Exact solutions of the Cauchy problem for the linearized KdV equation on metric star graphs

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(Dated: June 2, 2014)

We investigate the linearized KdV equation on metric star graphs with three semi-infinite bonds. Under some assumptions on the coefficients of the vertex conditions uniqueness of solutions is proven. Using the theory of potentials, we reduce the problem to systems of linear algebraic equations, show that these are uniquely solvable under conditions of the uniqueness theorem, and obtain explicit solution formulas which prove existence in the class of Schwartz functions and in Sobolev classes.

Keywords: linearized KdV, metric graphs, fundamental solutions, exact solutions of PDE.

I. INTRODUCTION

The Korteweg-de Vries (KdV) equation has attracted attention of both physical scientists and mathematicians, since it was found to admit soliton solutions and be able to model the propagation of solitary wave on the water surface, a phenomena first discovered by Scott Russell in 1834. The equation is also used, e.g., to model the unidirectional propagation of small amplitude long waves in nonlinear dispersive systems such as the ion-acoustic waves in a collisionless plasma, and the magnetosonic waves in a magnetized plasma etc [VS05]. The linearized KdV provides an asymptotic description of linear, undirectional, weakly dispersive long waves, for example, shallow water waves. In [Taf83] it is proven that via normal form transforms the solution of the KdV equation can be reduced to the solution of the linear KdV equation. Belashov and Vladimirov [VS05] numerically investigate evolution of the single disturbance $u(0,x) = u_0 \exp(-x^2/l^2)$ and show that in the limit $l \to 0$, $u_0 l^2 = const$, the solution of the KdV equation is qualitatively similar to the solution of linearized KdV equation. Boundary value problems on half lines are considered in [Dju79, FS99, BF08].

Here we investigate the linearized KdV equation on star graphs Γ with three semi-infinite bonds connected in one point, called vertex. The bonds are denoted by B_1, B_2 and B_3, the coordinate x_1 on B_1 is defined from $-\infty$ to 0, and the coordinates x_2 and x_3 on the bonds B_2 and B_3 from 0 to $+\infty$ such that on each bond the vertex corresponds to 0. On each bond we consider the linear equation

$$\left(\frac{\partial}{\partial t} + \sigma \frac{\partial^3}{\partial x_k^3}\right) u_k(x_k, t) = f_k(x, t), \quad t > 0, x_k \in B_k. \tag{1}$$

Below we will also use x instead of x_k $(k = 1, 2, 3)$. We investigate two initial value problems, namely the case $\sigma = 1$ which describes forward wave propagation, and the case $\sigma = -1$ describing backward wave propagation (see Remark 1 on this terminology), and using the method of potentials construct solutions formulas.

Backward wave propagation. On an interval, for $\sigma = -1$ one needs 3 boundary conditions (BC): two on the left end of the x-interval and one on the right end, (see, e.g., [Dju79, FL10] and references therein). For our graph we thus need to impose 5 BCs, which should provide also connection between the bonds. In detail, we require

$$u_1(0,t) = a_2 u_2(0,t) = a_3 u_3(0,t), \tag{2}$$

$$\frac{\partial u_1}{\partial x}(0,t) = b_2 \frac{\partial u_2}{\partial x}(0,t) = b_3 \frac{\partial u_3}{\partial x}(0,t), \tag{3}$$

$$\frac{\partial^2 u_1}{\partial x^2}(0,t) = a_2^{-1} \frac{\partial^2 u_2}{\partial x^2}(0,t) + a_3^{-1} \frac{\partial^2 u_3}{\partial x^2}(0,t) \tag{4}$$

for $t > 0$, where $a_k, b_k, k = 2, 3$, are nonzero constants. Moreover, we assume that the $f_k(x,t)$ and the initial conditions

$$u_k(x, 0) = u_{0k}(x), \quad x \in \bar{B}_k, \ (k = 1, 2, 3), \tag{5}$$

are sufficiently smooth enough and bounded, and that $u_{0,k}$ satisfies the vertex conditions (2–4).

These vertex conditions are not the only possible ones, and the main motivation for our choice is that they guarantee uniqueness of the solution, and, if the solutions decay (to zero) at infinity, the norm (energy) conservation.

