DIFFUSIVE STABILITY OF ROLLS IN THE TWO–DIMENSIONAL REAL AND COMPLEX SWIFT–HOHENBERG EQUATION

Hannes Uecker

Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, D–30167 Hannover[1]

Abstract We show the nonlinear stability of small bifurcating stationary rolls $u_{\varepsilon,\kappa}$ for the Swift–Hohenberg–equation on the domain $\mathbb{R}^2$. In Bloch wave representation the linearization around a marginal stable roll $u_{\varepsilon,\kappa}$ has continuous spectrum up to 0 with a locally parabolic shape at the critical Bloch vector 0. Using an abstract renormalization theorem we show that small spatially localized integrable perturbations decay diffusively to zero. Moreover we estimate the size of the domain of attraction of a roll $u_{\varepsilon,\kappa}$ in terms of its modulus and Fourier wavenumber. To explain the method we also treat the nonlinear stability of stationary rolls for the complex Swift–Hohenberg equation on $\mathbb{R}^2$.

1 Introduction

We investigate the nonlinear stability of stationary rolls $u_{\varepsilon,\kappa}$ for the Swift–Hohenberg–equation (SHE)

$$\partial_t u = -(1+\triangle)^2 u + \varepsilon^2 u - u^3, \quad t \geq 0, \quad x \in \mathbb{R}^2, \quad u = u(t,x) \in \mathbb{R}, \qquad (1)$$

and of stationary rolls $A_{\varepsilon,\kappa}$ for the complex Swift–Hohenberg equation (cSHE)

$$\partial_t A = -(1+\triangle)^2 A + \varepsilon^2 A - |A|^2 A, \ t \geq 0, \ x \in \mathbb{R}^2, \quad A = A(t,x) \in \mathbb{C}. \qquad (2)$$

In this introduction we first focus on the SHE, which is a phenomenological model for Bénard's problem at the onset of thermal convection, where u

[1]Current address: Mathematik VI, Universität Bayreuth, D-95447 Bayreuth

measures an averaged flow. See [Man92, chapter 8] for some modeling background and [Mie97b] for a partial linear stability analysis for the rolls in the full Bénard problem. For small $\varepsilon > 0$ there exist stationary roll solutions $u_{\varepsilon,\kappa}$ of (1), $\kappa \in (-\varepsilon, \varepsilon)$, which bifurcate from $(\varepsilon, u) \equiv (0,0)$. These rolls are of modulus $r \approx \sqrt{4(\varepsilon^2 - \kappa^2)/3}$ and independent of x_2 and periodic in x_1 with period $2\pi/k$, where $k = \sqrt{\kappa + 1}$. Letting $u(t,x) = u_{\varepsilon,\kappa}(x_1) + v(t,x)$, the perturbations v have to satisfy

$$\partial_t v = \mathcal{L} v + F(v), \tag{3}$$

where $\mathcal{L}$ is a linear operator, F contains the nonlinear terms, and $\mathcal{L}$ and F depend on ε, κ via $u_{\varepsilon,\kappa}$ and have coefficients which are $2\pi/k$–periodic in x_1.

The spectral stability analysis of the rolls is carried out in great detail in [Mie97a], extending the principle of reduced instability as stated in [Mie95] to provide stability results too, giving necessary and sufficient conditions on the amplitude and wavenumber of the rolls. The analysis is done in Bloch wave space and gives marginal stability, i.e., for spectrally stable rolls the spectrum of the operator $\mathcal{L}$ lies entirely in left complex half plane but extends continuously up to 0.

We recall these results and also the setup of Bloch waves in section 5. In our case, a Bloch wave is a function in the form $v(x) = e^{i\sigma\cdot x} V(x_1)$ with $V \in H^4(\mathcal{T}_{2\pi/k})$, where $\mathcal{T}_\alpha = \mathbb{R}/(\alpha\mathbb{Z})$ is the one dimensional torus of length α, and $\sigma \in \mathcal{T}_k \times \mathbb{R}$ is called the Bloch wave vector. Inserting a Bloch wave into the eigenvalue problem $\mathcal{L}v = \lambda v$ we obtain

$$B(\varepsilon,\kappa,\sigma)V \stackrel{\text{def}}{=} -(1 + (\partial_x + i\sigma_1)^2 - \sigma_2^2)^2 V + (\varepsilon^2 - 3u_{\varepsilon,\kappa}^2)V = \lambda V. \tag{4}$$

For every fixed $\sigma \in \mathcal{T}_k \times \mathbb{R}$ the eigenvalue problem (4) is self adjoint in $L^2(\mathcal{T}_{2\pi/k})$. This gives a discrete set of real eigenvalues $\{\lambda_j(\sigma) \in \mathbb{R} : j \in \mathbb{N}\}$, $\lambda_j(\sigma) \geq \lambda_{j+1}(\sigma) \to -\infty$ for $j \to \infty$. The main point is the identity

$$L^2 - \operatorname{spec}(\mathcal{L}) = \operatorname{closure}\Bigg(\bigcup_{\sigma \in \mathcal{T}_k \times \mathbb{R}} \operatorname{spec} B(\varepsilon,\kappa,\sigma) \Bigg).$$

As we will see, the stability of $u_{\varepsilon,\kappa}$ is then determined from the behavior of the smooth function $\sigma \mapsto \lambda_1(\sigma)$ for σ close to 0. In fact, we always have $\lambda_1(0) = 0$. This eigenvalue 0 comes from the translational invariance of the SHE. The associated (generalized) eigenvector is $\partial_{x_1} u_{\varepsilon,\kappa}$.

For a marginally stable roll $u_{\varepsilon,\kappa}$ the surface $\sigma \mapsto \lambda_1(\sigma)$ has a parabolic shape for σ close to 0, i.e.

$$\lambda_1(\sigma) = -c_1\sigma_1^2 - c_2\sigma_2^2 + \mathcal{O}(|\sigma|^4) \quad \text{with} \quad c_j = c_j(\varepsilon,\kappa) > 0, \; j = 1,2. \tag{5}$$

This shape suggests that solutions to the linear problem $v_t = \mathcal{L}v$ decay like solutions to the 2–dimensional linear diffusion equation $u_t = \triangle u$, $x \in \mathbb{R}^2, t \geq 0$,

$u(t,x) \in \mathbb{R}$, $u(0,x) = u_0(x)$. For this equation, respectively for the general d–dimensional case $x \in \mathbb{R}^d$, it is well known, that for integrable spatially localized initial conditions the solutions fulfill

$$\|u(t,\cdot) - \frac{U}{(4\pi t)^{d/2}} e^{-|\cdot|^2/(4t)}\|_{L^\infty} \leq C t^{-(d+1)/2}, \tag{6}$$

where $U = \int u_0(x)dx$. This behavior is called the diffusive stability of the trivial solution $u \equiv 0$. The renormalized solution $(4\pi t)^{d/2} u(t, \sqrt{t}x)$ converges towards the fixed point $Ue^{-|x|^2/4}$.

By means of renormalization theory [BK92, BKL94], the asymptotics (6), with U now given by some function $\mathcal{U}$ of the initial condition u_0, can be shown also for solutions of the d–dimensional nonlinear diffusion equation

$$u_t = \triangle u + u^{p_1}(\partial_{x_j} u)^{p_2}, t \geq 0,\ \ x \in \mathbb{R}^d, \tag{7}$$

provided that

$$d(p_1 + p_2 - 1) + p_2 > 2, \tag{8}$$

and the initial condition is sufficiently small. Therefore, nonlinearities for which (8) holds are called asymptotically irrelevant. The renormalization group approach converts the problem of large time asymptotics for (7) into the iterative process of solving (7) on a fixed time interval, followed by rescaling. We thus obtain a map in the space of initial data: the renormalization map. An estimate like (6) then becomes the problem of existence and stability of a fixed point for the renormalization map. As pointed out in [BKL94], see also [Gal94], the main advantage of this method is, that it works for a wide class of equations and systems and does not depend on special properties like the maximum principle.

Using renormalization theory, the nonlinear stability of the marginal stable rolls for the Swift–Hohenberg equation on $\mathbb{R}$ ($d = 1$) is proved in [Sch96]. The method has been further developed in [Sch97] to prove the nonlinear stability of Eckhaus–stable Taylor vortices in the Taylor Couette problem over an infinite cylinder. Higher order asymptotics for perturbations of rolls in the one–dimensional SHE were derived in [EWW97] using a time–continuous renormalization approach based on [Way97] to construct invariant manifolds.

In the present work we gather the ideas of [BK92, BKL94] and [Sch96] into the abstract Theorem 1, which we then apply to the stability problem for rolls in the SHE and the cSHE in two dimensions. For the real case we prove that perturbations v of a spectrally stable roll $u_{\varepsilon,\kappa}$, that are sufficiently small in a suitable Banach space, converge diffusively to zero, i.e.

$$\|v(t,x) - \frac{\alpha^*}{\sqrt{c_1 c_2}\ t} e^{-\frac{x_1^2}{4c_1 t} - \frac{x_2^2}{4c_2 t}} \partial_{x_1} u_{\varepsilon,\kappa}(x)\|_{L^\infty(\mathbb{R}^2)} \leq C t^{-3/2} \tag{9}$$

with c_1, c_2 from (5). The constant $C > 0$ depends only on the roll $u_{\varepsilon,\kappa}$, i.e. on (ε, κ), and $\alpha^* \in \mathbb{R}$ is a function of (ε, κ) and $v(0, \cdot)$. The precise result is stated in Theorem 20. With a slight abuse of notation, we occasionally write $\|u(x)\|_{L^\infty}$ for the L^∞ norm of functions $u : x \mapsto u(x)$.

For the nonlinear diffusion equation (7) it is an important point that the condition (8) is sharp. In fact, for $d(p_1 - 1) \leq 2$ positive but arbitrarily small initial condition to (7) can blow up in finite time in $L^1(\mathbb{R}^d)$, see for example [Wei81]. Condition (8) shows that a higher space dimension gives a lower minimal order for irrelevant nonlinearities. Formally one can see, that the same happens for the SHE where in two dimensions cubic terms become irrelevant. This better behavior of nonlinear diffusion equations in higher dimensions together with its consequences for the diffusive stability method is also discussed in [Sch98].

However, taking a roll $u_{\varepsilon,\kappa}$ as new origin for the SHE we obtain quadratic terms in the nonlinearity. But these quadratic terms can be controlled using the special structure of the equation, namely the translational invariance of the original problem. Condition (8) also shows, that nonlinear terms with derivatives, corresponding in Fourier space via $\mathcal{F}(\partial_{x_j} u)(\sigma) = \mathrm{i}\sigma_j(\mathcal{F}u)(\sigma)$ to vanishing coefficients at the Fourier wave vector $\sigma = 0$, are "more irrelevant" than nonlinear terms without derivatives. Although the nonlinearity $F(v)$ in (3) does not contain derivatives, the diffusive stability method works for the SHE, because the projection of the quadratic interaction of the critical modes onto the eigenspace associated to the critical eigenvalue $\lambda_1(\sigma)$ vanishes at the Bloch wave vector $\sigma = 0$. This crucial observation from [Sch96] for the one–dimensional case does not depend on the dimension, but only on the translational invariance of the original problem.

In fact, for the SHE and for similar translational invariant problems this vanishing of the projection of the critical nonlinear terms onto the critical eigenspace at $\sigma = 0$ can be seen by an abstract argument, relating the projection with the center manifold in case of a space of periodic functions, see Remark 27 and [Sch96, EWW97].

These arguments heuristically show that the nonlinearity vanishes up to a sufficiently high order. However, the arguments are formal. It is by means of the renormalization approach that we are able to prove the asymptotic irrelevance of the nonlinear terms rigorously and to obtain precise decay rates.

To explain the method in a somewhat simpler setting we also treat the nonlinear stability of stationary rolls $A_{\varepsilon,\kappa}$ for the complex Swift–Hohenberg equation (2). This is an instructive model problem, which nevertheless exhibits the same difficulties as the SHE. It has, compared to the SHE, an additional symmetry, namely the phase invariance $A \mapsto e^{\mathrm{i}\alpha} A$, $\alpha \in [0, 2\pi)$. Therefore the cSHE has an explicitly given three parameter family of stationary solutions in the form

$$A(x) = r e^{\mathrm{i}(\alpha + k_1 x_1 + k_2 x_2)}, \tag{10}$$

$$k \in \mathbb{R}^2 \text{ with } |k| \in (\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}),\ r^2 = \varepsilon^2 - (1-|k|^2)^2, \quad \alpha \in [0, 2\pi).$$

Without loss of generality we choose $(k_1, k_2) = (k, 0)$. By $A_{\varepsilon,\kappa}$ we denote the unique roll with $\alpha = 0$, where $k = \sqrt{\kappa + 1}$. We let $A(t,x) = A_{\varepsilon,\kappa}(x) + e^{\mathrm{i}kx_1} B(t,x)$ to obtain

$$\partial_t B = \mathcal{L} B + F(B), \tag{11}$$

where now $\mathcal{L}$ and F have constant coefficients. Thus (11) can be treated by Fourier transform. The linear problem $\partial_t B = \mathcal{L} B$ has been also analyzed in [Mie97a]. We recall these results in Section 3. In Section 4 we formulate a renormalization process for (11) in Fourier space and apply the abstract Theorem 1 to prove an analogous result to (9), namely that for suitably small $B(0,\cdot)$ the solution B of (11) satifies

$$||B(t,x) - \frac{\mathrm{i}\alpha^*}{\sqrt{c_1 c_2}\, t} e^{-\frac{x_1^2}{4c_1 t} - \frac{x_2^2}{4c_2 t}}||_{L^\infty(\mathbb{R}^2)} \leq C t^{-3/2} \text{ for } t \to \infty, \tag{12}$$

where $c_1, c_2, C > 0$ and $\alpha^* \in \mathbb{R}$, see Theorem 7. We remark that (12) is completely analogous to (9). In (12) the derivative $\partial_{x_1} A_{\varepsilon,\kappa} = \mathrm{i}k e^{\mathrm{i}kx_1}$ of the roll does not appear because it is hidden in the ansatz $A(t,x) = A_{\varepsilon,\kappa}(x) + e^{\mathrm{i}kx_1} B(t,x)$, which gave the constant coefficient system (11). As a remainder we have the factor i in (12).

