a neighborhood of the trivial solution $V = 0$. We do this by bringing the WEM in a form where a standard iteration scheme can be applied. In order to do so the regularity of the components of V has to be chosen properly, for instance n has to be chosen one time more regular than v.

Notation. The Sobolev space $H^m(\Omega)$ is the space of m-times weakly differentiable functions $\Omega \to \mathbb{R}$ equipped with the norm

$$\|u\|_{H^m(\Omega)} = \sum_{|j|=0}^{m} \|\partial_x^j u\|_{L^2(\Omega)} \qquad \text{with} \qquad \|u\|_{L^2(\Omega)}^2 = \int_\Omega |u(x)|^2 dx.$$

We shall also need fractional order Sobolev spaces and interpolation spaces. The symmetry $\mathcal{S}_3$ allows to extend the WEM periodically into the bounded direction and to expand the variables in Fourier series with respect to z, i.e we write for instance

$$n_2(x,y,z) = \int\int\sum_{k_3\in\mathbb{Z}} \hat{n}_2(k_1,k_2,k_3)e^{ik_1x+ik_2y+ik_3z}dk_1dk_2, \tag{22}$$

and similarly for $n_3,\ldots,\sigma$, and consider even in z extensions for $n_2, v_1, v_2, \sigma, \rho$ and odd in z extensions for n_3, v_3, ϕ. For $s\in\mathbb{R}$ we now define

$$\|\hat{u}\|^2_{l^2(s)} = \int\int\sum_{k_3\in\mathbb{Z}} |\hat{u}(k_1,k_2,k_3)|^2(1+|k_1|^2+|k_2|^2+|k_3|^2)^s dk_1dk_2.$$

Due to Parseval's identity there is a one to one relation and norm equivalence between physical and Fourier space, i.e., for all $m\in\mathbb{N}$ there is a constant $C>0$ such that

$$C^{-1}\|\hat{u}\|_{l^2(m)} \le \|u\|_{H^m} \le C\|\hat{u}\|_{l^2(m)}. \tag{23}$$

In the following we use H^m as abbreviation for $H^m(\mathbb{R}^2\times[0,2\pi])$ with periodic boundary conditions in the third variable.

Using (23) we also define H^s for every $s\ge 0$ as the space of functions in L^2 whose Fourier transform is in $l^2(s)$, equipped with the norm

$$\|u\|_{H^s} = \|\hat{u}\|_{l^2(s)}.$$

Finally we note that as a consequence of the periodic boundary conditions interpolation spaces ([LM72]) are easily characterized. For instance for $\Delta : H^{s+2}\to H^s$ we have

$$\begin{aligned}[H^s, H^{s+2}]_\theta &= \{u\ :\ (1+(-\Delta)^\theta)u\in H^s\} = \{u\ :\ (1+|k|^{2\theta})\hat{u}\in l^2(s)\}\\ &= \{u\ :\ \hat{u}\in l^2(s+2\theta)\} = H^{s+2\theta}.\end{aligned}$$

We may now state our main theorem.

Theorem 1.1. *Let $\theta\in(0,1)$ and $m\in\mathbb{N}$. There exists a $C_1>0$ such that for all initial conditions*

$$V_0 = (n_2,n_3,v_1,v_2,v_3,\sigma,\varrho)|_{t=0} \in [H^{m+3+2\theta}]^2\times[H^{m+2+2\theta}]^3\times[H^{m+2}]^2$$

with $\nabla\cdot v=0$ and $\|V_0\|_{[H^{m+3+2\theta}]^2\times[H^{m+2+2\theta}]^3\times[H^{m+2}]^2} \le C_1$ there exists a $T_0>0$ such that (18) has a unique mild solution

$$V\in C([0,T_0],[H^{m+3}]^2\times[H^{m+2}]^3\times[H^{m+2}]^2)\cap C^1([0,T_0],[H^{m+1}]^2\times[H^m]^3\times[H^{m+1}]^2)$$

with $V|_{t=0}=V_0$.

The additional regularity for $(n_2, n_3)|_{t=0}$ and $v|_{t=0}$ described by θ is needed to fulfill some compatibility conditions at $t = 0$ to apply maximal regularity to the quasilinear parabolic subsystem for (n_2, n_3) and v. As a consequence, (n_2, n_3) and v enjoy further regularity properties, e.g., they are Hölder continuous in time with values in $[H^{m+3}]^2 \times [H^{m+2}]^3$, while further regularity for the charge density ρ and the local conductivity σ are unclear, and we restrict to the simple formulation of Theorem 1.1.