Forward wave propagation. For $\sigma = 1$ we need four conditions at the vertex. We choose

$$u_1(0,t) = \alpha_2 u_2(0,t) = \alpha_3 u_3(0,t), \tag{6}$$

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = \beta_2 \left.\frac{\partial u_2}{\partial x}\right|_{x=0} + \beta_3 \left.\frac{\partial u_3}{\partial x}\right|_{x=0}, \tag{7}$$

$$\left.\frac{\partial^2 u_1}{\partial x^2}\right|_{x=0} = \beta_2 \left.\frac{\partial^2 u_2}{\partial x^2}\right|_{x=0} + \beta_3 \left.\frac{\partial^2 u_3}{\partial x^2}\right|_{x=0}. \tag{8}$$

The initial conditions

$$u_k(x,0) = u_{0k}(x), \quad x \in B_k \ k = 1,2,3, \tag{9}$$

are assumed sufficiently smooth and bounded functions, that satisfy the gluing conditions (6)–(8).

Remark 1. Our notion of backward ($\sigma = -1$) and forward ($\sigma = 1$) propagation and some interesting effects of propagation on the graph with three bonds is illustrated in Fig. 1. For $\sigma = 1$ the pertinent fundamental solution decays exponentially to the right, and the oscillatory tail moves to the left; this is called forward propagation as the maximum stays roughly stationary, and hence moves to the right relative to the tails. Ultimately, the name forward propagation then comes from the fact that bright solitons in the KdV equation $\partial_t u + \partial_x^3 u + 6u\partial_x u = 0$ move to the right, see panel (d). For $\sigma = -1$ it is vice versa.

If, e.g., we put a localized IC on bond 2, then oscillations move backwards to bond 1, but there is hardly any coupling from the vertex to bond 3, see (a). Interestingly, suitable choices of IC can lead to suppression of coupling from bonds 2 and 3 to bond 1 even numerically, see panel (b). This, however of course also depends on the coupling constants, see (c). These effects will also be studied based on the explicit solution formulas (19) and (29) in a future paper.

Acknowledgement. We gratefully thank the Volkswagen Stiftung for financial support.

II. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We easily get a-priori estimates for solutions and thus uniqueness.

Lemma 2. *Let $b_2^{-2} + b_3^{-2} \leq 1$. Then (1)–(5) with $\sigma = -1$ has at most one solution.*

Lemma 3. *Let $\beta_2^2 + \beta_3^2 \leq 1$. Then (1) with $\sigma = 1$ and (6)–(9) has at most one solution.*

Proof of Lemma 2. Multiply both sides of (1) by u_k, integrate on B_k, and sum up. Integration by parts yields

$$\frac{d}{dt}\|u\|_\Gamma^2 = 2\left(\frac{1}{b_2^2} + \frac{1}{b_3^2} - 1\right) u_{1x}^2(-0,t) + 2\sum_{k=1}^{3} \int\limits_{B_k} F_k(x,t) u_k(x,t) dx \leq 2\|F\|_\Gamma \|u\|_\Gamma,$$

i.e., $\frac{d}{dt}\|u\|_\Gamma \leq \|F\|_\Gamma$, where $\|v\|_\Gamma^2 = \sum_{k=1}^3 \int\limits_{B_k} v_k^2(x)dx$. Thus $\|u(x,t)\|_\Gamma \leq \|u(x,0)\|_\Gamma + \int\limits_0^t \|F(x,s)\|_\Gamma ds$. which implies uniqueness. □

Proof of Lemma 3. This is similar as the previous case. From (1), initial and vertex conditions we have

$$\frac{d}{dt}\|u\|_\Gamma^2 = 2\sum_k \int_{B_k} u_k(x,t)\partial_t u_k(x,t) dx = -2\sum_k \int_{B_k} u_k \partial_x^3 u_k dx + 2\sum_k \int_{B_k} u_k f_k dx \leq$$
$$\leq (\beta_2^2 - 1)(\partial_x u_2(0,t))^2 + 2\beta_2\beta_3 \partial_x u_2(0,t)\partial_x u_3(0,t) + (\beta_3^2 - 1)(\partial_x u_3(0,t))^2 + 2\|u\|_\Gamma \|\mathbf{f}\|_\Gamma \leq 2\|u\|_\Gamma \|\mathbf{f}\|_\Gamma.$$

