Erratum of

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In [1] we derived and justified coupled mode equations for the stationary 2D Gross-Pitaevskii equation with a non-separable periodic potential and proved rigorously the existence of gap solitons. The key technique was to consider the problem in Bloch variables. However, due to a wrong transfer of the main ansatz [1, (3.2)] from physical space to Bloch wave space [1, (3.10)] the paper contains a number of wrong or at least misleading formulas. The main results of [1] are correct, but we find it necessary to correct the ansatz and outline the subsequent changes in the analysis. In doing so we also want to remove some minor errors and inconsistencies. For readers' convenience we have incorporated all the corrections in the arXiv version of this paper [3].

1 Preliminary remarks

- Definition of the Fourier transform [1, (1.4)]. The normalization in the Fourier transform should be chosen differently, namely

$$\hat{\phi}(k) := (\mathcal{F}\phi)(k) := \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \phi(x)e^{-\mathrm{i}k\cdot x}\,\mathrm{d}x, \quad \phi(x) = (\mathcal{F}^{-1}\hat{\phi})(k) := \int_{\mathbb{R}^2} \hat{\phi}(k)e^{\mathrm{i}k\cdot x}\,\mathrm{d}k. \quad (1)$$

- Assumption A.2. This has to be replaced by assuming definiteness of the quadratic form given by $\begin{pmatrix} \partial^2_{k_1}\omega_{n_j}(k^{(j)}) & \partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)}) \\ \partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)}) & \partial^2_{k_2}\omega_{n_j}(k^{(j)}) \end{pmatrix}$, which is needed in order to ensure that the extrema at $k^{(j)}$ are quadratic. The new version reads:

  "**Assumption A.2** The quadratic form $\partial^2_{k_1}\omega_{n_j}(k^{(j)})x^2+2\partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)})xy+\partial^2_{k_2}\omega_{n_j}(k^{(j)})y^2$ defined by the Hessian of ω_{n_j} at $k = k^{(j)}$ is (positive or negative) definite."

  Consequently, Remark 3.2 should be modified to:

  "The definiteness in A.2 ensures that the extremum of ω_{n_j} at $k = k^{(j)}$ is quadratic and that the resulting CMEs are of second order. Unlike in the separable case [19] it is possible that $\partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)}) \neq 0$, which then leads to CMEs with mixed second order derivatives."

2 Asymptotic ansatz in Bloch variables and the formal derivation of coupled mode equations

We now come to the main correction needed in [1] and its consequences. The ansatz [1, (3.2)] reads

$$\begin{aligned} \phi(x) &= \varepsilon\phi^{(0)}(x) + \varepsilon^2\phi^{(1)}(x) + \varepsilon^3\phi^{(2)}(x) + \mathcal{O}(\varepsilon^4), \\ \varepsilon\phi^{(0)}(x) &= \varepsilon\sum_{j=1}^N A_j(y)u_{n_j}(k^{(j)};x), \qquad \omega = \omega_* + \varepsilon^2\Omega, \qquad y = \varepsilon x, \qquad 0 < \varepsilon \ll 1. \end{aligned} \tag{2}$$

The Bloch transform $\mathcal{T}$ of the ansatz for $\varepsilon\phi^{(0)}(x)$ is

$$\varepsilon\tilde{\phi}^{(0)}(k;x) = \frac{1}{\varepsilon}\sum_{j=1}^N p_{n_j}(k^{(j)};x)\sum_{m\in\mathbb{Z}^2}\hat{A}_j\left(\frac{k-k^{(j)}+m}{\varepsilon}\right)e^{\mathrm{i}m\cdot x} \tag{3}$$

with $k \in \mathbb{T}^2, x \in \mathbb{P}^2$, and where $\sum_{m\in\mathbb{Z}^2}$ was forgotten in [1]. The basic idea, however, remains: as $\hat{A}_j(p)$ is localized near $p = 0$, we approximate $\hat{A}_j\left(\frac{k-k^{(j)}+m}{\varepsilon}\right)$ by $\chi_{D_j}(k+m)\hat{A}_j\left(\frac{k-k^{(j)}+m}{\varepsilon}\right)$, where $D_j = \{k \in \mathbb{R}^2 : |k - k^{(j)}| < \varepsilon^r\}$ with $0 < r < 1$. Here we make a slight change of notation compared to [1, (3.12)] where we defined $D_j := \{k \in \mathbb{R}^2 : |k-k^{(j)}| < \varepsilon^r$ modulo 1 in each component$\}$. These "periodically wrapped" disks will now be denoted $\tilde{D}_j$.

Note that $k + m \in D_j$ with $k \in \mathbb{T}^2$ is possible only for $m \in \{m \in \mathbb{Z}^2 : -1 \le m_1, m_2 \le 1\}$. We define the set of $m-$values for which $k + m \in D_j$ for some $k \in \mathbb{T}^2$ by $M_j := \{m \in \mathbb{Z}^2 : k + m \in D_j$ for some $k \in \mathbb{T}^2\}$. In fact, for small ε only the following cases occur: $M_j = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$ if $k_1^{(j)} = 1/2$ and $k^{(j)} \ne (1/2, 1/2)$, $M_j = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ if $k_2^{(j)} = 1/2$ and $k^{(j)} \ne (1/2, 1/2)$, $M_j = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\}$ if $k^{(j)} = (1/2, 1/2)$, and $M_j = \{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\}$ if $k^{(j)} \in \mathrm{int}(\mathbb{T}^2)$. Thus the asymptotic ansatz in Bloch variables now reads

$$\begin{aligned} \tilde{\phi}(k;x) &= \frac{1}{\varepsilon}\tilde{\psi}^{(0)}(k;x) + \tilde{\psi}^{(1)}(k;x) + \varepsilon\tilde{\psi}^{(2)}(k;x) + \mathcal{O}(\varepsilon^2), \\ \tilde{\psi}^{(0)}(k;x) &= \sum_{j=1}^N p_{n_j}(k^{(j)};x)\sum_{m\in M_j}\chi_{D_j}(k+m)\hat{A}_j\left(\frac{k+m-k^{(j)}}{\varepsilon}\right)e^{\mathrm{i}m\cdot x}, \\ \omega &= \omega_* + \Omega\varepsilon^2, \qquad 0 < \varepsilon \ll 1, \end{aligned} \tag{4}$$

which replaces [1, (3.10)]. The difference between the leading order terms in (3) and in (4) is

$$\frac{1}{\varepsilon}\tilde{\psi}^{(0)}(k;x) - \varepsilon\tilde{\phi}^{(0)}(k;x) =: \sum_{j=1}^N \tilde{h}_j(k;x) \tag{5}$$

with

$$\begin{aligned} \tilde{h}_j(k;x) = \frac{1}{\varepsilon}p_{n_j}(k^{(j)};x)\Bigg[&\big(1-\chi_{D_j}(k+m)\big)\sum_{m\in M_j}\hat{A}_j\left(\frac{k-k^{(j)}+m}{\varepsilon}\right)e^{\mathrm{i}m\cdot x} \\ &+ \sum_{m\in\mathbb{Z}^2\setminus M_j}\hat{A}_j\left(\frac{k-k^{(j)}+m}{\varepsilon}\right)e^{\mathrm{i}m\cdot x}\Bigg]. \end{aligned} \tag{6}$$