For both cases, the SHE and the cSHE, the perturbations of the rolls have to be small in $H^2(\mathbb{R}^2, 3)$, where

$$H^m(\mathbb{R}^2, k) = \{u : \mathbb{R}^2 \to \mathbb{C} : (\partial^\alpha u)\rho^k \in L^2(\mathbb{R}^2)\ , \alpha \in \mathbb{N}_0^2, |\alpha| \leq m\},$$

with $\rho(x) = (1 + |x|^2)^{1/2}$. For notational convenience we mostly drop the domain $\mathbb{R}^2$ and write $H^m(k)$ for $H^m(\mathbb{R}^2, k)$. Note that Fourier transform

$$\hat{u}(\sigma) = (\mathcal{F}u)(\sigma) = \frac{1}{2\pi} \int e^{-\mathrm{i}\sigma\cdot x} u(x)\, \mathrm{d}x, \quad (\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{2\pi} \int e^{\mathrm{i}\sigma\cdot x} \hat{u}(\sigma)\, \mathrm{d}\sigma,$$

is an isomorphism between $H^m(k)$ and $H^k(m)$.

Finally we remark, that although for the SHE the existence and spectral stability of stationary rolls is established using bifurcation theory, the diffusive stability method itself does not rely on any bifurcation arguments. However, as in [Sch96], we want to estimate the size of the domain of attraction of a spectrally stable roll in terms of (ε, κ). Therefore we need to keep track of the bifurcation arguments from [Mie97a] for the spectral analysis of the rolls. This makes our analysis more lengthy than it would be for a single fixed spectrally stable roll.

2 An abstract frame

We provide the abstract Theorem 1. First however, we explain the idea of renormalization, considering the nonlinear diffusion equation (7), i.e. $u_t = \triangle u + u^{p_1}(\partial_{x_j} u)^{p_2}$, $x \in \mathbb{R}^d$, under the condition (8), i.e. $d(p_1+p_2-1)+p_2 > 2$.

2.1 The idea of renormalization

In the introduction we indicated that if (8) holds, then for small, spatially localized initial data the renormalized solution $t^{d/2}u(t,\sqrt{t}x)$ to (7) converges towards the fixed point $Ue^{-|x|^2/4}$, $U = \mathcal{U}(u_0)$, i.e. we have the asymptotics (6). To show this, the idea is as follows. We consider (7) in a Banach space X of functions $u : \mathbb{R}^d \to \mathbb{R}$, where X has to be suitably chosen, and introduce a discrete analogue of the scaling $\tilde{u}(t,x) = t^{d/2}u(t,\sqrt{t}x)$: for $L > 0$ we define scaling operators $\mathcal{R}_L : X \to X$, $(\mathcal{R}_L u)(x) = u(Lx)$. Clearly we have $(\mathcal{R}_L)^{-1} = \mathcal{R}_{1/L}$. For notational convenience we assume the initial condition for (7) to be given at time $t = 1$. We define $F(u) = u^{p_1}(\partial_{x_j} u)^{p_2}$, and for fixed $L > 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we let

$$u_n(T,y) = L^{nd}u(L^{2n}T, L^n y), \tag{13}$$

i.e. $u_n(T) = L^{nd}\mathcal{R}_{L^n}u(L^{2n}T)$. We obtain

$$\partial_T u_n = \mathcal{L}_n u_n + F_n(u_n), \quad \text{where} \tag{14}$$

$$F_n(u_n) = L^{n(2+d)}F(L^{-nd}\mathcal{R}_{1/L^n}u_n) = L^{n(2+d(1-p_1-p_2)-p_2)}u_n^{p_1}(\partial_{y_j}u_n)^{p_2}, \tag{15}$$

and $\mathcal{L}_n = \triangle$ for all n, which is the scale invariance of the linear diffusion equation $u_t = \triangle u$. Now we write the variation of constant formula for (14) as sequence of equations ($n \in \mathbb{N},\ T \in [1/L^2, 1]$)

$$u_n(T) = e^{\triangle(T-1/L^2)}L^d\mathcal{R}_L u_{n-1}(1) + \int_{1/L^2}^{T} e^{\triangle(T-\tau)}F_n(u_n(\tau))\,\mathrm{d}\tau. \tag{16}$$

This means that we consider the following process, starting with $n = 1$, which is equivalent to solving (7) on the time interval $(1,\infty)$:

> (16) is solved in $C([1/L^2,1],X)$. Then $L^d\mathcal{R}_L u_n(1)$ is taken as initial condition for $n+1$, i.e. $u_{n+1}(1/L^2) = L^d\mathcal{R}_L u_n(1)$.

To solve (16) we proceed as usual: we consider a second Banach space $\widetilde{X} \supset X$, such that on the one hand $F : X \to \widetilde{X}$ is well defined and continuous, and fullfills $\|F(u)\|_{\widetilde{X}} \le C\|u\|_X^{p_1+p_2}$, which in turn gives

$$\|F_n(u_n)\|_{\widetilde{X}} \le CL^{n(2+d(1-p_1-p_2)-p_2)}\|u_n\|_X^{p_1+p_2}. \tag{17}$$

On the other hand, we combine this with smoothing properties of the linear semigroup, i.e. with an estimate of the form $\|e^{\mathcal{L}_n T}\|_{L(\widetilde{X},X)} \leq C(1+T^{-1/2})$. Then for $\|u_n(1/L^2)\|_X$ sufficiently small we have a unique solution for (16).

Next we want to show, that if $\|u_0(1)\|_X$ is small, then $\|u_n(1/L^2)\|_X$ stays small for all $n \in \mathbb{N}$, where $L \in [L_0, L_0^2]$ for some $L_0 > 1$, and that for some $\alpha^* \in \mathbb{R}$ we have

$$\|u_n(1) - \alpha^* \psi\|_X \leq C L^{-n}. \tag{18}$$

Here $\psi(x) = e^{-|\cdot|^2/4}$ is the fixed point of the map $u \mapsto Ku = e^{\triangle(1-1/L^2)} L^d \mathcal{R}_L u$. The idea is, that in $F_n(u_n)$ the factor $L^{n(2+d(1-p_1-p_2)-p_2)}$ goes to 0 for $n \to \infty$, provided that $d(p_1+p_2-1)+p_2 > 2$. Therefore, if $\|u_n\|_X$ stays bounded, the whole nonlinearity vanishes for $n \to \infty$. Then, additionally to the above requirements, it remains to choose X in such a way, that the linear map K contracts X to the invariant subspace $\mathrm{span}\{\psi\}$.

The discrete convergence (18) gives for $t = L^{2n}$

$$\|u(t, \sqrt{t}\cdot) - \frac{\alpha^*}{t^{d/2}} \psi(\cdot)\|_X \leq C t^{-(d+1)/2}. \tag{19}$$

This also shows, that (in an obvious sense) we only need the parabolic shape of the spectrum of $\mathcal{L} = \triangle$ for σ close to 0. In fact, assume that X is such that Fourier transform is an isomorphism from X into a Banach space $\widehat{X}$. Then the formula

$$\mathcal{F}(\mathcal{R}_L u) = L^{-d}(\mathcal{R}_{1/L} \mathcal{F} u) \tag{20}$$

and (19) give $\|\widehat{u}(t, \cdot/\sqrt{t}) - \alpha^* \widehat{\psi}(\cdot)\|_{\widehat{X}} \leq C t^{-1/2}$, i.e., $\widehat{u}(t,\sigma)$ concentrates at $\sigma = 0$.

2.2 The abstract theorem

The following abstract theorem generalizes the above ideas and is additionally motivated as follows. We consider systems of the form $u_t = \mathcal{L}u + F(u)$. Assume that u can be decomposed as $u(t,x) = u^c(t,x) + u^s(t,x)$, where the so called diffusive part u^c behaves like the solution of a diffusion equation (with irrelevant nonlinearity), and u^s is linearly exponentially damped in time. Then introducing variables u_n and v_n similar to (13) we obtain a sequence of systems similar to (16).

In the following theorem we give assumptions on an abstract system of this kind, under which an asymptotic behavior in the sense of (18) holds. The results (12) and (9) will be proved as an application of the theorem to the cSHE and to the SHE.

Theorem 1 *Let $(X_n)_{n\in\mathbb{N}_0}, (Y_n)_{n\in\mathbb{N}_0}$ be sequences of Banach spaces. Let $Z_n = X_n \times Y_n$. For arbitrary $L > 1$ consider*

$$\left.\begin{aligned} u_n(T) &= e^{\mathcal{L}_n^c(T-1/L^2)} H_n^c u_{n-1}(1) + \int_{1/L^2}^T e^{\mathcal{L}_n^c(T-\tau)} F_n^c(u_n(\tau), v_n(\tau))\,\mathrm{d}\tau \\ v_n(T) &= e^{\mathcal{L}_n^s(T-1/L^2)} H_n^s v_{n-1}(1) + \int_{1/L^2}^T e^{\mathcal{L}_n^s(T-\tau)} F_n^s(u_n(\tau), v_n(\tau))\,\mathrm{d}\tau \end{aligned}\right\} \quad n\in\mathbb{N}, \qquad (21)$$

where $(u_0(1), v_0(1)) \in Z_0$, and $\mathcal{L}_n^c, \mathcal{L}_n^s, H_n^c, H_n^s, F_n^c, F_n^s$ may depend on L. Define $z_n(T) = (u_n(T), v_n(T))$. Let $(\widetilde{X})_{n\in\mathbb{N}_0}$ be a sequence of Banach spaces with $X_n \subset \widetilde{X}_n$ and assume that there exists a $C > 0$ such that for all $L > 1$ and all $n \in \mathbb{N}$ the following holds:

A1) *$(e^{\mathcal{L}_n^c T})_{T\geq 0}$ and $(e^{\mathcal{L}_n^s T})_{T\geq 0}$ are strongly continuous semigroups on X_n, Y_n, $\|e^{\mathcal{L}_n^c T}\|_{\widetilde{X}_n\to X_n} \leq C(1+T^{-1/2})$ and $\|e^{\mathcal{L}_n^s T}\|_{Y_n\to Y_n} \leq Ce^{-aL^{2n}T}$ where $a > 0$.*

A2) *$F_n^c \in C(Z_n, \widetilde{X}_n), F_n^s \in C(Z_n, Y_n)$, $F_n^c(0) = F_n^s(0) = 0$, and for all $z, \widetilde{z} \in B_1^{Z_n}(0)$ we have $\|F_n^c(z)-F_n^c(\widetilde{z})\|_{\widetilde{X}_n} \leq CL^{-n}(\|z\|_{Z_n} + \|\widetilde{z}\|_{Z_n})\|z-\widetilde{z}\|_{Z_n}$ and $\|F_n^s(z)-F_n^s(\widetilde{z})\|_{Y_n} \leq CL^n(\|z\|_{Z_n} + \|\widetilde{z}\|_{Z_n})\|z-\widetilde{z}\|_{Z_n}$.*

A3) *$\|H_n^c\|_{X_{n-1}\to X_n} \leq CL^{m_c}, \|H_n^s\|_{Y_{n-1}\to Y_n} \leq CL^{m_s}$ for some $m_c, m_s \geq 0$.*

A4) *There exist $\Pi_n \in \mathrm{Lin}(X_n, \mathbb{R}), \psi_n \in X_n$, $\|\Pi_n\|_{\mathrm{Lin}(X_n,\mathbb{R})} \leq C$, $\|\psi_n\|_{X_n} \leq C$, such that $\Pi_n\psi_n = 1$ and such that $\forall \theta_{n-1} \in X_{n-1}$ with $\Pi_{n-1}\theta_{n-1} = 0$ we have $\|e^{\mathcal{L}_n^c(1-1/L^2)} H_n^c(\alpha\psi_{n-1}+\theta_{n-1}) - \alpha\psi_n\|_{X_n} \leq CL^{-1}\|\theta_{n-1}\|_{X_{n-1}} + \alpha CL^{-n}$.*

Then there exist constants $\delta > 0$, $L_0 > 1$ such that for all $z_0(1) \in Z_0$ with $\|z_0(1)\|_{Z_0} \leq \delta$ and for all $L \in [L_0, L_0^2]$ there exists a unique solution $(z_n)_{n\in\mathbb{N}}$ of (21) with $z_n \in C([1/L^2, 1], Z_n)$. Moreover, there exist a $C > 0$ and a continuous function $\alpha^* : B_\delta^{Z_0}(0) \to \mathbb{R}$ with $|\alpha^*(z)| \leq C\|z\|_{Z_0}$, such that

$$\|u_n(1,\cdot) - \alpha^*(z_0(1))\psi_n(\cdot)\|_{X_n} \leq CL^{-n} \text{ and } \|v_n(1,\cdot)\|_{Y_n} \leq CL^{-n}. \qquad (22)$$

Remark 2 This rather complicated looking formulation of the theorem with sequences $(X_n), (\widetilde{X}_n)$ and (Y_n) of Banach spaces is due to the fact, that in the Bloch transformed SHE the wave vector domain changes on rescaling, see Remark 19. However, this is not essential for the theorem or for the applications. For the cSHE we can work in fixed Banach spaces $X, \widetilde{X}, Y$, see section 4. Then also the operators $\Pi_n \equiv \Pi$ (evaluation of the function at wave vector 0 in Fourier space) and $H_n^c = H_n^s \equiv \mathcal{R}_{1/L}$ (rescaling in Fourier space) are the same for all n.