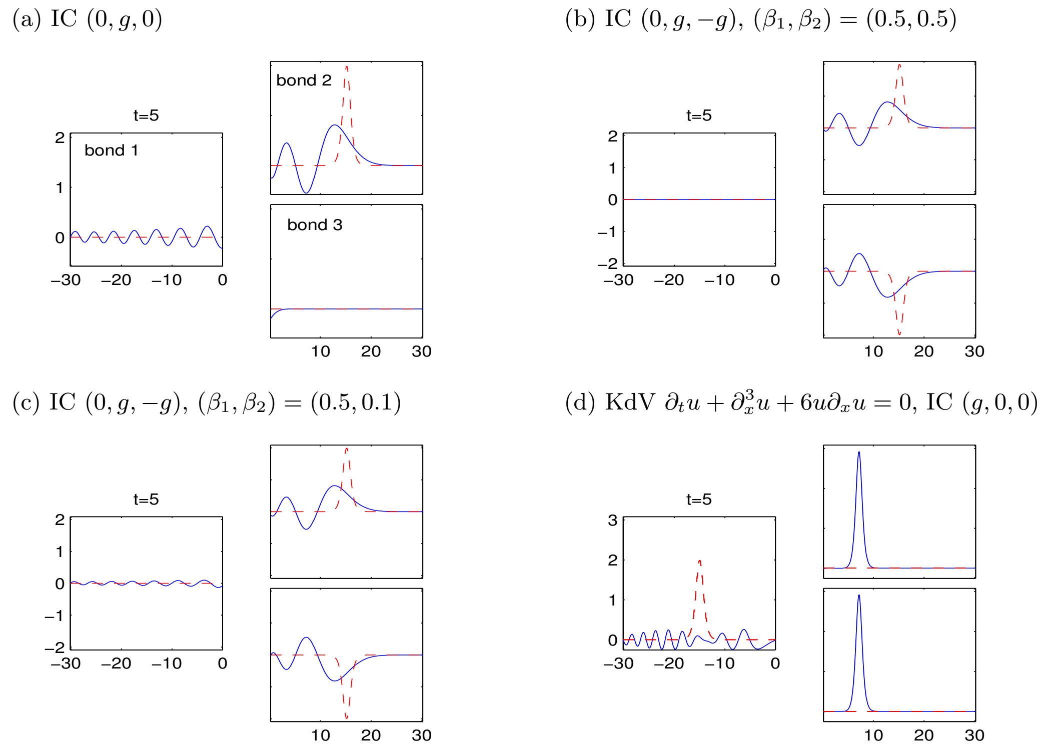

FIG. 1: (a)–(c) Numerical simulation of (1), $\sigma = 1$, (6)–(8), and $(\alpha_2, \alpha_3, \beta_2, \beta_3) = (1, 1, 0.5, 0.5)$ except where indicated otherwise. Initial conditions as indicated with $g(x) = 2\mathrm{sech}(\sqrt{2}(x - x_0)$ ($x_0 = -15$ on bond 1, $x_0 = 15$ on bonds 2,3). Solutions at $t = 5$ as full lines (blue), IC dashed (red). (d) Numerical simulation of the KdV equation, vertex parameters from (a), IC $(g, 0, 0)$. y-scale on bonds 2,3 always as on bond 1.

Thus $\|u\|_\Gamma \le \|u_0\|_\Gamma + \int_0^t \|\mathbf{f}\|dt$ which implies uniqueness. $\square$

We shall construct solutions and prove existence theorems for data from the Schwartz class of smooth decreasing functions, and for data in Sobolev classes. Let $S(B_k)$ be the Schwartz space of rapidly decaying functions on the closure of B_k, $k = 1, 2, 3$. We say $v(x, t) \in C^1([0, T]; S(B_k))$ $(T > 0)$ if v and $\frac{\partial v}{\partial t}$ in $C^0([0, T]; S(B_k))$.

Theorem 4. *Assume that* $\frac{1}{a_2^2} + \frac{1}{a_3^2} + \frac{1}{a_2 b_2} + \frac{1}{a_3 b_3} \ne -1$, $u_{0k}(x) \in S(B_k)$, $f_k(x, t) \in C^1([0, T]; S(B_k))$ *for some* $T > 0$ *and that* $u_{0k}^{(p)} \equiv \frac{\partial^{3p}}{\partial x^{3p}} u_{0k}(x)$ *and* $f_k^{(p)} = \frac{\partial^{3p}}{\partial x^{3p}} f_k(x, t)$ *satisfy (2)–(4) for any non negative integer* p*. Then (1)–(5) has a solution in* $C^1([0, T]; S(B_k))$.

A similar result holds in the case $\sigma = 1$.

Theorem 5. *Assume that* $\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_3} \ne -1$, $u_{0k}(x) \in S(B_k)$, $f_k(x, t) \in C^1([0, T]; S(B_k))$ *for some* $T > 0$ *and that* $\frac{\partial^{3p}}{\partial x^{3p}} u_k(x)$*, and* $\frac{\partial^{3p}}{\partial x^{3p}} f_k(x, t)$ *satisfy (6)–(8) for any non negative integer* p*. Then (1), (6)–(9) with* $\sigma = 1$ *has a solution in* $C^1([0, T]; S(B_k))$.