The chosen L^2 framework excludes spatially extended solutions, like periodic, quasi-periodic, or front solutions. Hence, w.r.t. the above mentioned Ginzburg-Landau approximation it would be desirable to generalize the L^2 framework to a $L^2_{l,u}$ framework which would include these solutions, cf. [Schn94]. Difficulties in $\mathbb{R}^2 \times (0, \pi)$ come from the non-smoothness of the symbol of the inverse Stokes operator or of the projection operator Q (cf. Lemma 2.3) onto the divergence free vector fields, cf. [SU07, Remark 4.6]. In spatial domains $\mathbb{R} \times (0, \pi)$ this problem can be avoided, but we are not aware of any literature which handle quasilinear hyperbolic systems in $L^2_{l,u}$ spaces.

The plan of the proof is as follows. In §2.1 we explain that the WEM is an evolutionary system for the variables collected in V, i.e. we eliminate the pressure term ∇p and express E in terms of V. In §2.2–§2.3 we extract the leading terms in the (n, v)-part and rewrite the balance laws as symmetric quasilinear hyperbolic systems in the sense of [Kat75], and show local existence for each of these subsystems. Then in §3 we formulate an iteration scheme for the full system and prove the convergence of the sequence constructed by the iteration scheme.

2 The structure of the WEM

2.1 The WEM as a dynamical system

To write the WEM as an evolutionary system in $V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma)$ we proceed as in [SU07], where also the three Lemmas below are proved (Lemmas A.2, A.3, A.4 in [SU07]). Essentially the proofs follow by explicit calculation from Fourier representation as (22) and (23). First we need to express E in terms of V. Therefore we have to solve

$$\rho \;=\; \sum_{k=1}^{3} \partial_k(\varepsilon_{km} E_m) = \sum_{k=1}^{3} \sum_{m=1}^{3} \partial_k \big[(\delta_{km} + \varepsilon_a n_k n_m)(E_0 \cos(\omega_0 t)\delta_{m3} - \partial_m \phi) \big]$$

with respect to ϕ under the boundary conditions $\phi|_{z=0,\pi} = 0$. We find

$$(M + G)\phi = F(n, \rho, E_0)$$

where

$$\begin{aligned} F(n,\rho,E_0) &= -\rho+\cos(\omega_0 t)\sum_{k=1}^{3}\sum_{m=1}^{3}\partial_k((\delta_{km}+\varepsilon_a n_k n_m)E_0\delta_{m3}) \ , \\ M\phi &= \Delta\phi+\varepsilon_a\partial_1^2\phi, \quad G\phi=\varepsilon_a\sum_{k=1}^{3}\sum_{m=1}^{3}\partial_k(n_k n_m\partial_m\phi)-\varepsilon_a\partial_1^2\phi \ . \end{aligned}$$

Lemma 2.1. *The operator M^{-1} is bounded from H^s into $\{\phi\in H^{s+2} \ : \ \phi=0 \text{ at } z=0,\pi\}$.*

Hence the electric potential ϕ satisfies $(1+GM^{-1})M\phi=F(n,\rho,E_0)$, where GM^{-1} is small for $\tilde{n}=n-(1,0,0)^T$ small. By using Neumann's series we formally obtain

$$\phi=M^{-1}(1+GM^{-1})^{-1}F(n,\rho,E_0). \tag{24}$$

Lemma 2.2. *For $s\geq 2$ and $\|V\|_{H^s}$ sufficiently small the operator $M^{-1}(1+GM^{-1})^{-1}$ is bounded from H^s into H^{s+2}.*

Next we focus on the hydrodynamic part of (18) and define the projection Q onto the divergence free vector fields by $w=Qf$, where w solves

$$w-\nabla p \;=\; f, \quad \nabla\cdot w=0, \quad \partial_z w_1=\partial_z w_2=w_3=0 \text{ at } z=0,\pi. \tag{25}$$

Lemma 2.3. *The projection Q is continuous from $[H^m]^3$ into $\{v\in[H^m]^3 \ : \ \nabla\cdot v=0\}$.*