As already outlined in the previous section, the idea of the theorem is as follows. The assumptions A1) and A2) assure that we can solve (21) for fixed n, provided that $\|z_n(1/L^2)\|_{Z_n}$ is sufficiently small. Here we combine the smoothing property of the diffusive semigroup $e^{\mathcal{L}_n^c T}$ with the estimate on the nonlinearity F_n^c in the weaker space $\widetilde{X}_n$. This accounts for a possible "derivative like structure" in F_n^c. Note that the different scaling behavior of the Lipschitz constants of F_n^c and F_n^s in A2) fits with the corresponding estimates for $e^{\mathcal{L}_n^c T}$ and $e^{\mathcal{L}_n^c T}$ in A1).

By A3) we can control $\|z_n(1/L^2)\|_{Z_n}$ in terms of $\|z_{n-1}(1)\|_{Z_{n-1}}$. Next, the integrals on the right hand side of (21) vanish for $n \to \infty$, provided that $\|z_n\|_{Z_n}$ stays small. Then A4) means, that the linear map $e^{\mathcal{L}_n^c(1-1/L^2)}H_n^c$ has the right contraction properties. This couples the properties of the operators $e^{\mathcal{L}_n^c(1-1/L^2)}$ and H_n^c. It is clear that the operator $e^{\mathcal{L}_n^s(1-1/L^2)}H_n^s$ is a contraction for L sufficiently large.

Remark 3 Consider (7) with $p_2 \geq 1$ and $d(p_1+p_2-1)+p_2 > 2$. We choose $X_n = H^m(\mathbb{R}^d,k)$ and $\widetilde{X}_n = H^{m-1}(\mathbb{R}^d,k)$ for all $n \in \mathbb{N}$, where $m = k = \lceil d/2+1+\delta \rceil$ for some $\delta > 0$, and $\lceil \alpha \rceil$ is the smallest integer greater than α. Define u_n by (13), $\mathcal{L}_n^c = \triangle$, $H_n^c = L^d\mathcal{R}_L$, and $v_n \equiv 0$, i.e. $Y_n = \{0\}$ for all $n \in \mathbb{N}$. Then A1) is obvious. For $F(u) = u^{p_1}(\partial_{x_j}u)^{p_2}$ we have $F \in C(X,\widetilde{X})$ with $\|F(u)\|_{\widetilde{X}} \leq \|u\|_X^{p_1+p_2}$ by the Sobolev imbedding theorem. With (15) we obtain A2). For $\alpha \in \mathbb{N}_0^d$ we find

$$\|\partial^\alpha(L^d\mathcal{R}_L u)\|_{H^0(\mathbb{R}^d,k)} \leq CL^{d/2+|\alpha|}\|\partial^\alpha u\|_{H^0(\mathbb{R}^d,k)}, \text{ and thus}$$
$$\|L^d\mathcal{R}_L u\|_{H^m(\mathbb{R}^d,k)} \leq CL^{d/2+m}\|u\|_{H^m(\mathbb{R}^d,k)}.$$

Hence we have A3) with $m_c = m + d/2$. For A4) we define $\psi_n = e^{-|\cdot|^2/4}$ and $\Pi_n u = \hat{u}(0)$ for all $n \in \mathbb{N}$, and consider the map $u \mapsto Ku = e^{\triangle(1-1/L^2)}L^d\mathcal{R}_L u$ in Fourier space. Using (20) this map becomes $\hat{u} \mapsto e^{-|\sigma|^2(1-1/L^2)}\mathcal{R}_{1/L}\hat{u}$, and with $\mathcal{F}\psi(\sigma) = \hat{\psi}(\sigma) = e^{-|\sigma|^2}$ we find $K\psi = \psi$. Next, for $\hat{\theta} \in H^k(\mathbb{R}^d,m)$ with $\hat{\theta}(0) = 0$ we have $|\hat{\theta}(\sigma/L) - \hat{\theta}(0)| \leq C|\sigma|L^{-1}\|\hat{\theta}\|_{H^k(\mathbb{R}^d,m)}$. This holds by the mean value theorem since $H^k(\mathbb{R}^d,m) \hookrightarrow C^1(\mathbb{R}^d)$. Here we need the regularity in Fourier space. Then

$$\begin{aligned}
&\|e^{-|\sigma|^2(1-1/L^2)}\mathcal{R}_{1/L}\hat{\theta}\|^2_{H^k(\mathbb{R}^d,m)} \\
&\leq C\int\Bigg\{\Bigg(\sum_{|\alpha|=0}^{k}|D^\alpha e^{-|\sigma|^2}|^2\ |\sigma|^2L^{-2}\|\hat{\theta}\|^2_{C^1(\mathbb{R}^d)} + \sum_{|\alpha|=0}^{k-1}|D^\alpha e^{-|\sigma|^2}|^2\ L^{-2}\|\hat{\theta}\|^2_{C^1(\mathbb{R}^2)} \\
&\qquad + \ldots + e^{-|\sigma|^2}\sum_{|\alpha|=k}|D^\alpha\hat{\theta}(\sigma/L)|^2L^{-2k}\Bigg)^2(1+|\sigma|^2)^k\Bigg\}\,\mathrm{d}\sigma \\
&\qquad\qquad \leq CL^{-2}\|\hat{\theta}\|^2_{C^1(\mathbb{R}^d)} + CL^{-2}\|\hat{\theta}\|^2_{H^k(\mathbb{R}^d,m)}.
\end{aligned}$$

This shows A4). Thus we have (22), and from $H^m(\mathbb{R}^d,k) \hookrightarrow L^\infty(\mathbb{R}^d)$ we conclude (6).

2.3 Proof of Theorem 1

Throughout this section we use the notations from Theorem 1 and assume A1)–A4). Moreover, we abbreviate $H_n(u,v) = (H_n^c u, H_n^s v)$ and $K_n^c = e^{\mathcal{L}_n^c(1-1/L^2)} H_n^c$. By (21) we obtain $(u_n(1), v_n(1))$ for $n \in \mathbb{N}$ from iterating the following process:

solve (21) on $[1/L^2, 1]$, take $H_{n+1}(u_n(1), v_n(1))$ as initial condition for $n+1$.

Without loss of generality we assume $m_c \geq m_s$. To estimate the solutions (u_n, v_n) on $[1/L^2, 1]$ we introduce

$$r_n(T) := \|u_n(T)\|_{X_n} + \|v_n(T)\|_{Y_n}, \quad R_n := \sup_{T \in [1/L^2, 1]} r_n(T).$$

Lemma 4 *Let $\alpha > m_c$. There exist $L_0, C \geq 1$ such that for all $L > L_0$ and for all $n \in \mathbb{N}$ the following holds. If $r_{n-1}(1) \leq L^{-\alpha}$, then there exists a unique solution $(u_n, v_n) \in C([1/L^2, 1], Z_n)$ of (21). This solution fulfills $R_n \leq C L^{-\alpha + m_c}$.*

Proof. Define $\delta = 2 r_n(1/L^2)$ where L is such that $\delta \leq C L^{-\alpha + m_c} \leq 1$. Then it is easy to see that the solution operator $S : z_n \mapsto \widetilde{z}_n$, defined via the variation of constant formula (21) by replacing $z_n = (u_n, v_n)$ on the left hand side by $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$ and omitting H_n^c, H_n^s maps

$$B_\delta := \{z : C([1/L^2], Z_n) : z(1/L^2) = H_n z_{n-1}(1) \sup_{T \in [1/L^2, 1]} \|z(T)\|_{Z_n} \leq \delta\}$$

into itself and is a contraction, provided that L is sufficiently large. □

Now suppose for a moment that $\sup_n R_n \leq R$ for some $R \in (0, 1]$. We let $u_n(1) = \alpha_n \psi_n + \theta_n$ with $\alpha_n \in \mathbb{R}$, $\Pi_n \theta_n = 0$, and $\phi_n = v_n(1)$. From (21) we obtain

$$\begin{aligned}
\alpha_n =& \alpha_{n-1}(1 + \mathcal{O}(L^{-n})) + \mathcal{O}(L^{-1} \|\theta_{n-1}\|_{X_{n-1}}) \\
& + \Pi_n \int_{1/L^2}^{1} e^{\mathcal{L}_n^c(1-\tau)} F_n^c(u_n(\tau), v_n(\tau)) \, d\tau, \\
\theta_n =& K_n^c(\alpha_{n-1}\psi_{n-1} + \theta_{n-1}) - \alpha_n \psi_n + \int_{1/L^2}^{1} e^{\mathcal{L}_n^c(1-\tau)} F_n^c(u_n(\tau), v_n(\tau)) \, d\tau, \\
\phi_n =& e^{\mathcal{L}_n^s(1-1/L^2)} H_n^s \phi_{n-1} + \int_{1/L^2}^{1} e^{\mathcal{L}_n^s(1-\tau)} F_n^s(u_n(\tau), v_n(\tau)) \, d\tau.
\end{aligned}$$

This gives

$$|\alpha_n - \alpha_{n-1}| \leq C L^{-n} |\alpha_{n-1}| + C L^{-1} \|\theta_{n-1}\|_{X_{n-1}} + C L^{-n} R_n^2, \tag{23}$$

$$\|\theta_n\|_{X_n} \leq C L^{-n} |\alpha_{n-1}| + C L^{-1} \|\theta_{n-1}\|_{X_{n-1}} + C L^{-n} R_n^2, \tag{24}$$

$$\|\phi_n\|_{Y_n} \leq C e^{-a L^{2n}} L^{m_s} \|\phi_{n-1}\|_{Y_{n-1}} + C L^{-n} R_n^2, \tag{25}$$

where in (25) we already used (23) and (24). Choosing L_0 sufficiently large, we find $\|\theta_n\|_{X_n}, \|\phi_n\|_{Y_n}, |\alpha_n - \alpha_{n-1}| \leq CL^{-n}$ for $L > L_0$. Thus, there exists an $\alpha^* = \alpha^*(u_0(1), v_0(1)) \in \mathbb{R}$ such that $|\alpha_n - \alpha^*| \leq CL^{-n}$. Hence we may conclude assertion (22) in Theorem 1, collecting the various C into C. The continuity of the function α^* and the estimate $|\alpha^*(z_0(1))| \leq C\|z_0(1)\|_{Z_0}$ follow from the continuity of the solution $z_n \in C([1/L^2, 1], Z_n)$ and from the above estimates.

It remains to show $\sup_n R_n \leq R$ for some $R \in (0, 1]$. By Lemma 4 it is sufficient to show that $r_n(1) \leq L^{-(m_c+\gamma)}$ for all $n \in \mathbb{N}$, for some $\gamma > 0$, $L > L_0$ such that $CL^{-\gamma} \leq 1/2$. Define $\delta_n = C|\alpha_n| + \|\theta_n\|_{X_n} + \|\phi_n\|_{Y_n}$ where C is chosen such that $r_n(1) \leq \delta_n$ and let $\delta_0 \leq L^{-\beta_0}$ with $\beta_0 = (\lceil m_c\rceil^2 + \lceil m_c\rceil)/2 + (m_c+1)\gamma$. By Lemma 4 and (23)–(25) we obtain $R_1 \leq CL^{-\beta_0+m_c}$ and thus

$$\delta_1 \leq \delta_0 + r_0(1)CL^{-1} + CL^{-1}R_1^2 \leq L^{-\beta_1},$$

where $\beta_1 = \beta_0 + 1 - m_c - \gamma$ for L sufficiently large. Next we have $R_2 \leq CL^{-\beta_1+m_c}$ and hence $\delta_2 \leq \delta_1 + r_1(1)CL^{-2} + CL^{-2}R_1^2 \leq L^{-\beta_2}$, where $\beta_2 = \beta_1 + 2 - m_c - \gamma$ for L sufficiently large. Proceeding inductively we find $\delta_n \leq L^{\beta_n}$ with $\beta_n = \beta_{n-1} + n - m_c - \gamma = \beta_0 + n(n+1)/2 - n(m_c + \gamma)$. Thus, for $\beta_0 = (\lceil m_c\rceil^2 + \lceil m_c\rceil)/2 + (m_c+1)\gamma$ we have $\beta_{\lceil m_c\rceil} \geq m_c + \gamma$. Then the condition $\delta_n \leq \delta_{n-1} + CL^{-n}R_n^2 + CL^{-n}R_n \stackrel{!}{\leq} 2\delta_{\lceil m_c\rceil}$ gives

$$\sum_{n=\lceil m_c\rceil+1}^{\infty} L^{-\gamma}L^{-n} = L^{-\lceil m_c\rceil-1-\gamma}\frac{1}{1-1/L} \leq L^{-m_c-\gamma},$$

which is fulfilled, like all the conditions above, for all $L > L_0$ for some $L_0 > 1$. Defining δ in Theorem 1 by $\delta = L_0^{-2\beta_0}$, we find $\sup_n R_n \leq 1$ and hence (22) for all $z_0 \in Z_0$ with $\|z_0\|_{Z_0} \leq \delta$ and for all $L \in [L_0, L_0^2]$. This completes the proof of Theorem 1. □

3 Spectral analysis for the rolls $A_{\varepsilon,\kappa}$ in the cSHE

We recall from [Mie97a] the spectral analysis for rolls $A_{\varepsilon,\kappa} = re^{ikx_1}$ in the cSHE, $r = \sqrt{\varepsilon^2 - \kappa^2}$, $\kappa = k^2 - 1$. In fact, we redo the analysis in some detail since we have a slightly different focus. In addition to characterizing the spectrally stable rolls we are interested in the shape of the spectrum for the linearization around these rolls.