We now estimate $\|\tilde{h}_j\|_{L^2(\mathbb{T}^2,H^s(\mathbb{P}^2))}$. In the first sum in (6) we have $|k+m-k^{(j)}|\geq \varepsilon^r$ while in the second sum $|k+m-k^{(j)}|\geq 1$ because $k+m\notin D_j$ for all $k\in\mathbb{T}^2$ if $m\in\mathbb{Z}^2\setminus M_j$. By the triangle inequality and the substitution $p=(k-k^{(j)}+m)/\varepsilon$ we obtain

$$\begin{aligned}\|\tilde{h}_j\|^2_{L^2(\mathbb{T}^2,H^s(\mathbb{P}^2))}\leq &\sum_{m\in M_j}\|p_{n_j}(k^{(j)};\cdot)e^{\mathrm{i}m\cdot\cdot}\|^2_{H^s(\mathbb{P}^2)}\int_{\substack{|p|>\varepsilon^{r-1}\\ p\in(\mathbb{T}^2-k^{(j)}+m)/\varepsilon}}|\hat{A}_j(p)|^2\,\mathrm{d}p\\ &+\sum_{m\in\mathbb{Z}^2\setminus M_j}\|p_{n_j}(k^{(j)};\cdot)e^{\mathrm{i}m\cdot\cdot}\|^2_{H^s(\mathbb{P}^2)}\int_{\substack{|p|>c\varepsilon^{-1}\\ p\in(\mathbb{T}^2-k^{(j)}+m)/\varepsilon}}|\hat{A}_j(p)|^2\,\mathrm{d}p\\ \leq &C\left[\int_{|p|>\varepsilon^{r-1}}|\hat{A}_j(p)|^2\,\mathrm{d}p+\int_{|p|>\varepsilon^{-1}}|\hat{A}_j(p)|^2\,\mathrm{d}p\right],\end{aligned}$$

where the H^s regularity of $p_{n_j}(k^{(j)};\cdot)$ is guaranteed if $V\in H^{s-2}_{\mathrm{loc}}(\mathbb{R}^2)$. By rewriting the right hand side as $C\left[\int_{|p|>\varepsilon^{r-1}}|\hat{A}_j(p)|^2\frac{(1+|p|)^{2s}}{(1+|p|)^{2s}}\,\mathrm{d}p+\int_{|p|>\varepsilon^{-1}}|\hat{A}_j(p)|^2\frac{(1+|p|)^{2s}}{(1+|p|)^{2s}}\,\mathrm{d}p\right]$ and taking the supremum of $(1+|p|)^{-2s}$ out of the integrals, we have

$$\|\tilde{h}_j\|_{L^2(\mathbb{T}^2,H^s(\mathbb{P}^2))}\leq C(\varepsilon^{s(1-r)}+\varepsilon^s)\|\hat{A}_j\|_{L^2_s(\mathbb{R}^2)}\leq C\varepsilon^{s(1-r)}\|\hat{A}_j\|_{L^2_s(\mathbb{R}^2)}. \tag{7}$$

For $r<1$ we thus have that $\varepsilon^{-1}\psi^{(0)}(x)$ approximates $\varepsilon\phi^{(0)}(x)$ up to $\mathcal{O}(\varepsilon^{s(1-r)})$ in the $H^s(\mathbb{R}^2)$ norm. Because $\|\varepsilon\phi^{(0)}\|_{H^s(\mathbb{R}^2)}=\mathcal{O}(1)$, this approximation is satisfactory.

Due to the corrected ansatz (4) we next need to reconsider the formal derivation of amplitude equations for $\hat{A}_j$. Applying $\mathcal{T}$ to the stationary Gross-Pitaevskii equation [1, (1.2)] yields

$$\left[\tilde{L}-\omega\right]\tilde{\phi}+\sigma\;\tilde{\phi}*_B\tilde{\bar{\phi}}*_B\tilde{\phi}=0, \tag{8}$$

on $(k;x)\in\mathbb{T}^2\times\mathbb{P}^2$, where $\tilde{L}(k;x)=(\mathrm{i}\partial_{x_1}-k_1)^2+(\mathrm{i}\partial_{x_2}-k_2)^2+V(x)$. Setting $p^{(j,m)}:=\frac{k+m-k^{(j)}}{\varepsilon}$, we have

$$\begin{aligned}\tilde{L}(k;x)&=\tilde{L}(k^{(j)}-m+\varepsilon p^{(j,m)};x)\\ &=\tilde{L}(k^{(j)}-m;x)-2\varepsilon\left[(\mathrm{i}\partial_{x_1}-k_1^{(j)}+m_1)p_1^{(j,m)}+(\mathrm{i}\partial_{x_2}-k_2^{(j)}+m_2)p_2^{(j,m)}\right]+\varepsilon^2\left[p_1^{(j,m)^2}+p_2^{(j,m)^2}\right].\end{aligned} \tag{9}$$

Substituting (4) in (8) and using (9), we obtain a hierarchy of equations on $x\in\mathbb{P}^2,k\in\mathbb{T}^2$ such that $k+m\in D_j, j\in\{1,\ldots,N\}$. Note that the combination of $k\in\mathbb{T}^2$ and $k+m\in D_j$ implies $m\in M_j$. The following hierarchy is thus for each $(j,m)\in\{1,\ldots,N\}\times M_j$.

$\mathcal{O}(\varepsilon^{-1}):\quad \hat{A}_j(p^{(j,m)})\left[\tilde{L}(k^{(j)}-m;x)-\omega_*\right](p_{n_j}(k^{(j)};x)e^{\mathrm{i}m\cdot x})=0,$

which is equivalent to $\hat{A}_j(p^{(j,m)})e^{\mathrm{i}m\cdot x}\left[\tilde{L}(k^{(j)};x)-\omega_*\right]p_{n_j}(k^{(j)};x)=0$ and thus holds by definition of $\omega_*=\omega_{n_j}(k^{(j)})$.