Since $n_1^2+n_2^2+n_3^2=1$ for our purposes it is sufficient to consider n_2 and n_3. Hence we finally consider

$$\begin{aligned} \partial_t n_2 &= \langle e_2,-(v\cdot\nabla)n+\omega\times n+\delta^{\perp}(\lambda An-h)\rangle \ , && (26)\\ \partial_t n_3 &= \langle e_3,-(v\cdot\nabla)n+\omega\times n+\delta^{\perp}(\lambda An-h)\rangle \ , && (27)\\ \partial_t v &= P_2^{-1}Q(-(v\cdot\nabla)v-\nabla\cdot(T^{visc}+\Pi)+\pi^2\rho E) \ , && (28)\\ \partial_t\rho &= -v\cdot\nabla\rho-P_1^{-1}\nabla\cdot(\mu E\sigma), && (29)\\ \partial_t\sigma &= -v\cdot\nabla\sigma-\alpha^2\pi^2\nabla\cdot(\mu E\rho)-\frac{r}{2}(2\sigma+\sigma^2-P_1\pi^2\alpha\rho^2), && (30) \end{aligned}$$

under the boundary conditions (17), i.e., $\partial_z n_2=n_3=\partial_z v_1=\partial_z v_2=v_3=\phi=0$ at $z=0,\pi$.

2.2 The quasilinear parabolic part

We start with the computation of the highest order derivative terms in the (n,v)–part of the system. Here and in the following $\star$ stands for terms with less derivatives or terms in which the highest derivative occurs nonlinearly.

We introduce the derivation $\tilde{n}$ of the director from the planar alignment by

$$n=(1+\tilde{n}_1,\ \tilde{n}_2,\ \tilde{n}_3)^T \ .$$

From $(1+\tilde n_1)^2 + \tilde n_2^2 + \tilde n_3^2 = 1$ we find $\tilde n_1 = \mathcal{O}(\tilde n_2^2 + \tilde n_3^2)$. Therefore

$$(\nabla\cdot n)^2 = (\partial_{x_2}\tilde n_2 + \partial_{x_3}\tilde n_3)^2 + \star, \quad \nabla\times n = (\partial_{x_2}\tilde n_3 - \partial_{x_3}\tilde n_2,\ -\partial_{x_1}\tilde n_3,\ \partial_{x_1}\tilde n_2)^T \ + \star\,,$$
$$n\cdot(\nabla\times n) = (\partial_{x_2}\tilde n_3 - \partial_{x_3}\tilde n_2) + \star \quad \text{and} \quad n\times(\nabla\times n) = (0,\ -\partial_{x_1}\tilde n_2,\ -\partial_{x_1}\tilde n_3)^T \ + \star\,.$$

Thus

$$2f = (\partial_{x_2}\tilde n_2 + \partial_{x_3}\tilde n_3)^2 + K_2((\partial_{x_1}\tilde n_2)^2 + (\partial_{x_1}\tilde n_3)^2) + \ K_3(\partial_{x_2}\tilde n_3 - \partial_{x_3}\tilde n_2)^2 + \star\,.$$

Moreover

$$\delta^\perp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \star,$$

and therefore to calculate h in (5) we only need to calculate rows 2 and 3 of $\frac{\partial f}{\partial(\nabla n)}$, i.e.,

$$2\,\frac{\partial f}{\partial(\nabla n)} = \begin{pmatrix} \star & \star & \star \\ 2K_2\partial_{x_1}n_2 & 2(\partial_{x_2}n_2 + \partial_{x_3}n_3) & -2K_3(\partial_{x_2}n_3 - \partial_{x_3}n_2) \\ 2K_2\partial_{x_1}n_3 & 2K_3(\partial_{x_2}n_3 - \partial_{x_3}n_2) & 2(\partial_{x_2}n_2 + \partial_{x_3}n_3) \end{pmatrix} + \star\,.$$

Thus,

$$2\nabla\cdot\frac{\partial f}{\partial(\nabla n)} = \begin{pmatrix} \star \\ 2K_2\partial_{x_1}^2\tilde n_2 + 2\partial_{x_2}^2\tilde n_2 + 2\partial_{x_3}\partial_{x_2}\tilde n_3 + 2K_3\partial_{x_3}^2\tilde n_2 - 2K_3\partial_{x_2}\partial_{x_3}\tilde n_3 \\ 2K_2\partial_{x_1}^2\tilde n_3 + 2K_3\partial_{x_2}^2\tilde n_3 - 2K_3\partial_{x_2}\partial_{x_3}\tilde n_2 + 2\partial_{x_2}\partial_{x_3}\tilde n_2 + 2\partial_{x_3}^2\tilde n_3 \end{pmatrix} \ + \ \star.$$