From now on we use the coordinates $x = (\xi, x_2)$ with $\xi = kx_1$ for the original x_1. Letting $A(t,x) = A_{\varepsilon,\kappa}(x) + e^{i\xi}B(t,x)$ we obtain

$$\partial_t B = \widetilde{\mathcal{L}}B + \widetilde{F}(B), \text{ where} \tag{26}$$

$$\widetilde{\mathcal{L}}B = -(1 + k^2(\partial_\xi + \mathrm{i})^2 + \partial_{x_2}^2)^2 B + \varepsilon^2 B - 2r^2 B - r^2\overline{B},$$

$$\widetilde{F}(B) = -2r|B|^2 - rB^2 - |B|^2 B.$$

By the ansatz $B(t,x) = w_1(t,x) + \mathrm{i}w_2(t,x),\ w_j(t,x) \in \mathbb{R},\ j = 1,2$ and separation into real and imaginary part we find, for $w = (w_1, w_2) \in \mathbb{R}^2$,

$$\partial_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathcal{L} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + F(w), \text{ where} \tag{27}$$

$$\mathcal{L} = \begin{pmatrix} -L_2^2 + 4k^4\partial_\xi^2 + \varepsilon^2 - 3r^2 & 4k^2\partial_\xi L_2 \\ -4k^2\partial_\xi L_2 & -L_2^2 + 4k^4\partial_\xi^2 + \varepsilon^2 - r^2 \end{pmatrix},$$

$$F(w) = -r \begin{pmatrix} 3w_1^2 + w_2^2 \\ 2w_1w_2 \end{pmatrix} - (w_1^2 + w_2^2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

with $L_2 = (k^2\partial_\xi^2 + \partial_{x_2}^2 - \kappa)$. We now solve the linear system $\partial_t w = \mathcal{L}w$ by complexification and Fourier transform. Letting $w = e^{\mathrm{i}\sigma\cdot x}W$ with constant vector $W \in \mathbb{C}^2$ we obtain, for each $\sigma \in \mathbb{R}^2$, the algebraic eigenvalue problem

$$M(\sigma)W = \lambda W, \quad \text{where } M(\sigma) = M(\varepsilon,\kappa,\sigma) = \begin{pmatrix} \rho + c & \mathrm{i}\nu \\ -\mathrm{i}\nu & \rho \end{pmatrix}, \tag{28}$$

$$\rho = -(\kappa + k^2\sigma_1^2 + \sigma_2^2)^2 - 4k^4\sigma_1^2 + \kappa^2,$$
$$\nu = -4k^2\sigma_1(\kappa + k^2\sigma_1^2 + \sigma_2^2), \text{ and } c = -2(\varepsilon^2 - \kappa^2).$$

The matrix $M(\sigma)$ is hermitian and the matrix–entries of $M(\sigma)$ depend analytically on $\sigma \in \mathbb{R}^2$. Thus, for $\varepsilon^2 > \kappa^2$, we obtain two analytical surfaces of real eigenvalues $\sigma \mapsto \lambda_{1,2}(\sigma)$ with $\lambda_1(\sigma) > \lambda_2(\sigma)$ for all σ, explicitly given by

$$\lambda_{1,2}(\sigma) = \rho(\kappa,\sigma) + \frac{c(\varepsilon,\kappa)}{2} \pm \sqrt{\frac{c(\varepsilon,\kappa)^2}{4} + \nu(\kappa,\sigma)^2}.$$

By $W^1(\sigma) \in \mathbb{C}^2$ and $W^2(\sigma) \in \mathbb{C}^2$ we denote the associated orthonormal eigenvectors. We have $\lambda_j(\sigma) = \lambda_j((-\sigma_1,\sigma_2)) = \lambda_j((\sigma_1,-\sigma_2)), j = 1,2$, and choose $W^j(\sigma)$ in a smooth way with $W^j(\sigma) = \overline{W^j(-\sigma_1,\sigma_2)} = W^j((\sigma_1,-\sigma_2))$. Since

$$M(0) = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$$

we have $\ker M(0) = \mathrm{span}\{(0,1)\}$, and the above convention gives

$$W^1(0) = (0,1). \tag{29}$$

Expanding λ_1 around $\sigma_0 = 0$ we obtain

$$\lambda_1(\sigma) = -c_1(\varepsilon,\kappa)\sigma_1^2 - c_2(\kappa)\sigma_2^2 + \mathcal{O}(|\sigma|^4), \text{ where} \tag{30}$$

$$c_1(\varepsilon,\kappa) = 2(1+\kappa)\left(2 + 3\kappa - 4\frac{\kappa^2(1+\kappa)}{\varepsilon^2 - \kappa^2}\right) \text{ and } c_2(\kappa) = 2\kappa.$$

Thus, the following conditions are necessary for the spectral stability of $A_{\varepsilon,\kappa}$,

$$c_1(\varepsilon,\kappa) \geq 0 \Leftrightarrow \varepsilon^2 \geq E_E^C(\kappa) = \kappa^2 \frac{6+7\kappa}{2+3\kappa} = 3\kappa^2 - \kappa^3 + \frac{3}{2}\kappa^4 + \mathcal{O}(\kappa^5), \qquad (31)$$

$$c_2(\kappa) \geq 0 \Leftrightarrow \kappa \geq 0. \qquad (32)$$

The curves $\varepsilon^2 = E_E^C(\kappa)$ and $\kappa = 0$ are called Eckhaus– respectively zigzag–instability curves.

The important point from [Mie97a] is, that although the $\mathcal{O}(|\sigma|^4)$ term in (30) contains positive fourth order terms, there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the conditions (31), (32) are also sufficient for the spectral stability of $A_{\varepsilon,\kappa}$. Hence $\widetilde{\mathcal{P}} = \{(\varepsilon,\kappa) : 0 < \varepsilon \leq \varepsilon_0,\ \kappa \geq 0,\ \varepsilon^2 \geq E_E^C(\kappa)\}$, is the parameter region of spectrally stable rolls.

This region is shown in figure 1a). The additional line $\kappa = \beta\varepsilon$ will be explained below, where we describe the behavior of the surface $(\sigma \mapsto \lambda_1(\sigma))$ in dependence of $(\varepsilon,\kappa) \in \widetilde{\mathcal{P}}$. To apply the diffusive stability method we need to split the domain $\mathbb{R}^2$ of wave vectors into two sets. In the first one, the set of center wave vectors, σ and $\lambda_1(\sigma)$ are close to zero, and the quadratic terms $-c_1(\varepsilon,\kappa)\sigma_1^2 - c_2(\kappa)\sigma_2^2$ in the expansion (30) dominate the shape of λ_1. Its complement is the second set, called the set of stable wave vectors, where $\lambda_1(\sigma)$ is below some (ε,κ)–dependent negative bound.

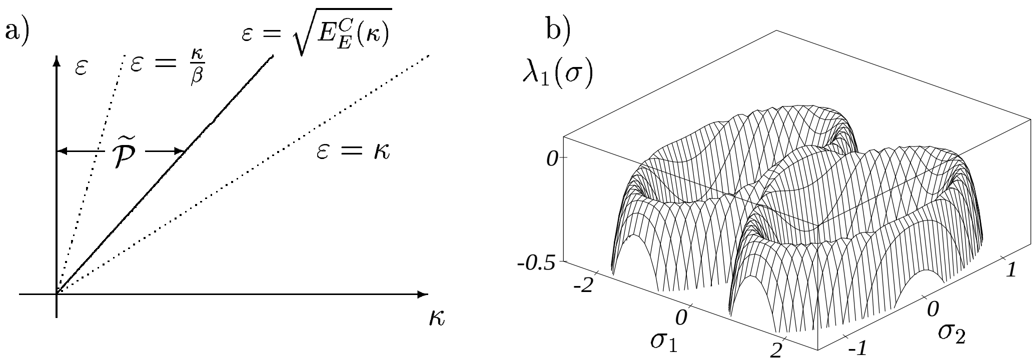

Figure 1: a) Stability of rolls in parameter–region, b) Plot of λ_1 for $(\varepsilon,\kappa) \in \widetilde{\mathcal{P}}$

To motivate our discussion we look at figure 1b), showing a global picture of $\lambda_1(\sigma)$ for some $(\varepsilon,\kappa) \in \widetilde{\mathcal{P}}$. We see that $\lambda_1(\sigma)$ is close to zero near two dangerous annuli $D_\pm$, which are centered at $\pm\sigma_1^* \approx 1$ and meet each other at $\sigma = 0$. These annuli are inherited from the case $(\varepsilon,\kappa) = (\kappa,\kappa)$ where λ_1 is given as

$$\lambda_1(\kappa,\kappa,\sigma) = \max\{-(1-k^2(\sigma_1-1)^2-\sigma_2^2)^2+\kappa^2,\ -(1-k^2(\sigma_1+1)^2-\sigma_2^2)^2+\kappa^2\},$$

and $\lambda_2(\kappa,\kappa,\sigma)$ is given by replacing max with min. Due to the symmetry of $\lambda_{1,2}$, we now concentrate on positive wave vectors, i.e. on $\sigma_1, \sigma_2 \geq 0$. We define the dangerous set $\mathcal{S}_0 = \{\sigma \in \mathbb{R}^2_+ : (\sigma_1 - 1)^2 + \sigma^2 = 1\}$. The matrix $M(\varepsilon, \kappa, \sigma)$ is a small perturbation of $M^\kappa(\sigma) = M(\kappa, \kappa, \sigma)$, i.e. $\|M(\varepsilon, \kappa, \sigma) - M^\kappa(\sigma)\| = \mathcal{O}(\varepsilon^2 - \kappa^2)$. Defining $\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} |a - b|$ we find $\lambda_1(\varepsilon, \kappa, \sigma) \leq -(\delta^2/2 - r) \leq -(\delta^2/2 - 2\varepsilon^2)$ for $\operatorname{dist}(\sigma, S_0) \geq \delta$ with $\delta^2 \geq 2\kappa^2$. In particular, choosing $\delta = 3\varepsilon$ we find $\lambda_1(\varepsilon, \kappa, \sigma) \leq -\varepsilon^2$ for $\operatorname{dist}(\sigma, S_0) \geq 3\varepsilon$.

It remains to study $\lambda_1(\varepsilon, \kappa, \sigma)$ for $\operatorname{dist}(\sigma, S_0) \leq \delta$. Therefore we let

$$C_1 = \{\sigma \in [2\delta, 3] \times [\sqrt{\delta}, 2] : \operatorname{dist}(\sigma, S_0) \leq \delta\},$$
$$C_2 = \{\sigma \in [0, 2\delta] \times [0, \sqrt{\delta}] : \operatorname{dist}(\sigma, S_0) \leq \delta\}.$$

In C_1, only $|\lambda_1(\varepsilon, \kappa, \sigma)|$ is small. By Liapunov–Schmidt reduction of the eigenvalue problem $(M(\sigma) - \lambda(\sigma)\mathrm{Id})W = 0$ one finds $\lambda_1(\varepsilon, \kappa, \sigma) \leq -\varepsilon^2/2$ in C_1, provided that $\varepsilon^2 \geq 3\kappa^2$.

In C_2, both eigenvalues are small. Writing the characteristic equation for $\lambda_{1,2}$ as $\lambda^2 + n_1\lambda + n_0 = 0$ with $n_1 = -2\rho - c$ and $n_0 = \det M(\varepsilon, \kappa, \sigma) = \rho(\rho + c) - \nu^2$ we have

$$\lambda_{1,2}(\sigma) \leq 0 \quad \Leftrightarrow \quad n_0 \geq 0 \quad \text{and} \quad n_1 \geq 0.$$

Since $M(\sigma) = M^H(\sigma)$ we have $n_1^2 \geq 4n_0$. Moreover, $n_1(\varepsilon, \kappa, 0) = \varepsilon^2 - \kappa^2 > 0$, and since n_1 can only change sign if $n_0 < 0$, it is sufficient to study the condition $n_0 \geq 0$. We have

$$n_0 = \sum_{k+l\leq 4} \mu_{kl}\sigma_1^{2l}\sigma_2^{2k},$$

with $\mu_{10} = 8(\varepsilon^2 - 3\kappa^2) + 4\kappa(5\varepsilon^2 - 13\kappa^2) - 28\kappa^4$ and $\mu_{01} = 4\kappa(\varepsilon^2 - \kappa^2)$. The conditions $\mu_{10} \geq 0$ and $\mu_{01} \geq 0$ yield again the necessary conditions (31) and (32). Moreover, for $(\varepsilon, \kappa) \in \widetilde{\mathcal{P}}$ we have

$$n_0(\varepsilon, \kappa, \sigma) \geq \mu_{10}\sigma_1^2 + \mu_{01}\sigma_2^2 + \frac{\varepsilon^2 - \kappa^2}{3}|\sigma|^4 \text{ for all } \sigma \in C_2.$$

From this we obtain $\lambda_1 < 0$ in C_2, and from $n_0 = \det M(\varepsilon, \kappa, \sigma) = \lambda_1\lambda_2$ we find $\lambda_1 \leq -(\mu_{10}\sigma_1^2 + \mu_{01}\sigma_2^2)/|\lambda_2|$ in C_2.