To treat the case of Sobolev data consider triples $\mathbf{v} = (v_1(x_1), v_2(x_2), v_3(x_3))$ defined on the graph. We suppose that $v_k \in S(B_k)$ and the functions $v_k^{(p)} \equiv \frac{\partial^{3p}}{\partial x^{3p}} v_k(x)$ satisfy vertex conditions (2)–(4) (vertex conditions (6)–(8)) for any non negative integer p and $k = 1, 2, 3$. We denote the set of all such triples by $S^-(\Gamma)$ $(S^+(\Gamma))$, and define $W^-(\Gamma)$ (or $W^+(\Gamma)$) as the closure of the set $S^-(\Gamma)$ (or $S^+(\Gamma)$) with respect to the norm $\|v\|_{3,\Gamma} = \sum_{k=1}^3 \|v_k\|_{H^3(B_k)}$.

Theorem 6. *Let* $\sigma = \pm 1$, $\mathbf{u} \equiv (u_{01}(x_1), u_{02}(x_2), u_{03}(x_3)) \in W^\pm(\Gamma)$, $\mathbf{f} \equiv (f_1(x_1, t), f_2(x_2, t), f_3(x_3, t)) \in L_\infty(0, T, W^\pm(\Gamma))$*, and assume that the conditions of the uniqueness Lemmas are fulfilled. Then (1)–(5) (resp. (1), (6)–(9)) has a unique solution in* $L_\infty(0, T, W^\pm(\Gamma))$.

First we construct exact solutions, using results from the theory of potentials for linearised KdV equation.

A. Some Preliminaries from Potentials Theory

The following functions are called fundamental solutions of the equation $u_t - u_{xxx} = 0$ (see [Cat59, Whi74, Dju79, Abd81])

$$U(x,t;\xi,\eta) = \begin{cases} \frac{1}{(t-\eta)^{1/3}} f\left(\frac{x-\xi}{(t-\eta)^{1/3}}\right), & t > \eta, \\ 0 & t \le \eta, \end{cases}, \qquad V(x,t;\xi,\eta) = \begin{cases} \frac{1}{(t-\eta)^{1/3}} \varphi\left(\frac{x-\xi}{(t-\eta)^{1/3}}\right), & t > \eta, \\ 0 & t \le \eta, \end{cases} \tag{10}$$

where $f(x) = \frac{\pi}{3^{1/3}} Ai\left(-\frac{x}{3^{1/3}}\right)$, $\varphi(x) = \frac{\pi}{3^{1/3}} Bi\left(-\frac{x}{3^{1/3}}\right)$, and $Ai(x)$ and $Bi(x)$ are the Airy functions. The functions $f(x)$ and $\varphi(x)$ satisfy $Z''(x) + \frac{x}{3} Z(x) = 0$. They are integrable and

$$\int_{-\infty}^{0} f(x)dx = \frac{\pi}{3}, \quad \int_{0}^{+\infty} f(x)dx = \frac{2\pi}{3}, \quad \int_{0}^{+\infty} \varphi(x)dx = 0. \tag{11}$$

We summarize some properties of potentials for (1) from [Dju79, Cat59]. For given ω, f and ϕ let

$$u(x,t) = \int_a^b U(x,t;\xi,0)\omega(\xi)d\xi, \qquad v(x,t) = \int_0^t \int_a^b U(x,t;\xi,\tau) f(\xi,\tau) d\xi d\tau,$$

$$w^{(1)}(x,t) = \int_\eta^t U_{x\xi}(x,\eta;a,t)\varphi(\eta)d\eta, \quad w^{(2)}(x,t) = \int_\eta^t V_{x\xi}(x,\eta;a,t)\varphi(\eta)d\eta.$$

Lemma 7. *a) Let $\omega \in BV([a,b])$. Then $u(x,t)$ satisfies $u_t - u_{xxx} = 0$ for $t > 0$ and*

$$\lim_{(x,y)\to(x_0,0)} u(x,t) = \begin{cases} \pi\omega(x_0) & \text{if} \quad x_0 \in (a,b) \\ 0 & \text{if} \quad x_0 \notin (a,b). \end{cases}$$

b) Let $f \in L^2((a,b)\times(0,T)$. Then $v(x,t)$ satisfies $u_t - u_{xxx} = \pi f(x,t)$ in $(a,b)\times(0,T]$, $T > 0$ and initial condition $u(x,0) = 0$, $x \in (a,b)$.
c) If $\varphi \in H^1(0,T)$, then $\lim\limits_{x\to a+0} w^{(1)}(x,t) = \frac{2\pi}{3}\varphi(y)$, $\lim\limits_{x\to a-0} w^{(1)}(x,t) = -\frac{\pi}{3}\varphi(y)$, $\lim\limits_{x\to a+0} w^{(2)}(x,t) = 0$.