$\mathcal{O}(\mathbf{1}):\quad \left[\tilde{L}(k^{(j)}-m;x)-\omega_*\right]\tilde{\psi}^{(1)}(k;x)$

$$\begin{aligned}&=2\hat{A}_j(p^{(j,m)})\left[p_1^{(j,m)}(\mathrm{i}\partial_{x_1}-k_1^{(j)}+m_1)+p_2^{(j,m)}(\mathrm{i}\partial_{x_2}-k_2^{(j)}+m_2)\right](p_{n_j}(k^{(j)};x)e^{\mathrm{i}m\cdot x})\\ &=2\hat{A}_j(p^{(j,m)})e^{\mathrm{i}m\cdot x}\left[p_1^{(j,m)}(\mathrm{i}\partial_{x_1}-k_1^{(j)})+p_2^{(j,m)}(\mathrm{i}\partial_{x_2}-k_2^{(j)})\right]p_{n_j}(k^{(j)};x)\end{aligned}$$

Similarly to [1] we conclude that for $k\in\mathbb{T}^2$ and $k+m\in D_j$ we have

$$\tilde{\psi}^{(1)}(k;x)=\hat{A}_j(p^{(j,m)})e^{\mathrm{i}m\cdot x}\sum_{l=1}^{2}p_l^{(j,m)}\partial_{k_l}p_{n_j}(k^{(j)};x). \tag{10}$$

$\mathcal{O}(\varepsilon)$: We have

$$\begin{aligned}
&\left[\tilde{L}(k^{(j)}-m;x)-\omega_*\right]\tilde{\psi}^{(2)}(k;x)\\
&=\Omega\hat{A}_j(p^{(j,m)})p_{n_j}(k^{(j)};x)e^{\mathrm{i}m\cdot x}+2\left[p_1^{(j,m)}(\mathrm{i}\partial_{x_1}-k_1^{(j)}+m_1)+p_2^{(j,m)}(\mathrm{i}\partial_{x_2}-k_2^{(j)}+m_2)\right]\tilde{\psi}^{(1)}(k;x)\\
&\quad-\left(p_1^{(j,m)^2}+p_2^{(j,m)^2}\right)\hat{A}_j(p^{(j,m)})p_{n_j}(k^{(j)};x)e^{\mathrm{i}m\cdot x}-\frac{\sigma}{\varepsilon^4}\chi_{D_j}(k+m)(\tilde{\psi}^{(0)}*_B\tilde{\psi}^{(0)}*_B\tilde{\bar{\psi}}^{(0)})(k;x)\\
&=\Omega\hat{A}_j(p^{(j,m)})p_{n_j}(k^{(j)};x)e^{\mathrm{i}m\cdot x}\\
&\quad-e^{\mathrm{i}m\cdot x}\sum_{l=1}^{2}\left[p_{n_j}(k^{(j)};x)-2(\mathrm{i}\partial_{x_l}-k_l^{(j)})\partial_{k_l}p_{n_j}(k^{(j)};x)\right]p_l^{(j,m)^2}\hat{A}_j(p^{(j,m)})\\
&\quad+2e^{\mathrm{i}m\cdot x}\left[(\mathrm{i}\partial_{x_1}-k_1^{(j)})\partial_{k_2}p_{n_j}(k^{(j)};x)+(\mathrm{i}\partial_{x_2}-k_2^{(j)})\partial_{k_1}p_{n_j}(k^{(j)};x)\right]p_1^{(j,m)}p_2^{(j,m)}\hat{A}_j(p^{(j,m)})\\
&\quad-\frac{\sigma}{\varepsilon^4}\chi_{D_j}(k+m)(\tilde{\psi}^{(0)}*_B\tilde{\psi}^{(0)}*_B\tilde{\bar{\psi}}^{(0)})(k;x)
\end{aligned} \tag{11}$$

using $\tilde{\psi}^{(1)}$ from (10). The nonlinear term has the form

$$\begin{aligned}
G_j(k;x):=&\frac{\sigma}{\varepsilon^4}\chi_{D_j}(k+m)(\tilde{\psi}^{(0)}*_B\tilde{\psi}^{(0)}*_B\tilde{\bar{\psi}}^{(0)})(k;x)=\frac{\sigma}{\varepsilon^4}\chi_{D_j}(k+m)\left[\sum_{\alpha=1}^{N}\xi_\alpha*_B\xi_\alpha*_B\xi_\alpha^c\right.\\
&\left.+2\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\xi_\alpha*_B\xi_\beta*_B\xi_\alpha^c+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\xi_\alpha*_B\xi_\alpha*_B\xi_\beta^c+\sum_{\substack{\alpha,\beta,\gamma=1\\ \alpha\neq\beta,\alpha\neq\gamma,\beta\neq\gamma}}^{N}\xi_\alpha*_B\xi_\beta*_B\xi_\gamma^c\right],
\end{aligned} \tag{12}$$

where $\xi_\alpha=\xi_\alpha(k;x):=p_{n_\alpha}(k^{(\alpha)};x)\sum_{m\in M_\alpha}\chi_{D_\alpha}(k+m)\hat{A}_\alpha\left(\frac{k+m-k^{(\alpha)}}{\varepsilon}\right)e^{\mathrm{i}m\cdot x}$ and $\xi_\alpha^c=\xi_\alpha^c(k;x):=\overline{p_{n_\alpha}}(k^{(\alpha)};x)\sum_{m\in M_\alpha}\chi_{-D_\alpha}(k-m)\hat{\bar{A}}_\alpha\left(\frac{k-m+k^{(\alpha)}}{\varepsilon}\right)e^{-\mathrm{i}m\cdot x}$. The last sum or the three last sums in (12) are absent if $N=2$ or $N=1$ respectively. $\xi_\alpha*_B\xi_\beta*_B\xi_\gamma^c$ consists of terms of the type

$$\begin{aligned}
g_{noq}(k;x)=&e^{\mathrm{i}(n+o-q)\cdot x}p_{n_\alpha}(k^{(\alpha)};x)p_{n_\beta}(k^{(\beta)};x)\overline{p_{n_\gamma}}(k^{(\gamma)};x)\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\chi_{D_\alpha}(k-r+n)\hat{A}_\alpha\left(\frac{k-r+n-k^{(\alpha)}}{\varepsilon}\right)\times\\
&\times\ \chi_{D_\beta}(r-s+o)\hat{A}_\beta\left(\frac{r-s+o-k^{(\beta)}}{\varepsilon}\right)\chi_{-D_\gamma}(s-q)\hat{\bar{A}}_\gamma\left(\frac{s-q+k^{(\gamma)}}{\varepsilon}\right)\,\mathrm{d}s\,\mathrm{d}r
\end{aligned} \tag{13}$$

with $n\in M_\alpha, o\in M_\beta$, and $q\in M_\gamma$. Clearly, the integration domains can be reduced to $r\in D_{2\varepsilon^r}(k^{(\beta)}-k^{(\gamma)}-o+q)$ and $s\in D_{\varepsilon^r}(-k^{(\gamma)}+q)$. The changes of variables $\tilde{s}:=(s+k^{(\gamma)}-q)/\varepsilon$ and $\tilde{r}:=(r-k^{(\beta)}+k^{(\gamma)}+o-q)/\varepsilon$ yield