We now explain the line $\kappa = \beta\varepsilon$, $0 < \beta < 1/\sqrt{3}$ in figure 1a). Choosing $\kappa = \beta\varepsilon$ we obtain $c_2 = 2\kappa = 2\beta\varepsilon \to 0$ for $\varepsilon \to 0$, but

$$c_1(\varepsilon, \beta\varepsilon) = 2(2 + 3\beta\varepsilon - \frac{4\beta^2}{1 - \beta^2}(1 + \beta\varepsilon)) = c_1^* + \mathcal{O}(\varepsilon), \; c_1^* = 4 - \frac{8\beta^2}{1 - \beta^2} > 0.$$

Analogously, $\mu_{10}(\varepsilon, \beta\varepsilon) = 8\varepsilon^2(1 - 3\beta^2) + \mathcal{O}(\varepsilon^3)$ but $\mu_{01}(\varepsilon, \beta\varepsilon) = 4\varepsilon^3\beta(1 - \beta^2)$. The dependence on (ε, κ) of the coefficients of the quadratic terms of $\lambda_1(\sigma)$ near $\sigma = 0$ is crucial in order to estimate the size of the attracted neigborhood of a spectrally stable roll. From now on we stick to the situation that $\kappa = \beta\varepsilon$, with $0 < \beta < 1/\sqrt{3}$. By this we can formulate our results in an easier way. For the linear problem we summarize them in the following lemma.

Lemma 5 *There exist continuous functions $a_j : (0, 1/\sqrt{3}) \to \mathbb{R}_+$, $j = 0, 1, 2$ and an $\varepsilon_0 > 0$ such that for all $\beta \in (0, 1/\sqrt{3})$ and for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let $\kappa = \beta\varepsilon$, then*

$$\begin{array}{lll} \lambda_1(\sigma) & = -c_1\sigma_1^2 - c_2\sigma_2^2 + \mathcal{O}(|\sigma|^4) & \\ & \leq -a_0(\beta)\sigma_1^2 - \varepsilon a_0(\beta)\sigma_2^2 & \textit{for } \sigma_1^2 + \varepsilon\sigma_2^2 \leq a_2(\beta)\varepsilon^2/2, \\ \lambda_1(\sigma) & \leq -2a_1(\beta)\varepsilon^2 & \textit{for } \sigma_1^2 + \varepsilon\sigma_2^2 \geq a_2(\beta)\varepsilon^2/4, \\ \lambda_2(\sigma) & \leq -2a_1(\beta)\varepsilon^2 & \textit{for all } \sigma \in \mathbb{R}^2. \end{array}$$

Remark 6 To apply the diffusive stability method for a fixed spectrally stable roll it is enough, that for given ε, κ the eigenvalue $\lambda_1(\sigma)$ has the quadratic expansion (30) around $\sigma = 0$, and that $\lambda_1 \leq -\gamma < 0$ for $|\sigma| \geq \delta > 0$ and $\lambda_2 \leq -\gamma$ for all σ. In particular, also in the region $E_E^C(\kappa) < \varepsilon^2 < 3\kappa^2$ the diffusive stability (12) holds. However, it becomes much more complicated to give estimates for the size of the domain of attraction in this region. In the situation $\kappa = \beta\varepsilon$ of Lemma 5 the size of the domain of attraction will depend on a_0 and a_1. We have the asymptotics

$$a_0(\beta), a_1(\beta) \to 0 \text{ for } \beta \to 0 \text{ or } \beta \to 1/\sqrt{3}. \tag{33}$$

Therefore our result will be worse for β close to 0 or β close to $1/\sqrt{3}$, i.e. for (ε, κ) close to the zigzag or the Eckhaus stability boundaries of $\widetilde{\mathcal{P}}$, see also Remark 8. The factors 1/2, 1/4 and 2 are arbitrary. They are needed later in the construction of the mode–filters and of the diffusive and stable semigroups, see in particular figure 2.

4 Nonlinear stability of the rolls $A_{\varepsilon,\kappa}$ in the cSHE

To prove (12) we formulate a renormalization process for (27) in Fourier space and apply Theorem 1. We continue to use coordinates $x = (\xi, x_2)$ with $\xi = kx_1$.

4.1 Refined scaling operators

In order to control the dependence of the constants in Theorem 1 on (ε, κ) we start by rescaling $\mathcal{F}w$ to obtain an ε–independent size of the set of center wave vectors. Therefore we introduce refined scaling operators $\mathcal{R}_{(L_1,L_2)} : H^m(k) \to H^m(k)$, $L_1, L_2 > 0$, $(\mathcal{R}_{(L_1,L_2)}u)(x) = u((L_1\xi, L_2x_2))$. Clearly we have $(\mathcal{R}_{L_1,L_2})^{-1} = \mathcal{R}_{(1/L_1,1/L_2)}$. Associated to the operator $\mathcal{R}_{(L_1,L_2)}$ in x–space, in Fourier space we have the operator $L_1^{-1}L_2^{-1}\ \mathcal{R}_{(1/L_1,1/L_2)} : H^k(m) \to H^k(m)$, due to the rule

$$\mathcal{F}(\mathcal{R}_{(L_1,L_2)}u) = L_1^{-1}L_2^{-1}(\mathcal{R}_{(1/L_1,1/L_2)}\mathcal{F}u). \tag{34}$$

In particular we will encounter the following two cases: first, for $L_1 = L_2 = L$ we write $\mathcal{R}_{1/L}$ for $\mathcal{R}_{(1/L,1/L)}$; second, for $L_2 = \sqrt{L_1}$ we write $\mathcal{R}^{1/L_1}$ for $\mathcal{R}_{(1/L_1,1/\sqrt{L_1})}$. This second case will occur for $L_1 = 1/\varepsilon$, due to the elliptic shape of the set of center wave vectors. To be explicit, as particular cases of $\mathcal{R}_{(1/L_1,1/L_2)}$ we define

$$(\mathcal{R}_{1/L}\hat{u})(\sigma) = \hat{u}((\sigma_1/L, \sigma_2/L)), \text{ and } (\mathcal{R}^{\varepsilon}\hat{u})(\sigma) = \hat{u}((\varepsilon\sigma_1, \sqrt{\varepsilon}\sigma_2)).$$

Likewise we define $\mathcal{R}^{\varepsilon}_{1/L} = \mathcal{R}_{(\varepsilon/L,\sqrt{\varepsilon}/L)}$. Straightforward calculations give

$$\mathcal{R}_{(1/L_1,1/L_2)}\hat{u} * \mathcal{R}_{(1/L_1,1/L_2)}\hat{v} = L_1 L_2 \mathcal{R}_{(1/L_1,1/L_2)}(\hat{u} * \hat{v}), \tag{35}$$

and, for $\alpha \in \mathbb{N}_0^2$,

$$\|\partial^{\alpha}(\mathcal{R}_{(1/L_1,1/L_2)}\hat{u})\|_{H^0(m)} \leq C\frac{\max\{L_1, L_2\}^m}{\min\{L_1, L_2\}^{|\alpha|}}\sqrt{L_1 L_2}\|\partial^{\alpha}\hat{u}\|_{H^0(m)}.$$

This in particular gives $\mathcal{R}_{1/L}\hat{u} * \mathcal{R}_{1/L}\hat{v} = L^2\mathcal{R}_{1/L}(\hat{u} * \hat{v})$ for $L > 0$ and

$$\|\mathcal{R}_{1/L}\hat{u}\|_{H^3(2)} \leq CL^3\|\hat{u}\|_{H^3(2)} \quad \text{for } L \geq 1. \tag{36}$$

4.2 The result

We assume the initial conditions for (26) to be given at time $1/\varepsilon^2$. We mostly drop the arguments ε and $\kappa = \beta\varepsilon$ of c_1 and c_2, and also the argument β of a_0, a_1 and a_2. By $\mathcal{P} = (0, 1/\sqrt{3}) \times (0, \varepsilon_0)$ we denote the set of parameters (β, ε) of spectrally stable rolls in Lemma 5.

Theorem 7 *There exist continuous functions $\delta, C : (0, 1/\sqrt{3}) \to \mathbb{R}_+$ and a continuous function $\mathcal{A} : \mathcal{P} \times H^3(2) \to \mathbb{R}$ such that for all $(\beta, \varepsilon) \in \mathcal{P}$ the following holds. Let $\kappa = \beta\varepsilon$, and let $B(t, x)$ be the solution of (26) with the initial condition $B(1/\varepsilon^2, \cdot) = B_0(\cdot) \in H^2(3)$ satisfying $\|\mathcal{R}^{\varepsilon}\mathcal{F}B_0\|_{H^3(2)} \leq \delta(\beta)$. Then we have*

$$\|B(t, x) - \frac{\mathrm{i}\mathcal{A}(\beta, \varepsilon, B_0)}{\sqrt{c_1 c_2}\; 4\pi t} e^{-\frac{\xi^2}{4c_1 t} - \frac{x_2^2}{4c_2 t}}\|_{L^{\infty}(\mathbb{R}^2)} \leq C(\beta)\varepsilon^{-3/2}t^{-3/2} \text{ for } t \to \infty. \tag{37}$$

Remark 8 In order to understand the condition $\|\mathcal{R}^{\varepsilon}\mathcal{F}B_0\|_{H^3(2)} \leq \delta(\beta)$ let $\mathcal{R}^{\varepsilon}\hat{B}_0 = \hat{H}$. Then $\|\hat{H}\|_{H^3(2)} \leq \delta(\beta)$ means that $\hat{B}_0 = \mathcal{R}^{1/\varepsilon}\hat{H}$ has to be concentrated in Fourier space over an ellipse, with the length of the σ_1–axis given by $\mathcal{O}(\varepsilon)$ and the length of the σ_2–axis given by $\mathcal{O}(\sqrt{\varepsilon})$. Using (34) we obtain

$$B_0 = \mathcal{F}^{-1}\mathcal{R}^{1/\varepsilon}\hat{H} = \varepsilon^{3/2}\mathcal{R}^{\varepsilon}H$$

by inverse Fourier transform. In other words, (37) holds for all initial conditions B_0 which can be written as $B_0(x) = \varepsilon^{3/2}H(\varepsilon\xi, \sqrt{\varepsilon}x_2)$ with $\|H\|_{H^2(3)} \leq$

$\delta(\beta)$. This in particular means, that the domain of attraction is of diameter $\mathcal{O}(\varepsilon^{3/2})$ in $L^\infty(\mathbb{R}^2)$. For the functions $\mathcal{A}, \delta, C$ we have $|\mathcal{A}(\beta, \varepsilon, B_0)| \leq C(\beta) \|\mathcal{R}^\varepsilon \mathcal{F} B_0\|_{H^3(2)}$ with $C(\beta)$ independent of ε, and

$$\delta(\beta) \to 0 \text{ and } C(\beta) \to \infty \quad \text{for} \quad \beta \to 0 \text{ or } \beta \to 1/\sqrt{3}. \tag{38}$$

The asymptotics (38) for $\delta(\beta)$ and $C(\beta)$ follow from $c_1(\varepsilon, \beta\varepsilon) \to 0$ for $\beta \to 1/\sqrt{3}$ and $\varepsilon \to 0$, respectively $c_2(\beta\varepsilon) \to 0$ for $\beta\varepsilon \to 0$. In both cases we have $a_0(\beta) \to 0$, and to check the assumptions in Theorem 1 we need estimates based on $\|e^{-a_0|\sigma|^2}\|_{L^2(\mathbb{R}^2)} \leq C(a_0) = \sqrt{2}\, a_0^{-1/2}$.

4.3 Proof of Theorem 7

In the following C denotes a generic constant, independent of β, ε, L and n, while $C(\beta)$ denotes a generic constant which depends on β. With u^{*p} denoting the p–times convolution, the Fourier transform of (27) gives

$$\partial_t \hat{w}(t,\sigma) = M(\sigma)\hat{w}(t,\sigma) + N(\hat{w})(t,\sigma), \text{ where} \tag{39}$$

$$N(\hat{w}) = -r \begin{pmatrix} 3\hat{w}_1^{*2} + \hat{w}_2^{*2} \\ 2\hat{w}_1 * \hat{w}_2 \end{pmatrix} - (\hat{w}_1^{*2} + \hat{w}_2^{*2}) * \begin{pmatrix} \hat{w}_1 \\ \hat{w}_2 \end{pmatrix}.$$

For the one-dimensional SHE one expects from the Ginzburg–Landau formalism [vH91, Sch94] and from the diffusive stability result for the associated solutions of the Ginzburg–Landau equation [BK92] that the attracted neighborhood of the rolls is of diameter $\mathcal{O}(\varepsilon)$ in $L^\infty(\mathbb{R}^1)$. Yet the above approach does not allow for a proof. Instead in [Sch96] before starting the renormalization process an appropriate scaling in Bloch space is done, which gives an ε–independent size of the set of center–modes. For the real or complex 2d-problem no Ginzburg–Landau formalism is available. But to estimate the size of the attracted neighborhood of $A_{\varepsilon,\kappa}$ we can proceed similar to [Sch96] and rescale $\hat{w}$ by $\mathcal{R}^\varepsilon$, i.e. we introduce $\widetilde{w} = \mathcal{R}^\varepsilon \hat{w}$. Then (39) turns into

$$\partial_t \widetilde{w} = (\mathcal{R}^\varepsilon M)\widetilde{w} + \mathcal{R}^\varepsilon N_3(\mathcal{R}^{1/\varepsilon}\widetilde{w}). \tag{40}$$

This in particular gives

$$(\mathcal{R}^\varepsilon \lambda_1)(\sigma) = -c_1\varepsilon^2\sigma_1^2 - (c_2/\varepsilon)\varepsilon^2\sigma_2^2 + \mathcal{O}(|(\varepsilon\sigma_1, \sqrt{\varepsilon}\sigma_2)|^4).$$

Hence, remembering that $c_1 = c^* + \mathcal{O}(\varepsilon)$ and $c_2 = 2\beta\varepsilon$, we see that the quadratic terms in σ_1 and σ_2 in $(\mathcal{R}^\varepsilon \lambda_1)$ scale identically in ε. The factor ε^2 will vanish upon rescaling in time.