Now we are ready to construct exact solutions of the considered problems. We suppose that initial data and source terms in each bond are sufficiently smooth and bounded function.

B. Exact solution in case $\sigma = -1$

We look for solution in the form

$$u_1(x,t) = \int_0^t U(x,t;0,\eta)\varphi_1(\eta)d\eta + F_1(x,t), \tag{12}$$

$$u_2(x,t) = \int_0^t U(x,t;0,\eta)\varphi_2(\eta)d\eta + \int_0^t V(x,t;0,\eta)\psi_2(\eta)d\eta + F_2(x,t), \tag{13}$$

$$u_3(x,t) = \int_0^t U(x,t;0,\eta)\varphi_3(\eta)d\eta + \int_0^t V(x,t;0,\eta)\psi_3(\eta)d\eta + F_3(x,t), \tag{14}$$

where $F_k(x,t) = \frac{1}{\pi}\int\limits_{B_k} U(x,t;\xi,0)u_k(\xi,0)d\xi + \frac{1}{\pi}\int\limits_0^t\int\limits_{B_k} U(x,t;\xi,\eta) f_k(\xi,\eta)d\xi d\eta$, $k = 1,2,3$, and the functions $\varphi_k(t)$, $k = 1,2,3$, $\psi_j(t)$, $j = 2,3$ are unknown. It is clear that the $F_k(x,t)$ and $u_k(x,t)$ satisfy (1) and the initial conditions

(5), and it remains to satisfy the vertex conditions (2–4), which now we rewrite in a more compact matrix form. For $\mathbf{w}(x) = (w_1(x_1), w_2(x_2), w_3(x_3)^T$ some function defined on Γ, let

$$\mathbf{w}(0) = (w_1(-0), w_2(+0), w_3(+0)^T, \qquad \partial^k \mathbf{w}(0) = ((\partial_x^k v_1)(-0,t), (\partial_x^k w_2)(+0,t), (\partial_x^k w_3)(+0,t))^T, k = 1,2.$$

The vertex conditions can be expressed by multiplying the vector $\mathbf{w}(0)$ with some constant row vector. For example, the vertex conditions (2) can be written as $(1, -a_2, 0)\mathbf{u}(0,t) = 0$ and $(1, 0, -a_3)\mathbf{u}(0) = 0$. Thus we rewrite the vertex conditions (2)–(4) as

$$V^- \mathbf{u}(0,t) \equiv V_0^- \mathbf{u}(0,t) + V_1^- \partial \mathbf{u}(0,t) + V_2^- \partial^2 \mathbf{u}(0,t) = 0, \tag{15}$$

where V_k^- are 5×3 matrices with the nonzero rows given by

$$\mu_1 := V_{0,1}^- = (\frac{1}{a_2}, -1, 0), \mu_2 := V_{0,2}^- = (\frac{1}{a_3}, 0, -1), \mu_3 := V_{1,3}^- = (\frac{1}{b_2}, -1, 0), \tag{16}$$

$$\mu_4 := V_{1,4}^- = (\frac{1}{b_3}, 0, -1), \mu_5 := V_{2,5}^- = (1, \frac{1}{a_2}, \frac{1}{a_3}). \tag{17}$$

Then setting $\mathbf{F} = (F_1, F_2, F_3)$ and $\phi = (\varphi_1, \varphi_2, \varphi_3)$, from (2, 3) we obtain the Abel integral equations

$$\int_0^t \frac{1}{(t-\eta)^{1/3}} \left[f(0)\mu_k \phi(\eta) - \varphi(0)\psi_{k+1}(\eta) \right] d\eta = -\mu_k \mathbf{F}(0,t), \quad k = 1,2,$$

$$\int_0^t \frac{1}{(t-\eta)^{2/3}} \left[f'(0)\mu_k \phi(\eta) - \varphi'(0)\psi_{k-1}(\eta) \right] d\eta = -\mu_k \partial_x \mathbf{F}, \quad k = 3,4.$$

with φ from (10). These can be written in terms of fractional integrals [RM08]

$$J_{(0,t)}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha < 1,$$

and solved using the inverse operators, i.e. the Riemann-Liouville fractional derivatives [RM08, Rah10] defined by

$$D_{(0,t)}^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau, \quad 0 < \alpha < 1.$$

From $D_{(0,t)}^{\alpha} J_{(0,t)}^{\alpha} = I$ we obtain the linear algebraic equations

$$f(0)\mu_k \phi(t) - \varphi(0)\psi_{k+1}(t) = -\frac{1}{\Gamma(1/3)} \mu_k D_{(0,t)}^{2/3} \mathbf{F}(0,t), \quad k = 1,2,$$

$$f'(0)\mu_k \phi(t) - \varphi'(0)\psi_{k-1}(t) = -\frac{1}{\Gamma(2/3)} \mu_k \partial D_{(0,t)}^{1/3} \mathbf{F}(0,t), \quad k = 3,4,$$