$$\begin{aligned}
g_{noq}(k;x)=&\varepsilon^4e^{\mathrm{i}(n+o-q)\cdot x}p_{n_\alpha}(k^{(\alpha)};x)p_{n_\beta}(k^{(\beta)};x)\overline{p_{n_\gamma}}(k^{(\gamma)};x)\times\\
&\int_{D_{2\varepsilon^{r-1}}\cap\frac{\mathbb{T}^2-k^{(\beta)}+k^{(\gamma)}+o-q}{\varepsilon}}\ \int_{D_{\varepsilon^{r-1}}\cap\frac{\mathbb{T}^2+k^{(\gamma)}-q}{\varepsilon}}\chi_{D_{\varepsilon^{r-1}}}\left(\frac{k-(k^{(\alpha)}+k^{(\beta)}-k^{(\gamma)})+n+o-q}{\varepsilon}-\tilde{r}\right)\times\\
&\hat{A}_\alpha\left(\frac{k-(k^{(\alpha)}+k^{(\beta)}-k^{(\gamma)})+n+o-q}{\varepsilon}-\tilde{r}\right)\chi_{D_{\varepsilon^{r-1}}}(\tilde{r}-\tilde{s})\hat{A}_\beta(\tilde{r}-\tilde{s})\chi_{D_{\varepsilon^{r-1}}}(\tilde{s})\hat{\bar{A}}_\gamma(\tilde{s})\,\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{r},
\end{aligned} \tag{14}$$

where $D_{\varepsilon^{r-1}} = \{p \in \mathbb{R}^2 : |p| < \varepsilon^{r-1}\}$.

Only those combinations of (n, o, q) which produce nonzero values of all the three characteristic functions in (13) for some $k, r, s \in \mathbb{T}^2$ are of relevance. Due to $\chi_{-D_\gamma}(s-q)$ we, therefore, require $q - k^{(\gamma)} \in \overline{\mathbb{T}^2} = [-1/2, 1/2]^2$, which ensures that $s - q \in -D_\gamma$ is satisfied by some $s \in \mathbb{T}^2$ for any $\varepsilon > 0$. The first condition is, thus,

$$s_0 := q - k^{(\gamma)} \in \overline{\mathbb{T}^2}. \tag{15}$$

Due to $\chi_{D_\beta}(r-s+o)$ we get the condition $s_0 - o + k^{(\beta)} \in \overline{\mathbb{T}^2}$, i.e.,

$$r_0 := s_0 - o + k^{(\beta)} \in \overline{\mathbb{T}^2}. \tag{16}$$

Finally, $\chi_{D_\alpha}(k-r+n)$ enforces $r_0 - n + k^{(\alpha)} \in \overline{\mathbb{T}^2}$, i.e.,

$$k_0 := r_0 - n + k^{(\alpha)} \in \overline{\mathbb{T}^2}. \tag{17}$$

Statements (15), (16), and (17) form the necessary condition

$$s_0 := q - k^{(\gamma)} \in \overline{\mathbb{T}^2}, \qquad r_0 := s_0 - o + k^{(\beta)} \in \overline{\mathbb{T}^2}, \quad \text{and} \quad k_0 := r_0 - n + k^{(\alpha)} \in \overline{\mathbb{T}^2} \tag{18}$$

for (13) (and thus (14)) not to vanish.

Another condition on (n, o, q) appears due to the factor $\chi_{D_j}(k+m)$ in G_j. From (14) it is clear that g_{noq} is supported on $k \in D_{\varepsilon^r}(k^{(\alpha)} + k^{(\beta)} - k^{(\gamma)} - n - o + q)$. The factor $\chi_{D_j}(k+m)$ thus annihilates all terms g_{noq} except those for which

$$k^{(\alpha)} + k^{(\beta)} - k^{(\gamma)} - n - o + q = k^{(j)} - m. \tag{19}$$

If (19) is satisfied, (14) becomes

$$\begin{aligned} g_{noq}(k;x) = \varepsilon^4 e^{\mathrm{i}(n+o-q)\cdot x} p_{n_\alpha}(k^{(\alpha)};x) p_{n_\beta}(k^{(\beta)};x) \overline{p_{n_\gamma}}(k^{(\gamma)};x) \times \\ \int\limits_{D_{2\varepsilon^{r-1}} \cap \frac{\mathbb{T}^2 - k^{(\beta)} + k^{(\gamma)} + o - q}{\varepsilon}} \int\limits_{D_{\varepsilon^{r-1}} \cap \frac{\mathbb{T}^2 + k^{(\gamma)} - q}{\varepsilon}} \chi_{D_{\varepsilon^{r-1}}}\left(\tfrac{k-k^{(j)}+m}{\varepsilon} - \tilde{r}\right) \times \\ \hat{A}_\alpha\left(\tfrac{k-k^{(j)}+m}{\varepsilon} - \tilde{r}\right) \chi_{D_{\varepsilon^{r-1}}}(\tilde{r} - \tilde{s}) \hat{A}_\beta(\tilde{r} - \tilde{s}) \chi_{D_{\varepsilon^{r-1}}}(\tilde{s}) \hat{\bar{A}}_\gamma(\tilde{s}) \,\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{r}. \end{aligned} \tag{20}$$

As a result, the term $A_\alpha A_\beta \bar{A}_\gamma$ will enters the $j-$th equation of the coupled mode system provided there exist $n \in M_\alpha, o \in M_\beta$ and $q \in M_\gamma$ such that (18) holds and such that (19) holds for some $m \in M_j$. Let us denote the set of (n, o, q) that satisfy (18) and (19) by $\mathcal{A}_{\alpha,\beta,\gamma,j,m}$.

The sum of the terms (20) over $(n, o, q) \in \mathcal{A}_{\alpha,\beta,\gamma,j,m}$ yields a double convolution integral over the full discs $\tilde{r} \in D_{2\varepsilon^{r-1}}$ and $\tilde{s} \in D_{\varepsilon^{r-1}}$, i.e.,

$$\begin{aligned} (\xi_\alpha *_B \xi_\beta *_B \xi_\gamma^c)(k;x) = \varepsilon^4 e^{\mathrm{i}(k^{(\alpha)} + k^{(\beta)} - k^{(\gamma)} - k^{(j)} + m)\cdot x} p_{n_\alpha}(k^{(\alpha)};x) p_{n_\beta}(k^{(\beta)};x) \overline{p_{n_\gamma}}(k^{(\gamma)};x) \times \\ \int_{D_{2\varepsilon^{r-1}}} \int_{D_{\varepsilon^{r-1}}} \chi_{D_{\varepsilon^{r-1}}}\left(\tfrac{k-k^{(j)}+m}{\varepsilon} - \tilde{r}\right) \hat{A}_\alpha\left(\tfrac{k-k^{(j)}+m}{\varepsilon} - \tilde{r}\right) \chi_{D_{\varepsilon^{r-1}}}(\tilde{r} - \tilde{s}) \hat{A}_\beta(\tilde{r} - \tilde{s}) \hat{\bar{A}}_\gamma(\tilde{s}) \,\mathrm{d}\tilde{s}\,\mathrm{d}\tilde{r}, \end{aligned} \tag{21}$$

where $e^{\mathrm{i}(n+o-q)\cdot x}$ was replaced by $e^{\mathrm{i}(k^{(\alpha)} + k^{(\beta)} - k^{(\gamma)} - k^{(j)} + m)\cdot x}$ due to (19).