In order to decompose $\widetilde{w}$ into a diffusive and an exponentially stable part we employ so called mode filters, which we define via multipliers in the wave vector σ. Here we need the following simple lemma.

Lemma 9 *Let $\widetilde{M} \in C_b^3(\mathbb{R}^2, \mathrm{Lin}(\mathbb{C}^2, \mathbb{C}^2))$. Then $\hat{w} \mapsto (\mathcal{R}_{(1/L_1,1/L_2)}\widetilde{M})\hat{w}$ defines a linear operator $\mathcal{R}_{(1/L_1,1/L_2)}M : [H^3(2)]^2 \to [H^3(2)]^2$. There exists a $C > 0$ such that for all $L_1, L_2 \geq 1$ we have*

$$\|(\mathcal{R}_{(1/L_1,1/L_2)}M)\hat{w}\|_{H^3(2)} \leq C\|\hat{w}\|_{H^3(2)}.$$

Proof. This follows directly from the definition of $H^3(2)$, since $(\mathcal{R}_{(1/L_1,1/L_2)}\widetilde{M})$ is uniformly bounded for all $L_1, L_2 \geq 1$. □

Next we define the family of projections

$$P_1(\sigma) : \mathbb{C}^2 \to \mathrm{span}(W^1(\sigma)), \quad P_1(\sigma)V = \langle V, W^1(\sigma)\rangle W^1(\sigma),$$

where $\langle V, W\rangle = V \cdot \overline{W}$. We choose a cut–off function $\chi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ with

$$\chi(\sigma) = \begin{cases} 1 & \text{for} \quad 0 \leq |\sigma| \leq a_2/4 \\ \in [0,1] & \text{for} \quad a_2/2 \leq |\sigma| \leq a_2/2 \\ 0 & \text{for} \quad a_2/2 \leq |\sigma| \end{cases}.$$

and define $\chi^{1/\varepsilon} = (\mathcal{R}^{1/\varepsilon}\chi)$. Because the eigenspace associated to $\lambda_1(\sigma)$ smoothly depends on σ, for every fixed ε the function $\widetilde{E}^c : \sigma \mapsto \chi^{1/\varepsilon}(\sigma)P_1(\sigma)$ is in $C_b^3(\mathbb{R}^2, \mathrm{Lin}(\mathbb{C}^2, \mathbb{C}^2))$. By Lemma 9 $\widetilde{E}^c$ defines the so–called central mode filter $E^c : [H^3(2)]^2 \to [H^3(2)]^2$, $(E^c\widetilde{w})(\sigma) = \chi^{1/\varepsilon}(\sigma)P_1(\sigma)\widetilde{w}(\sigma)$. We define the stable mode filter $E^s = \mathrm{Id} - E^c$. Since E^c and E^s are not projections it is useful to introduce auxiliary mode filters E^{ch} and E^{sh} defined via the multipliers $\widetilde{E}^{ch} : \sigma \mapsto \chi^{1/\varepsilon}(\sigma/2)P_1(\sigma)$ and $\widetilde{E}^{sh} : \mathrm{Id} - \chi^{1/\varepsilon}(2\sigma)P_1(\sigma)$. We then have $E^{ch}E^c = E^c$ and $E^{sh}E^s = E^s$. Finally there exist $C, \varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $\|\mathcal{R}^\varepsilon\widetilde{E}^*\|_{C_b^3(\mathbb{R}^2,\mathrm{Lin}(\mathbb{C}^2,\mathbb{C}^2))} < C$, where $\widetilde{E}^*$ denotes $\widetilde{E}^c, \widetilde{E}^{ch}, \widetilde{E}^s$ or $\widetilde{E}^{sh}$. Thus by Lemma 9 the following holds.

Lemma 10 *There exist $C, \varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $L \geq 1/\varepsilon$ we have $\|(\mathcal{R}^{1/L}E^*)\widetilde{w}\|_{H^3(2)} \leq C\|\widetilde{w}\|_{H^3(2)}$ for E^* equal to E^c, E^{ch}, E^s or E^{sh}.*

Now we define the center (diffusive) part $\widetilde{w}_c \in X$ and the stable (linearly exponentially damped) part $\widetilde{w}_s \in Y$ to be the solutions of

$$\begin{aligned} \partial_t\widetilde{w}_c &= \Lambda^c\widetilde{w}_c + N^c(\widetilde{w}_c, \widetilde{w}_s), \quad \widetilde{w}_c(1/\varepsilon^2) = (\mathcal{R}^\varepsilon E^c)\widetilde{w}(1/\varepsilon^2), \\ \partial_t\widetilde{w}_s &= \Lambda^s\widetilde{w}_s + N^s(\widetilde{w}_c, \widetilde{w}_s), \quad \widetilde{w}_s(1/\varepsilon^2) = (\mathcal{R}^\varepsilon E^s)\widetilde{w}(1/\varepsilon^2), \end{aligned} \tag{41}$$

where $N^*(\widetilde{w}_c, \widetilde{w}_s) = \mathcal{R}^\varepsilon E^* N(E^{ch}\mathcal{R}^{1/\varepsilon}\widetilde{w}_c + E^{sh}\mathcal{R}^{1/\varepsilon}\widetilde{w}_s))$ for $*$ equal c or s, and Λ^c and Λ^s are defined by

$$\begin{aligned} (\Lambda^c\widetilde{w})(\sigma) &= \begin{cases} (\mathcal{R}^\varepsilon\lambda_1 P_1)(\sigma)\widetilde{w}(\sigma) & \text{for } |\sigma| \leq a_2/2 \\ (\mathcal{R}^\varepsilon\lambda_1^c P_1)(\sigma)\widetilde{w}(\sigma) & \text{for } |\sigma| \geq a_2/2 \end{cases}, \\ (\Lambda^s\widetilde{w})(\sigma) &= \begin{cases} (\mathcal{R}^\varepsilon(M - \lambda_1 P_1 + \lambda_1^s P_1))(\sigma)\widetilde{w}(\sigma) & \text{for } |\sigma| \leq a_2/4 \\ (\mathcal{R}^\varepsilon M)(\sigma)\widetilde{w}(\sigma) & \text{for } |\sigma| \geq a_2/4 \end{cases}. \end{aligned} \tag{42}$$

Here $\lambda_1^c \in C^\infty(\mathbb{R}^2, \mathbb{R})$ is a smooth continuation of λ_1 for $\sigma_1^2 + \varepsilon\sigma_2^2 \geq \varepsilon^2 a_2/2$ such that $\lambda_1^c(\sigma) \leq -a_0\sigma_1^2 - \varepsilon a_0 \sigma_2^2$, and λ_1^s is a smooth continuation of λ_1 for $\sigma_1^2 + \varepsilon\sigma_2^2 \leq \varepsilon^2 a_2/4$ such $\lambda_1^s(\sigma) \leq -a_1\varepsilon^2$, see figure 2. Due to the construction we have $(\mathcal{R}^\varepsilon E^{ch})\widetilde{w}_c(t) = \widetilde{w}_c(t)$ and $(\mathcal{R}^\varepsilon E^{sh})\widetilde{w}_s(t) = \widetilde{w}_s(t)$ for all $t \geq 1/\varepsilon^2$. Therefore, if $(\widetilde{w}_c, \widetilde{w}_s)$ solves (41) then $\widetilde{w} = \widetilde{w}_c + \widetilde{w}_s$ solves (40).

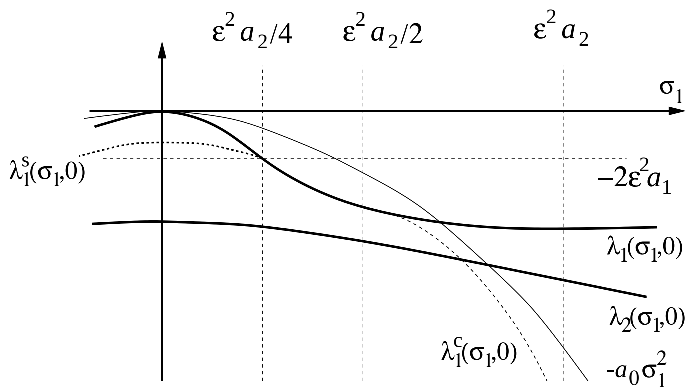

Figure 2: Construction of Λ^c and Λ^s, partial functions $\lambda_1(\sigma_1, 0)$, $\lambda_1^c(\sigma_1, 0)$ etc.

For $n \in \mathbb{N} \cup \{0\}$ and $L > 1$ we now introduce the variables

$$u_n(T, \Sigma) := \widetilde{w}_c(L^{2n}T/\varepsilon^2, \Sigma/L^n), \quad v_n(T, \Sigma) := L^n \widetilde{w}_s(L^{2n}T/\varepsilon^2, \Sigma/L^n), \tag{43}$$

Note that $u_0(1) + v_0(1) = \mathcal{R}^\varepsilon \hat{w}(1/\varepsilon^2) = \mathcal{O}(1)_{\varepsilon\to 0}$ in $[H^3(2)]^2$.

Remark 11 a) It would be more consistent to denote u_n by $\widehat{u}_n$, since u_n is the Fourier transform of the variable $\widetilde{u}_n(T) = L^{2n}\mathcal{R}_{L^n}\mathcal{F}^{-1}\widetilde{w}^c(L^{2n}T/\varepsilon^2)$, living in x–space. The same holds for v_n. However, for the rest of the section we work completely in Fourier space and therefore omit the $\widehat{\ }$.

b) The reason for the scalings (43) is that we expect the diffusive part $\widetilde{w}_c$ to decay temporally like $t^{-1} = L^{-2n}$, i.e. like solutions of the nonlinear diffusion equation in two dimensions with irrelevant nonlinearities. Therefore, because of quadratic interactions, the linearly exponentially damped part $\widetilde{w}_s$ should decay like L^{-4n}. Then $v_n = L^n\mathcal{R}_{1/L^n}\widetilde{w}_s$ still decays like L^{-3n}. The advantage of this scaling of $\widetilde{w}_s$ is, that in front of the mixed convolutions of $\widetilde{w}_c = \mathcal{R}_{L^n}u_n$ and $\widetilde{w}_s = L^{-n}\mathcal{R}_{L^n}v_n$ we get an additional factor L^{-n}.

c) We do not need normal form transformations similar to those in [Sch96, Sch97]. There the analogue of $\widetilde{w}_c$ measures the distance of the diffusive part to the quadratic approximation of the (formal) center–manifold. This removes the quadratic terms from the nonlinearity for the diffusive part. Here we do not need these transforms because of the faster decay of semigroup for the diffusive part. In terms of the renormalization process in Fourier space this better behavior in two dimensions is expressed by the factor L^{-2n} in $\mathcal{R}_{L^n}u*\mathcal{R}_{L^n}v = L^{-2n}\mathcal{R}_{L^n}(u*v)$, see (35), instead of L^{-n} in one space dimension.

Using the scalings (43) we write the variation of constant formula for (41) as

$$\begin{aligned} u_n(T) &= e^{\Lambda_n^c(T-1/L^2)}\mathcal{R}_{1/L}u_n(1/L^2) + \int_{1/L^2}^T e^{\Lambda_n^c(T-\tau)}N_n^c(u_n(\tau),v_n(\tau))\,\mathrm{d}\tau, \\ v_n(T) &= e^{\Lambda_n^s(T-1/L^{2n})}\mathcal{R}_{1/L}v_n(1/L^2) + \int_{1/L^2}^T e^{\Lambda_n^s(T-\tau)}N_n^s(u_n(\tau),v_n(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{44}$$

where $T \in [1/L^2]$, $n \in \mathbb{N}$, and

$$\begin{aligned} \Lambda_n^c &= L^{2n}\varepsilon^{-2}(\mathcal{R}_{1/L^n}\Lambda^c), \quad N_n^c(u_n,v_n) = L^{2n}\varepsilon^{-2}\mathcal{R}_{1/L^n}N_c(\mathcal{R}_{L^n}u_n, L^{-n}\mathcal{R}_{L^n}v_n), \\ \Lambda_n^s &= L^{2n}\varepsilon^{-2}(\mathcal{R}_{1/L^n}\Lambda^s), \quad N_n^s(u_n,v_n) = L^{3n}\varepsilon^{-2}\mathcal{R}_{1/L^n}N_s(\mathcal{R}_{L^n}u_n, L^{-n}\mathcal{R}_{L^n}v_n). \end{aligned}$$

The system (44) is in the form (21) with $H_n^c = H_n^s = \mathcal{R}_{1/L}$, $X_n = Y_n = [H^3(2)]^2$ and $\widetilde{X}_n = [H^3(1)]^2$, independent of n. From (36) we immediately see that assumption A3) on the operators H_n^c, H_n^s holds with $m_c = m_s = 3$. We define

$$\begin{aligned} &\Pi_n = \Pi \ \forall n \in \mathbb{N}, \text{ where } \Pi u = \Pi(u_1,u_2) = \mathrm{Re} u_2|_{\Sigma=0}, \\ &\text{and } \psi_n(\Sigma) = e^{-c_1\Sigma_1^2-(c_2/\varepsilon)\Sigma_2^2}(\mathcal{R}^{\varepsilon}_{1/L^n}\chi^{1/\varepsilon}W^1)(\Sigma), \end{aligned} \tag{45}$$

and check the assumptions A1), A2) and A4) of Theorem 1.