Using this expansion and $\langle e_j, \omega\times n + \delta^\perp(\lambda A n)\rangle = \star$ we find for the equations for $\partial_t\tilde n_2$ and $\partial_t\tilde n_3$ in Fourier space that

$$\partial_t\widehat{\begin{pmatrix}\tilde n_2\\ \tilde n_3\end{pmatrix}} = -2\begin{pmatrix} K_2k_1^2 + k_2^2 + K_3k_3^2 & k_2k_3 - K_3k_2k_3 \\ k_2k_3 - K_3k_2k_3 & K_2k_1^2 + K_3k_2^2 + k_3^2 \end{pmatrix}\widehat{\begin{pmatrix}\tilde n_2\\ \tilde n_3\end{pmatrix}} \ + \ \star\,.$$

This matrix turns out to be negative definite if $K_2 > 0$ and $K_3 > 0$.

Next we come to the equation for v. We proceed as above and compute the terms with highest derivatives which are linear. Since in the $\partial_t n$-equation no v terms played any role we have some lower triangular block structure and so in the $\partial_t v$ equation it is sufficient to consider the linear terms with highest derivative of v. All the rest will be denoted as above with $\star$. Hence, it is sufficient to analyse T^{visc} and in T^{visc} the $A_{k\ell}$ terms. We find

$$-T^{visc}_{ij} = (\alpha_1 + \alpha_5 + \alpha_6)\delta_{i1}\delta_{j1}A_{ij} + \alpha_4 A_{ij} + \star$$

so that

$$\nabla \cdot T^{visc} = \alpha_4 \Delta v + (\alpha_1 + \alpha_5 + \alpha_6) \begin{pmatrix} \partial_1^2 v_1 \\ 0 \\ 0 \end{pmatrix} + \star$$

where we used $\nabla \cdot v = 0$. Therefore the (n, v)–part is of the form

$$\begin{aligned} \partial_t \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} &= L_n \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} + G_n \ , \\ \partial_t v &= L_v v + G_v \ , \end{aligned}$$

where L_n is defined by its symbol in Fourier space

$$-\hat{L}_n = 2 \begin{pmatrix} K_2 k_1^2 + k_2^2 + K_3 k_3^2 & k_2 k_3 - K_3 k_2 k_3 \\ k_2 k_3 - K_3 k_2 k_3 & K_2 k_1^2 + K_3 k_2^2 + k_3^2 \end{pmatrix} ,$$

and

$$L_v v = P_2^{-1} Q \left[\alpha_4 \Delta v + (\alpha_1 + \alpha_5 + \alpha_6) \begin{pmatrix} \partial_1^2 v_1 \\ 0 \\ 0 \end{pmatrix} \right] , \tag{31}$$

and where G_n and G_v stand for the remaining terms. L_n and L_v generate analytic semigroups which later allow to control G_n and G_v by optimal regularity results.

Lemma 2.4. *Let $\theta \geq 0$ and $m \in \mathbb{N}$. a) The operator $L_n : [H^{m+2}]^2 \to [H^m]^2$ defines an analytic semigroup e^{tL_n} in $[H^m]^2$ satisfying*

$$\|e^{tL_n} u\|_{[H^{m+2\theta}]^2} \leq C(1 + t^{-\theta}) \|u\|_{[H^m]^2} .$$

b) The operator $L_v : Q[[H^{m+2}]^3] \to Q[[H^m]^3]$ defines an analytic semigroup e^{tL_v} in $Q[[H^m]^3]$ satisfying

$$\|e^{L_v t} u\|_{[H^{m+2\theta}]^3} \leq C(1 + t^{-\theta}) \|u\|_{[H^m]^3} .$$