We return now to equation (11) for $\tilde{\psi}^{(2)}$ on $k \in (D_j - m) \cap \mathbb{T}^2$. Its solvability condition is $L^2(\mathbb{P}^2)$-orthogonality to $\mathrm{Ker}(\tilde{L}(k^{(j)} - m; x) - \omega_*) = \mathrm{span}\{\cup_l p_{n_l}(k^{(j)}; x)e^{\mathrm{i}m\cdot x} \text{ s.t. } \omega_{n_l}(k^{(j)}) = \omega_*\}$. Clearly, the dimension of the kernel is at most N and the value N is attained in the case $k^{(1)} = \ldots = k^{(N)}$.

The factors $e^{\mathrm{i}(k^{(\alpha)}+k^{(\beta)}-k^{(\gamma)}-k^{(j)})\cdot x}$ in (21) after multiplication by the complex conjugate of $p_{n_l}(k^{(j)}; x)e^{\mathrm{i}m\cdot x} \in \mathrm{Ker}(\tilde{L}(k^{(j)} - m; x) - \omega_*)$ are new compared to [1, (3.19)]. These affect values of the coefficients of the nonlinear terms in the CMEs. In the linear terms in (11) the factor $e^{\mathrm{i}m\cdot x}$ is canceled in the inner product with $p_{n_l}(k^{(j)}; x)e^{\mathrm{i}m\cdot x}$. The range of $p^{(j,m)}$ is a different section of the disc $D_{\varepsilon^{r-1}}$ for each m. The section is $(1/|M_j|)$-th of the full disc so that these $|M_j|$ conditions build one equation in $p \in D_{\varepsilon^{r-1}}$.

The resulting CMEs in Fourier variables $p \in D_{\varepsilon^{r-1}}$ are

$$\Omega \hat{A}_j - \left(\frac{1}{2}\partial_{k_1}^2 \omega_{n_j}(k^{(j)})p_1^2 + \frac{1}{2}\partial_{k_2}^2 \omega_{n_j}(k^{(j)})p_2^2 + \partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)})p_1 p_2 \right) \hat{A}_j - \hat{\mathcal{N}}_j = 0, \tag{22}$$

$j \in \{1, \ldots, N\}$, where $\hat{\mathcal{N}}_j(p^{(j,m)}) = \langle G_j(\varepsilon p^{(j,m)} + k^{(j)} - m; \cdot), p_{n_j}(k^{(j)}; \cdot)e^{\mathrm{i}m\cdot\cdot}\rangle_{L^2(\mathbb{P}^2)}$.

For sufficiently smooth A_j we can neglect the contribution to $\hat{A}_j$ from $p \in \mathbb{R}^2 \setminus D_{\varepsilon^{r-1}}$ or, for simplicity, assume that the $\hat{A}_j$ satisfy (22) also there. Equation (22) is then posed on $p \in \mathbb{R}^2$. Performing the inverse Fourier transform yields

$$\Omega A_j + \left(\frac{1}{2}\partial_{k_1}^2 \omega_{n_j}(k^{(j)})\partial_{y_1}^2 + \frac{1}{2}\partial_{k_2}^2 \omega_{n_j}(k^{(j)})\partial_{y_2}^2 + \partial_{k_1}\partial_{k_2}\omega_{n_j}(k^{(j)})\partial_{y_1}\partial_{y_2} \right) A_j - \mathcal{N}_j = 0. \tag{23}$$

2.1 CMEs for the Potential (1.3) in [1]

As an example for the calculation of $\xi_\alpha *_B \xi_\beta *_B \xi_\gamma^c$ in (21) and hence of $\mathcal{N}_j$ in (23) we consider the case $\omega_* = s_3$. This is also the only case, where the resulting CMEs (values of their coefficients) need to be corrected in [1].

CMEs near $\omega_* = s_3$: Here $N = 2, n_1 = n_2 = 2, k^{(1)} = X$ and $k^{(2)} = X'$. We have thus $M_1 = \{(0,0)^T, (1,0)^T\}$ and $M_2 = \{(0,0)^T, (0,1)^T\}$. We carry out a straightforward sweep through all the possible combinations (n, o, q, m) for both $j = 1$ and $j = 2$ (performed using a Matlab script) to determine those that satisfy (18) and (19). The results are summarized in Table 1, and the resulting CMEs are

$$\begin{aligned} \left[\Omega + \alpha_1 \partial_{y_1}^2 + \alpha_2 \partial_{y_2}^2\right] A_1 - \sigma \left[\gamma_1 |A_1|^2 A_1 + \gamma_2 (2|A_2|^2 A_1 + A_2^2 \bar{A}_1)\right] &= 0, \\ \left[\Omega + \alpha_2 \partial_{y_1}^2 + \alpha_1 \partial_{y_2}^2\right] A_2 - \sigma \left[\gamma_1 |A_2|^2 A_2 + \gamma_2 (2|A_1|^2 A_2 + A_1^2 \bar{A}_2)\right] &= 0, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\partial_{k_1}^2 \omega_2(X) = \frac{1}{2}\partial_{k_2}^2 \omega_2(X'), \quad \alpha_2 = \frac{1}{2}\partial_{k_2}^2 \omega_2(X) = \frac{1}{2}\partial_{k_1}^2 \omega_2(X'), \\ \gamma_1 &= \langle p_2(X;\cdot)^2, p_2(X;\cdot)^2\rangle_{L^2(\mathbb{P}^2)} = \langle p_2(X';\cdot)^2, p_2(X';\cdot)^2\rangle_{L^2(\mathbb{P}^2)} \\ &= \|p_2(X;\cdot)\|_{L^4(\mathbb{P}^2)}^4 = \|p_2(X';\cdot)\|_{L^4(\mathbb{P}^2)}^4, \\ \gamma_2 &= \langle e^{\mathrm{i}(1,-1)^T\cdot\cdot} p_2(X;\cdot)^2, p_2(X';\cdot)^2\rangle_{L^2(\mathbb{P}^2)} = \langle e^{\mathrm{i}(-1,1)^T\cdot\cdot} p_2(X';\cdot)^2, p_2(X;\cdot)^2\rangle_{L^2(\mathbb{P}^2)} \\ &= \langle |p_2(X;\cdot)|^2, |p_2(X';\cdot)|^2\rangle_{L^2(\mathbb{P}^2)}. \end{aligned}$$