In order to check A2) we use the following idea. The formula (35) shows, that higher degree of nonlinearity gives higher powers of L^{-1} upon rescaling with $\mathcal{R}_L$. Similarly, derivatives in x–space, corresponding to vanishing coefficients in Fourier space at $\sigma = 0$ via $\mathcal{F}(\partial_{x_j}u)(\sigma) = \mathrm{i}\sigma_j(\mathcal{F}u)(\sigma)$, give higher powers in L^{-1}, cf. (15). The nonlinearity $F(w)$ does not contain derivatives of w, but for the projection of $N(\hat{w})$ onto the eigenspace associated to $\lambda_1(\sigma)$ we will get a vanishing coefficient at the critical wave vector 0, i.e. a "derivative–like structure". To account for this, we later need the following lemma.

Lemma 12 *Let $\widetilde{K} \in C_b^3(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ and assume $|\widetilde{K}(\sigma,\tau)| \le C(|\sigma|+|\tau|)^\gamma$. For $a, b \in H^3(2)$ define*

$$(\mathcal{R}_{1/L}K)(a,b)(\sigma) = \int_{\mathbb{R}^2}(\mathcal{R}_{1/L}\widetilde{K}(\sigma,\sigma-m))a(\sigma)b(\sigma-m)\,\mathrm{d}m.$$

Then $(\mathcal{R}_{1/L}K)$ is a bilinear mapping from $[H^3(2)]^2$ into $H^3(2)$. There exists a $C > 0$ such that for all $L > 1$ we have

$$\|(\mathcal{R}_{1/L}K)(a,b)\|_{H^3(2-\gamma)} \le CL^{-\min\{\gamma,1\}}\|a\|_{H^3(2)}\|b\|_{H^3(2)}.$$

Proof. The well–definedness and bilinearity of $\mathcal{R}_{1/L}K$ is clear. For brevity we only prove the estimate for $\|(\mathcal{R}_{1/L}K)(a,b)\|_{H^0(2-\gamma)}$. The estimate then holds for the derivatives of $(\mathcal{R}_{1/L}K)(a,b)$, using $\partial_{\sigma_j} f(\sigma/L) = \partial_j f(\sigma/L)/L$, where ∂_j means the derivative with respect to the j-th variable. We have

$$\begin{aligned}
&\|(\mathcal{R}_{1/L}K)(a,b)\|^2_{H^0(2-\gamma)} \\
&\quad = \int \Big(\int K(\frac{\sigma}{L}, \frac{\sigma - m}{L}) a(\sigma) b(\sigma - m)\,\mathrm{d}m \Big)^2 (1+|\sigma|^2)^{2-\gamma}\,\mathrm{d}\sigma \\
&\quad \le C L^{-2\gamma} \int \Big(\int \Big\{ a(\sigma) b(\sigma - m) |\sigma|^{\gamma} (1+|\sigma|^2)^{(2-\gamma)/2} \\
&\qquad\qquad + b(\sigma - m) a(\sigma) |\sigma - m|^{\gamma} (1 + |\sigma - m|^2)^{(2-\gamma)/2} \Big\} \,\mathrm{d}m \Big)^2 \mathrm{d}\sigma \\
&\quad \le C L^{-2\gamma} \Big(\|a\|^2_{H^0(2)} \|b\|^2_{L^1} + \|b\|^2_{H^0(2)} \|a\|^2_{L^1} \Big) \\
&\quad \le C L^{-2\gamma} \|a\|^2_{H^3(2)} \|b\|^2_{H^3(2)},
\end{aligned}$$

where we used $\sup_{\sigma\in\mathbb{R}^2} \frac{|\sigma|^{\gamma}}{(1+|\sigma|^2)^{\gamma/2}} \le 1$, $\|a * (b\,(1+|\cdot|^2))\|_{L^2} \le \|a\|_{L^1}\|b\|_{H^0(2)}$ and $\|b\|_{L^1} \le C\|b\|_{H^3(2)}$. $\square$

Lemma 13 *For all $\beta \in (0, 1/\sqrt{3})$ there exists a constant $C(\beta) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the assumptions A1) and A2) hold with $C = C(\beta)$.*

Proof. Using Lemma 5, A1) follows from $\|e^{\Lambda_n^c T} u_n(\Sigma)\|_{\mathbb{C}^2} \le e^{-a_0|\Sigma|^2 T}\|u_n(\Sigma)\|_{\mathbb{C}^2}$ and $\|e^{\Lambda_n^s T} v_n(\Sigma)\|_{\mathbb{C}^2} \le e^{-a_1 L^{2n} T}\|v_n(\Sigma)\|_{\mathbb{C}^2}$. The well–definedness and continuity statements in A2 are clear. We estimate $\|N_n^s(u_n, v_n)\|_{H^3(2)}$ and $\|N_n^c(u_n, v_n)\|_{H^3(1)}$. The Lipschitz estimates in A2) then follow completely analogous. For notational simplicity we assume $(\mathcal{R}^{\varepsilon}_{1/L^n} E^{ch}) u_n = u_n$ and $(\mathcal{R}^{\varepsilon}_{1/L^n} E^{sh}) v_n = v_n$. Writing $(u_{n,1}, u_{n,2})$ and $(v_{n,1}, v_{n,2})$ for the components of u_n and v_n, and using the formula (35) and $r = \sqrt{\varepsilon^2 - \kappa^2} = \mathcal{O}(\varepsilon)$ we obtain

$$\begin{aligned}
&\|N_n^s(u_n, v_n)\|_{H^3(2)} = L^{3n}\varepsilon^{-2} \|\mathcal{R}^{\varepsilon}_{1/L^n} E^c N(\mathcal{R}^{1/\varepsilon}_{L^n}(u_n + L^{-n} v_n))\|_{H^3(2)} \\
&\quad = r L^{3n}\varepsilon^{-2} \Big\| \ (1 - (\mathcal{R}^{\varepsilon}_{1/L^n} \chi^{1/\varepsilon} P_1)) \mathcal{R}^{\varepsilon}_{1/L^n} \Big\{ \\
&\qquad \begin{pmatrix} 3(\mathcal{R}^{1/\varepsilon}_{L^n}(u_{n,1} + L^{-n} v_{n,1}))^{*2} + (\mathcal{R}^{1/\varepsilon}_{L^n}(u_{n,2} + L^{-n} v_{n,2}))^{*2} \\ 2(\mathcal{R}^{1/\varepsilon}_{L^n}(u_{n,1} + L^{-n} v_{n,1})) * (\mathcal{R}^{1/\varepsilon}_{L^n}(u_{n,2} + L^{-n} v_{n,2})) \end{pmatrix} + \mathcal{O}(3) \Big\} \Big\|_{H^3(2)} \\
&\quad \le C \varepsilon^{1/2} L^n \big(\|u_n\|_{H^3(2)} + L^{-n}\|v_n\|_{H^3(2)} \big)^2,
\end{aligned}$$

where the the $\mathcal{O}(3)$–term stands for the third order convolutions in $N_n^s(u_n, v_n)$.

Estimating N_n^c in $\widetilde{X} = H^3(1)$ we find $\|N_n^c\|_{H^3(1)} = L^{2n}\varepsilon^{-2}\|s_1\|_{H^3(1)} + \mathcal{O}(\varepsilon^{1/2} L^{-n})$, where the second term comes from estimating the mixed convolutions of $\mathcal{R}^{1/\varepsilon}_{L^n} u_n$ with $L^{-n}\mathcal{R}^{1/\varepsilon}_{L^n} v_n$ in $H^3(2)$. The term s_1 contains the quadratic terms in u_n, given by

$$s_1(\Sigma) = -\frac{r\varepsilon^{3/2}}{L^{2n}}(\mathcal{R}^\varepsilon_{1/L^n}\chi^{1/\varepsilon}P_1)(\Sigma)\begin{pmatrix} (3u^{*2}_{n,1}+u^{*2}_{n,2})(\Sigma) \\ (2u_{n,1}*u_{n,2})(\Sigma)\end{pmatrix}$$

$$= -\frac{r\varepsilon^{3/2}}{L^{2n}}\left\langle(\mathcal{R}^\varepsilon_{1/L^n}W^1)(\Sigma), \begin{pmatrix} (3u^{*2}_{n,1}+{u_{n,2}}^{*2})(\Sigma) \\ (2u_{n,1}*u_{n,2})(\Sigma)\end{pmatrix}\right\rangle(\mathcal{R}^\varepsilon_{1/L^n}\chi^{1/\varepsilon}W^1)(\Sigma).$$

Now defining

$$\widetilde{K}_1(\sigma,\sigma-m) = \left\langle W^1(\sigma), \begin{pmatrix} 3W^1_1(\sigma-m)W^1_1(m)+W^1_2(\sigma-m)W^1_2(m) \\ 2W^1_1(\sigma-m)W^1_2(m)\end{pmatrix}\right\rangle,$$

and letting $u_n(T,\Sigma) = a_n(T,\Sigma)\mathcal{R}^\varepsilon_{1/L^n}W^1(\Sigma)$ with $a_n(T,\Sigma)\in\mathbb{C}$ we obtain

$$s_1(T,\Sigma) = -\frac{r\varepsilon^{3/2}}{L^{2n}}(\mathcal{R}^\varepsilon_{1/L^n}K_1)(a_n(T),a_n(T))(\Sigma)\mathcal{R}^\varepsilon_{1/L^n}\chi^{1/\varepsilon}W^1(\Sigma).$$

Here $(\mathcal{R}^\varepsilon_{1/L^n}K_1)$ is defined as in Lemma 12, i.e.

$$(\mathcal{R}^\varepsilon_{1/L^n}K_1)(a,b)(\Sigma) = \int_{\mathbb{R}^2}(\mathcal{R}^\varepsilon_{1/L^n}\widetilde{K}_1(\Sigma,\Sigma-M))a(\Sigma-M)a(M)\,\mathrm{d}M.$$

We have $\widetilde{K}_1\in C^3_b(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{C})$ with $\widetilde{K}_1(0,0)=0$, since $W^1(0)=(0,1)^T$, confirm (29). Hence we gain a factor $C\varepsilon^{1/2}L^{-n}$ in $H^3(1)$ using Lemma 12 with $\gamma=1$. Thus we obtain $\|s_1\|_{H^3(1)}\le C\varepsilon^3L^{-3n}\|u_n*u_n\|_{H^3(2)}$, where again we used $r=\mathcal{O}(\varepsilon)$. This completes the proof of Lemma 13. □

Lemma 14 *For all $\beta\in(0,1/\sqrt{3})$ there exists a constant $C(\beta)>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ the assumption A4) holds with $C=C(\beta)$ and Π_n,ψ_n defined by (45).*

Proof. $\Pi\in\mathrm{Lin}([H^3(2)]^2,\mathbb{R})$ and $\|\psi_n\|_{H^3(2)}\le C(\beta)$ is clear. It remains to show the estimates

a) $\|e^{\Lambda^c_n(1-1/L^2)}\mathcal{R}_{1/L}\psi_{n-1}-\psi_n\|_{H^3(2)}\le C(\beta)L^{-n}$,

b) $\|e^{\Lambda^c_n(1-1/L^2)}\mathcal{R}_{1/L}\theta\|_{H^3(2)}\le C(\beta)L^{-1}\|\theta\|_{H^3(2)}\quad\forall\,\theta\in[H^3(2)]^2$ with $\Pi\theta=0$.

a) We split the set $\{\sigma\in\mathbb{R}^2:\sigma_1^2+\varepsilon\sigma_2^2\le a_2(\beta)\varepsilon^2\}$ of center wave vectors into two sets Q_1 and Q_2, defined by $\lambda_1(\sigma)+c_1\sigma_1^2+c_2\sigma_2^2\le 0$ for $\sigma\in Q_1$ and $\lambda_1(\sigma)+c_1\sigma_1^2+c_2\sigma_2^2\ge 0$ for $\sigma\in Q_2$. The left hand side of a) is given by $(\|*\|^2_{H^3(Q_{1,n},2)}+\|*\|^2_{H^3(Q_{2,n},2)})^{1/2}$, where $*$ is the integrand, and $\|*\|_{H^3(Q_{j,n},2)}$ means integration over $Q_{j,n}:=\{\Sigma\in\mathbb{R}^2:(\varepsilon\Sigma_1/L^n,\sqrt{\varepsilon}\Sigma_2/L^n)\in Q_j\}$, $j=1,2$. Using $L^{2n}\varepsilon^{-2}(\mathcal{R}^\varepsilon_{1/L^n}\lambda_1)(\Sigma)+c_1\Sigma_1^2+(c_2/\varepsilon)\Sigma_2^2\le 0$ for $\Sigma\in Q_{1,n}$ and hence $|\exp\big((L^{2n}\varepsilon^{-2}\mathcal{R}^\varepsilon_{1/L^n}\lambda_1(\Sigma)+c_1\Sigma_1^2+(c_2/\varepsilon)\Sigma_2^2)(1-1/L^2)\big)-1|\le C(\beta)|\Sigma|^4L^{-2n}$

for $\Sigma \in Q_{1,n}$ we obtain $\|e^{\Lambda_n^c(1-1/L^2)}\mathcal{R}_{1/L}\psi_{n-1} - \psi_n\|^2_{H^0(Q_{1,n},2)} \leq C(\beta)L^{-2n}$. For $\| * \|_{H^0(Q_{2,n},2)}$ we obtain the same estimate by factorizing

$$e^{L^{2n}\varepsilon^{-2}\mathcal{R}^\varepsilon_{1/L^n}\lambda_1(\Sigma)(1-1/L^2)}e^{-c_1\Sigma_1^2/L^2-(c_2/\varepsilon)\Sigma_2^2/L^2} - e^{-c_1\Sigma_1^2-(c_2/\varepsilon)\Sigma_2^2}$$

the other way round. Clearly, the estimates also hold with derivatives.