Proof. The result follows from the fact that under the chosen boundary conditions the problem can be extended periodically into the bounded z-direction such that the estimate is a consequence of the representations of L_n and L_v in Fourier space and (23). Since $\|e^{t\hat{L}_n}\|_{\mathbb{R}^{2\times 2}} \leq e^{-\tilde{C}t|k|^2}$ for a $\tilde{C} \geq 0$, we have that

$$\begin{aligned} &\|e^{tL_n} u\|_{[H^{m+2\theta}]^2} \leq C \|e^{t\hat{L}_n} \hat{u}\|_{[l^2(m+2\theta)]^2} \leq C \|e^{-\tilde{C}t|k|^2} |\hat{u}|\|_{l^2(m+2\theta)} \\ &\quad \leq C \sup_k |e^{-\tilde{C}t|k|^2} (1+k^2)^\theta| \, \|\hat{u}\|_{(l^2(m))^2} \leq C(1+t^{-\theta}) \|\hat{u}\|_{(l^2(m))^2} \leq C(1+t^{-\theta}) \|u\|_{(H^m)^2} . \end{aligned}$$

Similarly the estimate for e^{tL_v} follows. □

In order to apply an iteration scheme to solve the quasilinear problem for (n_2, n_3) and v coupled to the hyperbolic problem for (ρ, σ) we shall need maximal regularity results. Therefore we first study the linear inhomogeneous problems

$$\partial_t(n_2, n_3) = L_n(n_2, n_3) + f_n \tag{32}$$

and

$$\partial_t v = L_v v + f_v. \tag{33}$$

Given, e.g., $f_v \in C^{0,\theta}([0, T_0], X)$ with $0 < \theta < 1$, where X is some Banach space, maximal regularity means that $\partial_t v$ and $L_v v$ enjoy the same regularity as f_v. Additional to the natural assumption that $v_0 = v_{t=0} \in D(L_v)$, the crucial point to obtain such maximal regularity results are compatibility conditions at $t = 0$, namely

$$L_v v_0 + f_v(0) \in D_{L_v}(\theta, \infty). \tag{34}$$

Here the real interpolation space $D_{L_v}(\theta, \infty)$ is the set of all $v \in X$ such that $t^{1-\theta}\|L_v e^{tL_v} v\|_X$ is bounded as $t \to 0$, see, e.g., [Lun95]. From Lemma 2.4 we see that for, e.g., $X = [H^m]^3$ and $v \in [H^{m+2\theta}]^3$ we have

$$\|L_v \mathrm{e}^{tL_v} v\|_X \le C\|\mathrm{e}^{tL_v} v\|_{[H^{m+2}]^3} \le C(1 + t^{-(1-\theta)})\|v\|_{[H^{m+2\theta}]^3},$$

and since clearly these estimates are sharp we thus have $D_{L_v}(\theta, \infty) = [H^{m+2\theta}]^3$. The problem for $\partial_t(n_2, n_3)$ can be analyzed in the same manner, and for later reference we note the following lemma.

Lemma 2.5. *For all $\theta \in (0, 1)$, $m \ge 0$ and $T_0 > 0$ there exists a $C_2 > 0$ such that the following holds.*

a) If $f_n \in C^{0,\theta}([0, T_0], [H^{m+1}]^2)$ and $L_n(n_2, n_3)|_{t=0} + f_n(0) \in [H^{m+1+2\theta}]^2$, then there exists a unique solution $(n_1, n_2) \in C^{0,\theta}([0, T_0],\ [H^{m+3}]^2) \cap C^{1,\theta}([0, T_0],\ [H^{m+1}]^2)$ of (32) which is bounded in this space by $C_2(\|L_n(n_2, n_3)|_{t=0} + f_n(0)\|_{[H^{m+1+2\theta}]^2} + \|f_n\|_{C^{0,\theta}([0,T_0],[H^{m+1}]^2)} + \|(n_2, n_3)|_{t=0}\|_{[H^{m+3+2\theta}]^2})$.

b) If $f_v \in C^{0,\theta}([0, T_0],\ Q[[H^m]^3])$ and $L_v v|_{t=0} + f_v(0) \in [H^{m+2\theta}]^3$, then there exists a unique solution $v \in C^{0,\theta}([0, T_0],\ Q[[H^{m+2}]^3]) \cap C^{1,\theta}([0, T_0],\ Q[[H^m]^3])$ of (33) with norm bounded in this space by $C_2(\|L_v v|_{t=0} + f_v(0)\|_{[H^{m+2\theta}]^3} + \|f_v\|_{C^{0,\theta}([0,T_0],[H^m]^3)} + \|v|_{t=0}\|_{[H^{m+2+2\theta}]^3})$.