From the vertex condition (4) we get

$$-\frac{\pi}{3}\varphi_1(t) - \frac{1}{a_2}\frac{2\pi}{3}\varphi_2(t) - \frac{1}{a_3}\frac{2\pi}{3}\varphi_3(t) = -\mu_5 \partial^2 F(0,t).$$

We rewrite this system of linear equations in the matrix form

$$M^- \Phi(t) = -\frac{1}{\Gamma(1/3)} V_0^- D_{0,t}^{2/3} F(0,t) - \frac{1}{\Gamma(2/3)} V_1^- D_{0,t}^{1/3} \partial F(0,t) - \frac{3}{\pi} V_2^- \partial^2 F(0,t), \tag{18}$$

$$M^- = \begin{pmatrix} \frac{f(0)}{a_2} & -f(0) & 0 & -\varphi(0) & 0 \\ \frac{f(0)}{a_3} & 0 & -f(0) & 0 & -\varphi(0) \\ \frac{f'(0)}{b_2} & -f'(0) & 0 & -\varphi'(0) & 0 \\ \frac{f'(0)}{b_3} & 0 & -f'(0) & 0 & -\varphi'(0) \\ -1 & -\frac{2}{a_2} & -\frac{2}{a_3} & 0 & 0 \end{pmatrix}.$$

One can easily compute, that $\det(M^-) = \frac{4\pi^2}{9}\left(\frac{1}{a_2^2}+\frac{1}{a_3^2}+\frac{1}{a_2b_2}+\frac{1}{a_3b_3}+1\right)$. Under the conditions of uniqueness Lemma 2, $\det(M^-) \geq \pi^2/3$, and solving (18) and using the representation of solutions (12)–(14) we get

$$\mathbf{u}(x,t) = \int_0^t \mathbf{U}(x,t-\tau)(M^-)^{-1}\left[-\frac{1}{\Gamma(1/3)}V_0^- D_{0,\tau}^{2/3}F(0,\tau) - \frac{1}{\Gamma(2/3)}V_1^- D_{0,\tau}^{1/3}\partial F(0,\tau) - \frac{3}{\pi}V_2^-\partial^2 F(0,\tau)\right]d\tau + F(x,t), \tag{19}$$

where

$$\mathbf{U}(x,t-\tau) = \begin{pmatrix} U(x,t;0,\tau) & 0 & 0 & 0 & 0 \\ 0 & U(x,t;0,\tau) & 0 & V(x,t;0,\tau) & 0 \\ 0 & 0 & U(x,t;0,\tau) & 0 & V(x,t;0,\tau) \end{pmatrix}.$$

Using the properties of fractional derivatives we rewrite the solution in the form

$$\mathbf{u}(x,t) = F(x,t) + S^0(x,t) * F(0,t) + S^1(x,t) * \partial F(0,t) + S^2(x,t) * \partial^2 F(0,t),$$

where

$$S^0(x,t) = -\frac{1}{\Gamma(1/3)}D_{(0,t)}^{2/3}\mathbf{U}(x,t)(M^-)^{-1}V_0^-, \quad S^1(x,t) = -\frac{1}{\Gamma(2/3)}D_{(0,t)}^{1/3}\partial\mathbf{U}(x,t)(M^-)^{-1}V_1^-, \text{ and}$$
$$S^2(x,t) = -\frac{3}{\pi}\partial^2\mathbf{U}(x,t)(M^-)^{-1}V_2^-$$

are scattering matrices that correspond to the vertex conditions (2), (3) and (4), respectively.

C. Exact solution in case $\sigma = +1$

The construction of exact solutions in the case $\sigma = 1$ is similar as in the case of $\sigma = -1$. Let

$$u_1(x,t) = \int_0^t U(0,t;x,\tau)\gamma_1(\tau)d\tau + \int_0^t V(0,t;x,\tau)\tilde{\gamma}_1(\tau)d\tau + \tilde{F}_1(x,t), \tag{20}$$

$$u_k(x,t) = \int_0^t U(0,t;x,\tau)\gamma_k(\tau)d\tau + \tilde{F}_k(x,t), \quad k = 2,3, \tag{21}$$

where $\gamma_1(t), \gamma_2(t), \gamma_3(t)$, and $\tilde{\gamma}_1(t)$ are unknown density functions, and

$$\pi \cdot F_k(x,t) = \int_{B_k} U(y,t;x,0)u_{0k}(y)dy + \int_0^t \int_{B_k} U(y,t;x,\tau)f_k(y,\tau)dyd\tau. \tag{22}$$