term	$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$	j	$k^{(\alpha)}+k^{(\beta)}$	$(n,o,q)^T$ satisfying (18) and (19)		coefficient of
in $\mathcal{N}_j$			$-k^{(\gamma)}-k^{(j)}$	$m=M_j(:,1)$	$m=M_j(:,2)$	the term in $\sigma\mathcal{N}_j$
$\|A_1\|^2A_1$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	1	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&0\\1&0\end{pmatrix}$	$\langle p_2(X,\cdot)^2,p_2(X,\cdot)^2\rangle$
		2	$\begin{pmatrix}1/2\\-1/2\end{pmatrix}$	/	/	0
$\|A_2\|^2A_2$	$\begin{pmatrix}2\\2\\2\end{pmatrix}$	1	$\begin{pmatrix}-1/2\\1/2\end{pmatrix}$	/	/	0
		2	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\\0&1\end{pmatrix}$	$\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix}$	$\langle p_2(X',\cdot)^2,p_2(X',\cdot)^2\rangle$
$\|A_1\|^2A_2$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$,	1	$\begin{pmatrix}-1/2\\1/2\end{pmatrix}$	/	/	0
	$\begin{pmatrix}2\\1\\1\end{pmatrix}$	2	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\\1&0\end{pmatrix}$	$2\langle \|p_2(X,\cdot)\|^2,\|p_2(X',\cdot)\|^2\rangle$
$\|A_2\|^2A_1$	$\begin{pmatrix}1\\2\\2\end{pmatrix}$,	1	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}$	$2\langle \|p_2(X,\cdot)\|^2,\|p_2(X',\cdot)\|^2\rangle$
	$\begin{pmatrix}2\\1\\2\end{pmatrix}$	2	$\begin{pmatrix}1/2\\-1/2\end{pmatrix}$	/	/	0
$A_1^2A_2^*$	$\begin{pmatrix}1\\1\\2\end{pmatrix}$	1	$\begin{pmatrix}1/2\\-1/2\end{pmatrix}$	/	/	0
		2	$\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$	$\langle e^{\mathrm{i}(1,-1)^T\cdot\cdot}p_2(X,\cdot)^2,p_2(X',\cdot)^2\rangle$
$A_2^2A_1^*$	$\begin{pmatrix}2\\2\\1\end{pmatrix}$	1	$\begin{pmatrix}-1\\1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\\1&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\\1&0\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}$	$\langle e^{\mathrm{i}(-1,1)^T\cdot\cdot}p_2(X',\cdot)^2,p_2(X,\cdot)^2\rangle$
		2	$\begin{pmatrix}-1/2\\1/2\end{pmatrix}$	/	/	0

Table 1: Calculation of the nonlinearity terms for the CME near $\omega_* = s_3$. $M_j(:,l)$ denotes the $l-$th vector in M_j.

The identities in α_1, α_2 and γ_1 hold due to [1, (2.6)]. The equalities in γ_2 yield $\gamma_2 \in \mathbb{R}$ and follow from the fact that $u_2(X,x) = e^{\mathrm{i}x_1/2}p_2(X;x)$ and $u_2(X',x) = e^{\mathrm{i}x_2/2}p_2(X';x)$ are real. In detail

$$\int_{\mathbb{P}^2} e^{\mathrm{i}x_1}p_2(X;x)^2 e^{-\mathrm{i}x_2}\overline{p_2(X';x)}^2\,\mathrm{d}x = \int_{\mathbb{P}^2} u_2(X;x)^2\overline{u_2(X';x)}^2\,\mathrm{d}x$$
$$= \int_{\mathbb{P}^2} u_2(X;x)^2 u_2(X';x)^2\,\mathrm{d}x = \int_{\mathbb{P}^2} u_2(X';x)^2\overline{u_2(X;x)}^2\,\mathrm{d}x = \int_{\mathbb{P}^2} e^{\mathrm{i}x_2}p_2(X';x)^2e^{-\mathrm{i}x_1}\overline{p_2(X;x)}^2\,\mathrm{d}x.$$

The CMEs (24) are thus identical to [1, (3.4)] derived in physical variables. Numerically, $\alpha_1 \approx 2.599391, \alpha_2 \approx 0.040561, \gamma_1 \approx 0.090082,$ and $\gamma_2 \approx 0.003032$. Note that the coefficient of the last term in each equation in (24) has thus changed compared to [1, §3.2.2.3].

3 Justification

For the justification of the CME, in [1, §4] we used the family of diagonalization operators

$$\mathcal{D}(k)_{k\in\mathbb{T}^2} : H^s(\mathbb{P}^2) \ni \widetilde{\phi}(k;\cdot) \mapsto \vec{\widetilde{\phi}}(k) \in \ell^2_s, \quad \vec{\widetilde{\phi}}_n(k) = \left\langle \widetilde{\phi}(k;\cdot), p_n(k;\cdot)\right\rangle_{L^2(\mathbb{P}^2)}$$

to analyze (8) in the space $\mathcal{X}^s := L^2(\mathbb{T}^2, \ell^2_s)$ with norm $\|\vec{\widetilde{\phi}}\|^2_{\mathcal{X}^s} = \int_{\mathbb{T}^2}\sum_{n\in\mathbb{N}}|\tilde{\phi}_n(k)|^2(1+n)^s\,\mathrm{d}k$. Similar to the corrected ansatz (4), after diagonalization we now use, instead of [1, (4.8)],

$$\vec{\widetilde{\phi}}(k) = \varepsilon^{-1}\vec{\widetilde{\eta}}^{(0)}_{\mathrm{LS}}(k) + \vec{\widetilde{\psi}}(k), \text{ where } \varepsilon^{-1}\vec{\widetilde{\eta}}^{(0)}_{\mathrm{LS}}(k) = \varepsilon^{-1}\sum_{j=1}^N e_{n_j}\sum_{m\in M_j}\hat{B}_j\left(\frac{k+m-k^{(j)}}{\varepsilon}\right) \tag{25}$$

with $\operatorname{supp}\hat{B}_j \subset D_{\varepsilon^{r-1}}, 0 < r < 1$, $\widetilde{\psi}_{n_j}(k) = 0$ for $k \in K_c := \cup\{\widetilde{D}_l : l \in \{1,\dots,N\}, n_l = n_j\}$, and where $\widetilde{D}_l = D_l$ wrapped periodically onto $\mathbb{T}^2$. Note that in general $\varepsilon^{-1}\vec{\widetilde{\eta}}^{(0)}_{\mathrm{LS}}(k)$ is not the diagonalization of an ansatz of the form (4), except at $k = k^{(j)}$, since in the diagonalization we have $p_n(k;\cdot)$ instead of $p_n(k^{(j)},\cdot)$. This was overlooked in [1]. However, we have

$$\varepsilon^{-1}\tilde{\eta}^{(0)}_{\mathrm{LS}}(k,x) = \varepsilon^{-1}\tilde{\psi}^{(0)}_{\mathrm{LS}}(k,x) + \tilde{\rho}(k,x) \tag{26}$$

with

$$\varepsilon^{-1}\tilde{\psi}^{(0)}_{\mathrm{LS}}(k,x) = \varepsilon^{-1}\sum_{j=1}^{N} p_{n_j}(k^{(j)},x)\sum_{m\in M_j}\hat{B}_j\left(\frac{k+m-k^{(j)}}{\varepsilon}\right)\mathrm{e}^{\mathrm{i}m\cdot x},$$

and where for $\hat{B}_j \in L^2_s$ with $s \geq 1$

$$\|\tilde{\rho}\|_{L^2(\mathbb{T}^2,H^s(\mathbb{P}^2))} \leq C\varepsilon\sum_{j=1}^{N}\|\hat{B}_j\|_{L^2_s}. \tag{27}$$