Proof. These are consequences of Lemma 2.4 and optimal regularity theory. See, e.g., [Lun95, Theorem 4.3.1] or [Sin85]. □

2.3 The balance laws

The equations (29), (30) for $\partial_t \rho$ and $\partial_t \sigma$ are of different type than (26)–(28). They are balance laws, hence quasilinear hyperbolic and not quasilinear parabolic. Nevertheless there is some damping in the (ρ, σ)-part due to the -2σ term in the σ equation.

Again we concentrate on the terms with highest derivatives, i.e.

$$\partial_t \rho = -v_j \partial_j \rho - P_1^{-1} \partial_j (a_j \sigma) \ , \tag{35}$$

$$\partial_t \sigma = -v_j \partial_j \sigma - \alpha^2 \pi^2 \partial_j (a_j \rho) + G_\sigma \ , \tag{36}$$

where we used Einstein's sum convention ($a_i b_i = \sum_i a_i b_i$) and the abbreviation

$$a = \mu E,$$

and where G_σ stands for the remaining terms, which are semilinear. We shall assume that the coefficients v_j and a_j and hence also the n–dependent coefficients in G_σ are sufficiently smooth and later relate this to the smoothness of V. Setting

$$u = \begin{pmatrix} \rho \\ \sigma \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_j = \begin{pmatrix} v_j & P_1^{-1} a_j \\ \alpha^2 \pi^2 a_j & v_j \end{pmatrix} \text{ for } j = 1, 2, 3,$$

and

$$f(t, x, u) = \begin{pmatrix} 0 \\ G_\sigma \end{pmatrix} - \begin{pmatrix} P_1^{-1} (\nabla \cdot a) \sigma \\ \alpha^2 \pi^2 (\nabla \cdot a) \rho \end{pmatrix},$$

(35),(36) becomes

$$A_0 \partial_t u + \sum_{j=1}^{3} A_j \partial_{x_j} u = f(t, x, u), \qquad u|_{t=0} = u_0, \tag{37}$$

which is almost of the form (Q) in [Kat75], where however the matrices $A_j \in \mathbb{R}^{2\times 2}$ are not yet symmetric. Clearly, the system can be symmetrized, e.g., by setting $\tilde{u} = (s_1 \rho, s_2 \sigma)$ and choosing s_1, s_2 to fulfill $s_1/s_2 = \sqrt{\alpha^2 \pi^2 P_1}$, but we omit this obvious step. To apply [Kat75, Theorem IV and Remark 5.1b] to (37) resp. its symmetrized version it is sufficient to ensure that $A_j \in C([0, T_0], H^s(\Omega, \mathbb{R}^{2\times 2}))$ and $f_0 \in L^\infty([0, T_0], H^s(\Omega, \mathbb{R}^2)) \cap C([0, T_0], L^2(\Omega, \mathbb{R}^2))$, where $f_0(t, x) = f(t, x, u^*(x))$ for some fixed $u^* \in H^s(\Omega, \mathbb{R}^2)$, which is chosen sufficiently close to the initial condition u_0. This gives conditions on $(\rho, \sigma)|_{t=0}, v$ and a. For simplicity we continue to consider G_σ as inhomogeneity and obtain

Lemma 2.6. *Let $s > 5/2$ and $T_0 > 0$. Then for all $C_1 > 0$ there exists a $C_2 > 0$ such that the following holds. Let $(\rho, \sigma)|_{t=0} \in [H^s]^2$, $v \in C([0, T_0], Q[[H^s]^3])$, $a \in C([0, T_0], [H^{s+1}]^3)$ and $G_\sigma \in C([0, T_0], H^s)$, with norms bounded in these spaces by C_1. Then there exists a unique solution*

$$(\rho, \sigma) \in C([0, T_0],\ [H^s]^2) \cap C^1([0, T_0],\ [H^{s-1}]^2)$$

of (35) and (36) with norm bounded in this space by $C_2 \| (\rho, \sigma)|_{t=0} \|_{[H^s]^2}$.