Clearly, the u_k defined by (20)–(21) satisfy (1) with $\sigma = 1$ and the initial condition (9), and again it remains to derive equations for the γ_j from the vertex conditions (6)–(8), which we rewrite as

$$V\mathbf{u}(0,t) \equiv V_0\mathbf{u}(0,t) + V_1\partial\mathbf{u}(0,t) + V_2\partial^2\mathbf{u}(0,t) = 0, \tag{23}$$

where V_k^- are 4×3 matrices with nonzero rows given by

$$\nu_1 := V_{0,1} = (1,-\alpha_2,0), \nu_2 := V_{0,2} = (1,0,-\alpha_3), \nu_3 := V_{1,3} = (1,-\beta_2,-\beta_3), \nu_4 := V_{3,4} = (1,-\alpha_2^{-1},-\alpha_3^{-1}). \tag{24}$$

Then setting $\gamma(t) = (\gamma_1(t),\gamma_2(t),\gamma_3(t))$, from (6) we have the Abel equation

$$\int_0^t \frac{1}{(t-\tau)^{1/3}}\left[f(0)\nu_k\gamma(\tau) + \varphi(0)\tilde{\gamma}_1(\tau)\right]d\tau = -\nu_k\mathbf{F}(0,t) \quad k = 1,2.$$

From this, and analogically from the second vertex condition, we obtain

$$f(0)\nu_k\gamma(t) + \varphi(0)\tilde{\gamma}_1(t) = -\frac{1}{\Gamma(1/3)}\nu_k D_{(0,t)}^{2/3}[\mathbf{F}(0,t)], \quad k = 1,2, \tag{25}$$

$$f'(0)\nu_3\gamma(t) + \varphi'(0)\tilde{\gamma}_1(t) = -\frac{1}{\Gamma(2/3)}\nu_3 D_{(0,t)}^{1/3}[\partial\mathbf{F}(0,t)], \tag{26}$$

and from the vertex condition (8) we have

$$2\gamma_1(t) + \alpha_2^{-1}\gamma_2(t) + \alpha_3^{-1}\gamma_3(t) = -\frac{3}{\pi}\nu_4\partial^2\mathbf{F}(0,t). \tag{27}$$

The system (25)–(27) can be written in matrix form as

$$M\tilde{\gamma} = V_0\frac{1}{\Gamma(1/3)}D_{(0,t)}^{2/3}\mathbf{F}(0,t) + \frac{1}{\Gamma(2/3)}V_1 D_{(0,t)}^{1/3}\partial\mathbf{F}(0,t) + V_2\partial^2\mathbf{F}(0,t), \tag{28}$$

where $\tilde{\gamma(t)} = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \tilde{\gamma}_1(t))$, and

$$M = \begin{pmatrix} f(0) & -\alpha_2 f(0) & 0 & \varphi(0) \\ f(0) & 0 & -\alpha_3 f(0) & \varphi(0) \\ -f'(0) & \beta_2 f'(0) & \beta_3 f'(0) & -\varphi'(0) \\ -2 & -\frac{1}{\alpha_2} & -\frac{1}{\alpha_3} & 0 \end{pmatrix}.$$

We have det $(M) = \dfrac{\pi}{3}f(0)\alpha_2\alpha_3\left[\dfrac{1}{\alpha_2^2} + \dfrac{1}{\alpha_3^2} + 1 + \dfrac{\beta_2}{\alpha_2} + \dfrac{\beta_3}{\alpha_3}\right]$, and using some elementary inequalities this can be estimated as

$$1 - \frac{1}{4}(\beta_2^2 + \beta_3^2) \le \det(M) \le \frac{2}{\alpha_2^2} + \frac{2}{\alpha_3^2} + \frac{1}{4}(\beta_2^2 + \beta_3^2) + 1.$$

Thus, under the conditions of the uniqueness Lemma 3, $\det(M) \ge 3/4$, which proves unique solvability of (25)–(27) with respect to $\tilde{\gamma(t)} = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \tilde{\gamma}_1(t))^T$. Solving, and using (20), (21) we find the solution of (1), (6)–(9) with $\sigma = 1$ in the form

$$\mathbf{u}(x,t) = \int_0^t \mathbf{U}(x,t-\tau)M^{-1}\left\{V_0\frac{1}{\Gamma(1/3)}D_{(0,\tau)}^{2/3}\mathbf{F}(0,\tau) + \frac{1}{\Gamma(2/3)}V_1 D_{(0,\tau)}^{1/3}\partial\mathbf{F}(0,\tau) + V_2\partial^2\mathbf{F}(0,\tau)\right\}d\tau + \mathbf{F}(\mathbf{x},\mathbf{t}), \tag{29}$$

where

$$\mathbf{U}(x,t-\tau) = \begin{pmatrix} U(0,t;x,\tau) & 0 & 0 & V(0,t;x,\tau) \\ 0 & U(0,t;x,\tau) & 0 & 0 \\ 0 & 0 & U(0,t;x,\tau) & 0 \end{pmatrix}.$$