This follows from writing $k = k^{(j)} - m + (k+m-k^{(j)})$, expanding

$$p_{n_j}(k,x) = p_{n_j}(k^{(j)}-m,x) + \nabla_k p_{n_j}(k^{(j)}_\star,x)\cdot(k+m-k^{(j)})$$

with $k^{(j)}_{\star,l} \in [\min(k^{(j)}_l - m_l, k_l), \max(k^{(j)}_l - m_l, k_l)]$, $l = 1,2$, and using $p_{n_j}(k^{(j)}-m,x) = p_{n_j}(k^{(j)},x)\mathrm{e}^{\mathrm{i}m\cdot x}$, which yields

$$\tilde{\rho}(k,x) = \varepsilon^{-1}\sum_{j=1}^{N}\sum_{m\in M_j}\hat{B}_j\left(\tfrac{k+m-k^{(j)}}{\varepsilon}\right)(k+m-k^{(j)})\cdot\nabla_k p_{n_j}(k^{(j)}_\star,x).$$

To prove (27), we may fix some (of the finitely many) j, m. Since the $\omega_j(k)$ are simple for $k \in \tilde{D}_j$, we have $\sup_{k\in\tilde{D}_j}\|\nabla_k p_{n_j}(k,\cdot)\|_{H^s(\mathbb{P}^2)} \leq C$, and it remains to estimate

$$\begin{aligned}\left\|\varepsilon^{-1}\hat{B}_j\left(\tfrac{k+m-k^{(j)}}{\varepsilon}\right)|k+m-k^{(j)}|\right\|^2_{L^2(\mathbb{T}^2)} &= \varepsilon^{-2}\int_{k\in\mathbb{T}^2}\left|\hat{B}_j\left(\tfrac{k+m-k^{(j)}}{\varepsilon}\right)\right|^2|k+m-k^{(j)}|^2\,\mathrm{d}k \\ \leq C\varepsilon^2\int_{K\in\mathbb{R}^2}|\hat{B}_j(K)|^2|K|^2\,\mathrm{d}K &\leq C\varepsilon^2\|\hat{B}_j\|^2_{L^2_s} \end{aligned} \tag{28}$$

for $s \geq 1$. Inequality (27) thus follows.
The Lyapunov–Schmidt equations [1, (4.9),(4.10)] now become

$$\frac{1}{\varepsilon}(\omega_{n_j}(k) - \omega_* - \varepsilon^2\Omega)\hat{B}_j\left(\frac{k+m-k^{(j)}}{\varepsilon}\right) = -\sigma\chi_{D_j}(k)\tilde{g}_{n_j}(k), \quad j = 1,\ldots,N,\ m\in M_j, \tag{29}$$

$$(\omega_n(k) - \omega_* - \varepsilon^2\Omega)\widetilde{\psi}_n(k) = -\sigma\left(1 - \sum_{j=1}^{N}\chi_{\tilde{D}_j}(k)\delta_{n,n_j}\right)\tilde{g}_n(k), \quad n\in\mathbb{N}. \tag{30}$$

The next step in [1] is Lemma 4.5, which states that for $\hat{\mathbf{B}} = (\hat{B})_{j=1,\ldots,N} \in L^2_s(D_{\varepsilon^{r-1}})$ and $\vec{\tilde{\psi}} \in \mathcal{X}^s$ with $s > 1$ we have

$$\begin{aligned}\|\vec{\tilde{g}}\|_{\mathcal{X}^s} \leq C\Big(&\varepsilon^2\big(\sum_{j=1}^{N}\|\hat{B}_j\|_{L^2_s(D_{\varepsilon^{r-1}})}\big)^3 + \varepsilon^2\big(\sum_{j=1}^{N}\|\hat{B}_j\|_{L^2_s(D_{\varepsilon^{r-1}})}\big)^2\|\vec{\tilde{\psi}}\|_{\mathcal{X}^s} \\ &+ \varepsilon\big(\sum_{j=1}^{N}\|\hat{B}_j\|_{L^2_s(D_{\varepsilon^{r-1}})}\big)\|\vec{\tilde{\psi}}\|^2_{\mathcal{X}^s} + \|\vec{\tilde{\psi}}\|^3_{\mathcal{X}^s}\Big).\end{aligned} \tag{31}$$

This Lemma stays as it is, but the proof needs some updates. First, for going back to physical space and estimating $g = |\phi|^2\phi = |\varepsilon^{-1}\psi_{\rm LS}^{(0)} + \rho + \psi|^2(\varepsilon^{-1}\psi_{\rm LS}^{(0)} + \rho + \psi)$, we need to use (27). We need to estimate terms of the types

$$\left(\varepsilon^{-1}\psi_{\rm LS}^{(0)}\right)^3,\ \left(\varepsilon^{-1}\psi_{\rm LS}^{(0)}\right)^2 f,\ \varepsilon^{-1}\psi_{\rm LS}^{(0)} f,\ \rho^2\psi,\ \rho\psi^2,\ \rho^3,\ \text{and } \psi^3. \tag{32}$$

First note that $\varepsilon^{-1}\psi_{\rm LS}^{(0)}(x) = \varepsilon\sum_{j=1}^N B_j(\varepsilon x)u_{n_j}(k^{(j)};x)$. Below we implicitly use $\|B_j(\varepsilon\cdot)u_{n_j}(k^{(j)};\cdot)\|_{H^s} \le \|u_{n_j}(k^{(j)};\cdot)\|_{C^{\lceil s\rceil}}\|B_j(\varepsilon\cdot)\|_{H^s}$, which holds by interpolation, see, e.g., [2, §4.2]. Next, $\|u_{n_j}(k^{(j)};\cdot)\|_{C^{\lceil s\rceil}} \le C\|u_{n_j}(k^{(j)};\cdot)\|_{H^{\lceil s\rceil+1+\delta}} \le C\|V\|_{H_{\rm loc}^{\lceil s\rceil-1+\delta}}$ for all $\delta > 0$, where the first inequality holds by Sobolev embedding and the second one by the differential equation.
In estimating all except the first term in (32) we use the Banach algebra property [1, Lemma 4.2]. For the first two terms we need to estimate $\|(\varepsilon B_j(\varepsilon\cdot))^n\|_{H^s}$ for $n = 2, 3$. We have