Remark 2.7. The results in [Kat75] are essentially based on a priori estimates and an iteration scheme. It is instructive to review these a priori estimates for (35), (36), which are the basis for any iteration scheme. For $s \geq 0$ and with $\int \cdot = \int \cdot \, \mathrm{d}x$ we find

$$\begin{aligned}
\frac{1}{2}\,\partial_t \int (\partial_k^s \rho)^2 &= -\int (\partial_k^s \rho)\partial_k^s(v_j \partial_j \rho) - \int (\partial_k^s \rho) P_1^{-1} \partial_k^s \partial_j (a_j \sigma) \\
&= -\int \frac{1}{2}\,\partial_j((\partial_k^s \rho)^2) v_j + s.t. - \int P_1^{-1} a_j (\partial_k^s \rho)(\partial_k^s \partial_j \sigma) + s.t. \\
&= +\int \frac{1}{2}\,(\partial_k^s \rho)^2 (\partial_j v_j) + s.t. - \int P_1^{-1} a_j (\partial_k^s \rho)(\partial_k^s \partial_j \sigma) + s.t. \\
&= -\int P_1^{-1} a_j (\partial_k^s \rho)(\partial_k^s \partial_j \sigma) + s.t.\ ,
\end{aligned}$$

with multiindex k and where s.t. stands here and in the following for semilinear terms, i.e. for terms with s or less derivatives acting on ρ, σ. Similarly, we find

$$\begin{aligned}
\frac{1}{2}\,\partial_t \int (\partial_k^s \sigma)^2 &= -\int (\partial_k^s \sigma)\partial_k^s(v_j \partial_j \sigma) - \int (\partial_k^s \sigma)\partial_k^s(\alpha^2 \pi^2 \partial_j (a_j \rho)) + \int (\partial_k^s \sigma)(\partial_k^s G_\sigma) \\
&= -\int \alpha^2 \pi^2 a_j (\partial_k^s \sigma)(\partial_k^s \partial_j \rho) + s.t. = \int \alpha^2 \pi^2 a_j (\partial_k^s \partial_j \sigma)(\partial_k^s \rho) + s.t.\ .
\end{aligned}$$

Thus we have

$$\frac{1}{2}\,\partial_t \left[\alpha^2 \pi^2 \int (\partial_k^s \rho)^2 + P_1^{-1} \int (\partial_k^s \sigma)^2 \right] = s.t.\ , \tag{38}$$

which shows that the energy contained in the highest derivatives is conserved up to lower order semilinear terms, and this also shows the essential symmetry of (35),(36).

3 The full system and the iteration scheme

To prove Theorem 1.1 we now combine the optimal regularity theory of [Sin85, Lun95] for quasilinear parabolic equations and the existence theory of [Kat75] for quasilinear hyperbolic systems. In combining the two methods one has to be careful since in contrast to solutions of quasilinear parabolic systems, solutions of quasilinear hyperbolic systems in general are not Hölder–continuous in time, see [Kat75, Sec.5.3] for some counter–examples.

The idea is to find solutions by the iteration scheme

$$\begin{aligned}
\partial_t (n_2, n_3)_{i+1} &= L_n (n_2, n_3)_{i+1} + G_n(V_i), \\
\partial_t v_{i+1} &= L_v v_{i+1} + G_v(V_i), \\
\partial_t \rho_{i+1} &= -\sum_{j=1}^{3} (v_j)_i \partial_j \rho_{i+1} - P_1^{-1} \sum_{j=1}^{3} \partial_j ((a_j)_i \sigma_{i+1}) \\
\partial_t \sigma_{i+1} &= -\sum_{j=1}^{3} (v_j)_i \partial_j \sigma_{i+1} - \alpha^2 \pi^2 \sum_{j=1}^{3} \partial_j ((a_j)_i \rho_{i+1}) + G_\sigma(V_i)\ .
\end{aligned} \tag{39}$$

Thus it remains to choose the space for V in such a way that for given V_i we have that $G_n(V_i), G_v(V_i)$ fulfill the assumptions of Lemma 2.5 and that v_i, a_i and $G_\sigma(V_i)$ fulfill the assumptions of Lemma 2.6. Therefore we note the following Lemma, where we add the parameter θ to deal with the compatibility conditions in Lemma 2.5.