Using the properties of fractional derivatives we can rewrite solution (29) in terms of scattering matrices

$$\mathbf{u}(x,t) = \mathbf{S}^0(x,t) * \mathbf{F}(0,t) + \mathbf{S}^1(x,t) * \partial\mathbf{F}(0,\tau) + \mathbf{S}^3(x,t) * \partial^2\mathbf{F}(0,\tau) + \mathbf{F}(\mathbf{x},\mathbf{t}), \tag{30}$$

with scattering matrices

$$\mathbf{S}^0(x,t) = \frac{1}{\Gamma(1/3)}D_{(0,t)}^{2/3}\mathbf{U}(x,t)M^{-1}V_0, \quad \mathbf{S}^1(x,t) = \frac{1}{\Gamma(2/3)}D_{(0,t)}^{1/3}\mathbf{U}(x,t)M^{-1}V_1, \quad \mathbf{S}^2(x,t) = \frac{3}{\pi}\mathbf{U}(x,t)M^{-1}V_2 \tag{31}$$

corresponding to the vertex conditions (6), (7) and (8), respectively.

D. Proof of existence theorems

Proof of Theorems 4 and 5. According to the theory of potentials [Cat59, CK02] the solutions constructed in the previous sections and their x-derivatives up to second order are continuous functions in the closure of each bond of the graph. Now consider the functions $v_k(x_k,t)$ that are solutions of the considered problem with initial conditions $v_k(x,0) = \frac{d^3}{x^3}u_{0k}(x)$, and with f_k replaced by $\frac{\partial^3}{x^3}f_k$, $k = 1,2,3$. We set

$$\tilde{u}_k(x,t) = \int_0^x \frac{(x-y)^2}{2}v_k(y,t)dy + g_{2k}(t)x^2 + g_{1k}(t)x + g_{0k}(t), \quad k = 1,2,3,$$

where $g_{2k}(t) = \int\limits_0^t \left[\frac{\partial^2 f}{\partial x^2} - \frac{\sigma}{2}\frac{\partial^2 v}{\partial x^2}\right](0,\tau)d\tau + \frac{1}{2}\frac{d^2 u_{0k}}{dx^2}(0)$, $g_{1k}(t) = \int\limits_0^t \left[-\frac{\partial f}{\partial x} + \sigma\frac{\partial v}{\partial x}\right](0,\tau)d\tau - \frac{du_{0k}}{dx}(0)$, $g_{0k}(t) = \int\limits_0^t \left[f(0,\tau) - \sigma v(0,\tau)\right]d\tau + u_{0k}(0)$, $k = 1,2,3$. It is easy to check that the $\tilde{u}_k(x,t)$, $k = 1,2,3$ are solutions of the original problem. Thus, by uniqueness we get $\tilde{u}_k(x,t) = u_k(x,t)$. Noting that $\frac{\partial^3 u_k}{\partial x^3}u(x,t) = v_k(x,t)$ we conclude that the functions $u_k(x,t), k = 1,2,3$ and their x-derivatives of any order are continuous functions in the closure of B_k, with $k = 1,2,3$, respectively.

Now we consider the half lines corresponding to each bond separately. Notice that $u_k(x,t)$ is a solution of linearized KdV equation on the half line B_k and satisfies compatibility conditions at the point $x = 0,\ t = 0$. Applying Theorem 1.1 from [FS99] we get that these solutions define a C^1 map from $[0,T]$ into $S(\bar{B}_k)$. □

Proof of Theorem 6. Above we proved the estimate

$$\|u(\cdot,t)\|_\Gamma \le \|u_0\|_\Gamma + \|f(\cdot,t)\|_\Gamma.$$

Note that for the function v constructed above the same estimate is hold. Summing up these two estimates we have

$$\|u(\cdot,t)\|_{3,\Gamma} \le \|u_0\|_{3,\Gamma} + \|f(\cdot,t)\|_{3,\Gamma}. \tag{32}$$

By construction, $S^{\pm}(\Gamma)$ is dense in $W^{\pm}(\Gamma)$. This, together with a-priori estimate (32) proves Theorem 5. □

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