$$\begin{aligned}
\|(\varepsilon B_j(\varepsilon\cdot))^n\|_{H^s}^2 &= \int (1+|k|)^{2s}|\mathcal{F}((\varepsilon B_j(\varepsilon\cdot))^n)(k)|^2\,\mathrm{d}k\\
&\le C\left[\int |\mathcal{F}((\varepsilon B_j(\varepsilon\cdot))^n)(k)|^2\,\mathrm{d}k + \varepsilon^{2n-4}\int |k|^{2s}\left|\widehat{B_j^n}\left(\frac{k}{\varepsilon}\right)\right|^2\mathrm{d}k\right]\\
&\le C\left[\|(\varepsilon B_j(\varepsilon\cdot))^n\|_{L^2}^2 + \varepsilon^{2n-2+2s}\int |K|^{2s}|\widehat{B_j^n}(K)|^2\,\mathrm{d}K\right]\\
&\le C\left[\varepsilon^{2(n-1)}\|B_j\|_{L^\infty}^{2(n-1)}\|\varepsilon B_j(\varepsilon\cdot)\|_{L^2}^2 + \varepsilon^{2n-2+2s}\|B_j^n\|_{H^s}^2\right]\\
&\le C\left[\varepsilon^{2(n-1)}\|B_j\|_{H^s}^{2(n-1)}\|B_j\|_{L^2}^2 + \varepsilon^{2(n-1)+2s}\|B_j\|_{H^s}^{2n}\right]
\end{aligned}$$

and hence

$$\|(\varepsilon B_j(\varepsilon\cdot))^n\|_{H^s} \le C\varepsilon^{n-1}\|B_j\|_{H^s}^n \text{ for } n = 1,2,3. \tag{33}$$

Note that for $n \ge 2$ this is much better than the naive estimate $\|(\varepsilon B_j(\varepsilon\cdot))^n\|_{H^s} \le C\|\varepsilon B_j(\varepsilon\cdot)\|_{H^s}^n \le C\|B_j\|_{H^s}^n$ based on (33) with $n = 1$. Next, for the third term in (32) we get

$$\|\varepsilon B_j(\varepsilon\cdot)f(\cdot)\|_{H^s} \le C\varepsilon\|B_j\|_{H^s}\|f\|_{H^s}, \tag{34}$$

and this together with (33) can be used to prove (31). To show (34), we start with

$$\|\varepsilon B_j(\varepsilon\cdot)f(\cdot)\|_{H^s} \le \|\varepsilon B_j(\varepsilon\cdot)f(\cdot)\|_{L^2} + C\left\||k|^s\left(\frac{1}{\varepsilon}\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right).*\hat{f}(\cdot)\right)\right\|_{L^2}$$

The first term is estimated as $\|\varepsilon B_j(\varepsilon\cdot)f(\cdot)\|_{L^2} \le \varepsilon\|B_j\|_\infty\|f\|_{L^2} \le \varepsilon\|B_j\|_{H^s}\|f\|_{L^2}$, and for the second we note that $w(k) \le \varepsilon w(\frac{k-l}{\varepsilon}) + w(l)$ where $w(k) = |k|^s$. Thus, similarly to the proof of [1, Lemma 4.2], we have, using Young's inequality,

$$\begin{aligned}
&\left\|w(k)\left(\frac{1}{\varepsilon}\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)*\hat{f}(\cdot)\right)\right\|_{L^2} \le C\left\|\left|w\left(\frac{\cdot}{\varepsilon}\right)\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\right|*|\hat{f}(\cdot)| + \left|\frac{1}{\varepsilon}\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\right|*|w(\cdot)\hat{f}(\cdot)|\right\|_{L^2}\\
&\quad\le C\left[\left\|w\left(\frac{\cdot}{\varepsilon}\right)\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2}\|\hat{f}\|_{L^1} + \left\|\frac{1}{\varepsilon}\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^1}\|w\hat{f}\|_{L^2}\right].
\end{aligned}$$

Now $\|w\left(\frac{\cdot}{\varepsilon}\right)\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\|_{L^2} \le \varepsilon\|\hat{B}_j\|_{L_s^2}$, $\left\|\frac{1}{\varepsilon}\hat{B}_j\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^1} \le C\varepsilon\|\hat{B}_j\|_{L_s^2}$, and $\|\hat{f}\|_{L^1} \le C\|\hat{f}\|_{L_s^2}$ (see[1, (4.3)]) yield (34).

The 4th, 5th and 7th term in (32) are estimated simply using [1, Lemma 4.2] and (27). The 6th term is treated the same way and is bounded by $\varepsilon^3\big(\sum_{j=1}^N \|\hat{B}_j\|_{L^2_s(D_{\varepsilon^{r-1}})}\big)^3$ so that it is of higher order than the first term on the right hand side of (31).

After (the proof of) Lemma 4.5 we note that [1, (4.13)] is proved by a straightforward application of the Banach fixed point theorem to [1, (4.10)]. Subsequently, some more but rather obvious corrections are needed, namely:

- Below [1, (4.13)] we get "The term $\varepsilon^{-1}\tilde{\psi}^{(0)}_{\rm LS}(k,x)$ in (26) corresponds to $\varepsilon^{-1}\tilde{\psi}^{(0)}(k,x)$ in the ansatz (4) used in the formal derivation of the CME".

- In the sequel, the remainder $\varepsilon^{\tilde{r}}\hat{R}_j(p)$ in [1, (4.14)] then also contains terms coming from $\tilde{\rho}$ in (26) which can be estimated using (27) so that $\tilde{r} = \min\{3r-1, 2-2r, 1\}$.

- In Theorem 4.6 of [1] the error (27) needs to be included in [1, (4.15)] so that the right hand side reads $C_2(\varepsilon^{2-2r}+\varepsilon)$. Note that in [1, (4.15)] we use that $\varepsilon\sum_{j=1}^N B_j(\varepsilon\cdot)u_{n_j}(k^{(j)};\cdot) = \varepsilon^{-1}\psi^{(0)}_{\rm LS}(x)$, which holds due to the truncated support of $\hat{B}_j$.

- The persistence proof in [1, §4.3] needs no corrections due to the rather abstract arguments used, and Theorem 4.9 as well as Corollary 4.10 stay correct with $\tilde{r} = \min\{3r-1, 2-2r\}$ because for $1/3 < r < 1$ we have $\min\{3r-1, 2-2r, 1\} = \min\{3r-1, 2-2r\}$.

  We have, however, been able to prove persistence with a weaker and more natural definition of reversibility than the one in [1, Def. 4.7]. In the new argument one no longer needs to assume reversibility of the whole lowest order approximation $\phi^{(0)}$. For details see [3].

The numerical results in [1, §5] need not be changed as computations were performed for CMEs not affected by the corrections.

4 Concluding Remarks

The main issue was the inclusion of $\sum_{m\in M_j}$ in the ansatz (4) and the subsequent updates in the formal derivation of the CME (23). The ansatz (25) for the justification also inherits the $\sum_{m\in M_j}$ which has resulted in some corrections in the sequel.

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