Lemma 3.1. *For $m \geq 1$ and $\theta \in [0,1)$ the nonlinearity $G = (G_n, G_v, 0, G_\sigma)$ is locally Lipschitz continuous from $[H^{m+3+2\theta}]^2 \times [H^{m+2+2\theta}]^3 \times [H^{m+2}]^2$ into $[H^{m+1+2\theta}]^2 \times [H^{m+2\theta}]^3 \times [H^{m+2}]^2$.*

Proof. We have $\omega \in H^{m+1+2\theta}$ by (4), $f \in H^{m+2+2\theta}$ by (6), $E \in H^{m+3}$ by (7) and Lemma 2.2, $h \in H^{m+1+2\theta}$ by (5), $\delta_{ij}^{\perp} - \delta_{ij} \in H^{m+3+2\theta}$ by (13), $A \in H^{m+1+2\theta}$ by (9), $m \in H^{m+1+2\theta}$ by (11), $T^{visc} \in H^{m+1+2\theta}$ by (10), and $\Pi \in H^{m+2+2\theta}$ by (12). Therefore the right hand side G_n of (1) is in $H^{m+1+2\theta}$ and the right hand side G_v of (2) is in $H^{m+2\theta}$. We have $\varepsilon_{ij} - \delta_{ij} \in H^{m+3+2\theta}$ and $\mu_{ij} - \delta_{ij} \in H^{m+3+2\theta}$. From $\rho \in H^{m+2}$ we immediately find that the terms collected in G_σ are in H^{m+2}. □

Proof of Theorem 1.1. To use the iteration scheme (39) we need to satisfy, in each step $i \mapsto i+1$,

a) the regularity of the initial data, and the compatibility conditions

$$L_n(n_2, n_3)|_{t=0} + f_n(0) \in [H^{m+1+2\theta}]^2 \text{ and } L_v v|_{t=0} + f_v(0) \in [H^{m+2\theta}]^3.$$

b) the conditions on v, a and G_σ in Lemma 2.6, with $s = m+2$;

c) the conditions $G_n \in C^{0,\theta}([0,T_0],[H^{m+1}]^2)$ and $G_v \in C^{0,\theta}([0,T_0],Q[[H^m]^3])$ in Lemma 2.5.

Since the initial data are always the same, a) follows from Lemma 3.1 with $\theta > 0$ chosen in Theorem 1.1.

It is clear that Lemma 3.1 also holds for functions continuous resp. Hölder continuous in time with values in the respective Sobolev spaces. Thus, $G_\sigma \in C([0,T_0],H^{m+2})$, and with $v \in C([0,T_0],H^{m+2})$ and $a = \mu E \in C^{(}[0,T_0],H^{m+3})$ we immediately have b).

Finally we need to check that the lack of Hölder continuity in time of the solutions (ρ_i, σ_i) does not cause problems for $G_n(V_i), G_v(V_i)$. The idea is to trade some spatial differentiability of (ρ, σ) for Lipschitz continuity in time. In detail, from (35) we find that

$$\begin{aligned}\|\rho(t+\delta) - \rho(t)\|_{H^m} &= \left\| \int_t^{t+\delta} \partial_t \rho \, \mathrm{d}\tau \right\|_{H^m} = \left\| \int_t^{t+\delta} -\sum_{j=1}^3 v_j \partial_j \rho - P_1^{-1} \sum_{j=1}^3 \partial_j(a_j \sigma) \, \mathrm{d}\tau \right\|_{H^m} \\ &\leq C\delta \Big(\|v\|_{C([0,T_0],[H^m]^3)} \|\rho\|_{C([0,T_0],H^{m+1})} \\ &\qquad + (\|\rho\|_{C([0,T_0],H^{m+1})} + \|n\|_{C([0,T_0],[H^{m+2}]^2)}) \|\sigma\|_{C([0,T_0],H^{m+1})} \Big)\end{aligned}$$

and similarly for σ. Hence

$$V \in C^{0,\theta}([0,T_0],[H^{m+3}]^2) \times C^{0,\theta}([0,T_0],[H^{m+2}]^3) \times C([0,T_0],[H^{m+2}]^2)$$

implies $\sigma, \rho \in C^{0,1}([0,T_0],H^m)$, where $C^{0,1}([0,T_0],X) \subset C^{0,\beta}([0,T_0],X)$ denotes Lipschitz continuity in time with values in X. Thus we obtain c).

For small C_1 from Theorem 1.1 we obtain a small Lipschitz constant in Lemma 3.1 which by using Lemma 2.5 and Lemma 2.6 with small $T_0 > 0$ implies the convergence of the iteration scheme. Therefore, we are done. □

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