b) Without loss of generality we again assume that $(\mathcal{R}^\varepsilon_{1/L^{n-1}}E^{ch})\theta = \theta$. By the mean value theorem we have $|\theta_i(\Sigma/L)| \leq |\Sigma|L^{-1}\|\theta_i\|_{C^1(\mathbb{R}^2,\mathbb{C})}$ for the components θ_i of θ. This holds since $\theta_1(0) = \alpha W_1^1(0) = 0$ and $\theta_2(0) = \Pi\theta = 0$. Thus

$$\begin{aligned}
&\|e^{\Lambda_n^c(1-1/L^2)}\mathcal{R}_{1/L}\theta\|^2_{H^3(2)} \\
&\leq C\int\Big\{\Big(\sum_{|\alpha|=0}^{3} |\partial_\Sigma^\alpha e^{L^{2n}\varepsilon^{-2}\mathcal{R}^\varepsilon_{1/L^n}\lambda_1^c(\Sigma)}|^2|\Sigma|^2\|\theta\|^2_{C^1(\mathbb{R}^2,\mathbb{C})}L^{-2} \\
&\qquad + \sum_{|\alpha|=0}^{2} |\partial_\Sigma^\alpha e^{L^{2n}\varepsilon^{-2}\mathcal{R}^\varepsilon_{1/L^n}\lambda_1^c(\Sigma)}|^2\|\theta\|^2_{C^1(\mathbb{R}^2,\mathbb{C})}L^{-2} + \ldots + \\
&\qquad + e^{L^{2n}\varepsilon^{-2}\mathcal{R}^\varepsilon_{1/L^n}\lambda_1^c(\Sigma)}\sum_{|\alpha|=3}|\partial_\Sigma^\alpha\theta(\frac{\Sigma}{L})|^2L^{-6}\Big)^2(1+|\Sigma|^2)^2\chi(\frac{\Sigma}{L^n})^2\Big\}\,\mathrm{d}\Sigma \\
&\qquad\qquad \leq C(\beta)L^{-2}\|\theta\|^2_{C^1(\mathbb{R}^2,\mathbb{C})} + C(\beta)L^{-2}\|\theta\|^2_{H^3(2)}.
\end{aligned}$$

The first term in the last line comes from $\|e^{-a_0|\cdot|^2}|\cdot|^2\|_{H^3(2)} \leq C(\beta)$. In the second term we also obtain the factor L^{-2}, because we loose L^{-2} on substituting $\widetilde{\Sigma} = \Sigma/L$ in the integral for the derivatives of θ of order α with $|\alpha| = 2, 3$. Estimate b) now follows from $\|\theta\|_{C^1(\mathbb{R}^2)} \leq C\|\theta\|_{H^3(2)}$. □

Thus, all the assumptions of Theorem 1 are fulfilled, and we conclude that there exist constants $\delta = \delta(\beta) > 0, C = C(\beta) > 0$ and $L_0 > 1$ such that for all $L \in [L_0, L_0^2]$ and for all $z_0(1) = (u_0(1), v_0(1)) \in [H^3(2)]^4$ with $\|z_0(1)\|_{H^3(2)} \leq \delta$ we have $\|u_n(1) - \alpha^*(z_0(1))\psi_n\|_{H^3(2)} \leq CL^{-n}$, $\|v_n(1)\|_{H^3(2)} \leq CL^{-n}$, where $\alpha^* : \mathcal{P}\times[H^3(2)]^4 \to \mathbb{R}$ is continuous and fulfills $|\alpha^*(\beta,\varepsilon,z_0(1))| \leq C(\beta)\|z_0(1)\|_{H^3(2)}$. Let $\mathcal{A}(\beta,\varepsilon,w_1+\mathrm{i}w_2) = \alpha^*(\beta,\varepsilon,(\mathcal{R}^\varepsilon E^c\mathcal{F}(w_1,w_2),\mathcal{R}^\varepsilon E^s\mathcal{F}(w_1,w_2)))$. Then $\mathcal{A} : \mathcal{P}\times H^3(2) \to \mathbb{R}$ is continuous with $|\mathcal{A}(\beta,\varepsilon,B_0)| \leq C(\beta)\|\mathcal{R}^\varepsilon\mathcal{F}B_0\|_{H^3(2)}$.

Next define $\psi(\Sigma) = e^{-c_1\Sigma_1^2-(c_2/\varepsilon)\Sigma_2^2}W^1(0)$. Then $\|\psi_n - \psi\|_{H^3(2)} \leq CL^{-n}$, and hence $\|u_n(1) - \alpha^*\psi\|_{H^3(2)} \leq CL^{-n}$. By (34) we have for $t = L^{2n}/\varepsilon^2$,

$$\begin{aligned}
L^{2n}\varepsilon^{-3/2}\mathcal{R}^{1/\varepsilon}_{L^n}B(t,\cdot) &= L^{2n}\varepsilon^{-3/2}\mathcal{R}^{1/\varepsilon}_{L^n}(w_1(t,\cdot)+\mathrm{i}w_2(t,\cdot)) \\
&= \mathcal{F}^{-1}\Big(u_{n,1}(1,\cdot)+L^{-n}v_{n,1}(1,\cdot)+\mathrm{i}(u_{n,2}(1,\cdot)+L^{-n}u_{n,2}(1,\cdot))\Big),
\end{aligned}$$

and with $(\mathcal{F}^{-1}\psi)(x) = \frac{1}{\sqrt{c_1(c_2/\varepsilon)}\ 4\pi} e^{-\frac{\xi^2}{4c_1} - \frac{x_2^2}{4(c_2/\varepsilon)}} (0,1)^T$ we obtain

$$\|L^{2n}\varepsilon^{-3/2}\mathcal{R}^{1/\varepsilon}_{L^n} B(t,x) - \frac{\mathrm{i}\alpha^*}{\sqrt{c_1(c_2/\varepsilon)}\ 4\pi} e^{-\frac{\xi^2}{4c_1} - \frac{x_2^2}{4(c_1/\varepsilon)}} \|_{H^2(3)} \leq C(\beta) L^{-n}$$

$$\Leftrightarrow \quad \| \underbrace{L^{2n}\varepsilon^{-2}}_{=t} \mathcal{R}^{1/\varepsilon}_{L^n} B(t,x) - \frac{\mathrm{i}\alpha^*}{\sqrt{c_1 c_2}\ 4\pi} e^{-\frac{\xi^2}{4c_1} - \frac{x_2^2}{4(c_2/\varepsilon)}} \|_{H^2(3)} \leq C(\beta) \underbrace{\varepsilon^{-1/2} L^{-n}}_{=\varepsilon^{-3/2} t^{-1/2}}.$$

Due to the embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ this gives upon rescaling with $\mathcal{R}^{\varepsilon}_{1/L^n}$ (which does not change the L^∞–norm) and multiplication with $t^{-1} = \varepsilon^2/L^{2n}$

$$\|B(t,x) - \frac{\mathrm{i}\alpha^*}{\sqrt{c_1 c_2}\ 4\pi t} e^{-\frac{\xi^2}{4c_1 t} - \frac{x_2^2}{4c_2 t}} \|_{L^\infty} \leq C(\beta)\varepsilon^{-3/2} t^{-3/2}.$$

This completes the proof of Theorem 7. □

5 Existence of rolls and spectral properties for the SHE

In this section we recall from [Mie97a] the setup of Bloch waves and the results on the existence and spectral stability of rolls in the two–dimensional SHE.

We search for stationary solutions u of (1) that bifurcate from $(\varepsilon, u) \equiv (0,0)$ and are independent of x_2 and periodic in x_1 with period $2\pi/k$. Hence we search for u in the form $u(x) = U(\xi)$, where $\xi = kx_1 \in \mathcal{T}_{2\pi}$. Here and in the following we denote by $\mathcal{T}_\alpha$ the one dimensional torus of length α, i.e. $\mathcal{T}_\alpha = \mathbb{R}/_{\alpha\mathbb{Z}}$. For explicity we take $\mathcal{T}_\alpha = (-\alpha/2, \alpha/2)$. In [Mie95], see also [CE90, Theorem 17.1], the problem for U,

$$-(1 + (1+\kappa)\partial_\xi^2)^2 U + \varepsilon^2 U - U^3 = 0, \quad U \in H^4(\mathcal{T}_{2\pi}), \tag{46}$$

dependent on the parameter ε, is treated by Liapunov–Schmidt–reduction around $(\varepsilon, U) \equiv (0,0)$. We cite Theorem 3.1 from [Mie97a]:

Theorem 15 *There exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and for all $\kappa \in (-\varepsilon, \varepsilon)$ there is a unique solution $U_{\varepsilon,\kappa}$ of (46), which is even in ξ and positive at $\xi = 0$. This solution has the expansion*

$$U_{\varepsilon,\kappa}(\xi) = a_1 \cos(\xi) + a_3 \cos(3\xi) + \mathcal{O}(\widetilde{a}^5) \ \textit{for}\ (\varepsilon,\kappa) \to 0,$$

$\widetilde{a} = \widetilde{a}(\varepsilon,\kappa) = \sqrt{\frac{4}{3}(\varepsilon^2 - \kappa^2)}$, $a_1 = \widetilde{a} + \widetilde{a}^3/512 + \mathcal{O}(\widetilde{a}^4)$, $a_3 = -\widetilde{a}^3/256 + \mathcal{O}(\widetilde{a}^4)$. *Moreover we have $U_{\varepsilon,\kappa}(\pi + \xi) = -U_{\varepsilon,\kappa}(\xi)$.*

We continue to use the coordinates $x = (\xi, x_2)$, keeping in mind that $\xi = kx_1$ for the original x_1. Letting $u(t,x) = U_{\varepsilon,\kappa}(\xi) + v(t,x)$ we obtain

$$\partial_t v = \mathcal{L}v + F(v), \quad \text{where} \tag{47}$$

$$\mathcal{L} = \mathcal{L}(\varepsilon,\kappa)v = -(1+k^2\partial_\xi^2+\partial_{x_2}^2)^2 v+\varepsilon^2 v-3U_{\varepsilon,\kappa}^2 v \quad \text{and} \quad F(v) = -3U_{\varepsilon,\kappa}v^2-v^3.$$

We need to solve the eigenvalue problem $\mathcal{L}v = \lambda v$, where the linear operator $\mathcal{L}$ has periodic coefficients due to the periodicity of $u_{\varepsilon,\kappa}$. In fact, the term $U_{\varepsilon,\kappa}^2$ is π–periodic in ξ since $U_{\varepsilon,\kappa}(\xi+\pi) = -U_{\varepsilon,\kappa}(\xi)$. However, in difference to [Mie97a], we treat $\mathcal{L}$ as 2π–periodic. Thus we can not refer directly to the results from [Mie97a]. Instead, we give a sketch of the analysis and in Lemma 16 we summarize the results.

Hence treating $\mathcal{L}$ as 2π–periodic, we now explain the Bloch decomposition for the operator $\mathcal{L}$, using the general frame from [Mie97a, section 2]. Defining the translation operators T_y by $(T_y u)(x) = u(x-y)$, the periodicity of $\mathcal{L}$ is characterized by the lattice group

$$\mathcal{G} := \{l \in \mathbb{R}^2 : T_l\mathcal{L} = \mathcal{L}T_l\} = 2\pi\mathbb{Z}\times\mathbb{R}.$$

Note that $\mathcal{L}$ leaves invariant the space of functions of the form $e^{\mathrm{i}\sigma\cdot x}W(\sigma,x)$ with $W(\sigma,x+l) = W(\sigma,x)$ for all $l\in\mathcal{G}$. In our problem this in particular means that $W(\sigma,\cdot)$ is independent of x_2. Therefore we will drop x_2 below, but for a moment we stay to the general notation. We define the periodicity domain $\mathcal{T} := \mathbb{R}^2/_{\mathcal{G}} = \mathcal{T}_{2\pi}\times\{0\}$, the dual lattice group $\mathcal{G}^* := \{h\in\mathbb{R}^2 : h\cdot l \in 2\pi\mathbb{Z} \ \ \forall l\in\mathcal{G}\} = \mathbb{Z}\times\{0\}$, and the wave number domain $\mathcal{T}^* := \mathbb{R}^2/_{\mathcal{G}^*} = \mathcal{T}_1\times\mathbb{R}$. The space $L^2(\mathcal{T}^*, L^2(\mathcal{T}))$ is called Bloch space, and a function u given in the form $u(x) = e^{\mathrm{i}\sigma\cdot x}U(x)$ with $\sigma\in\mathcal{T}^*$ and $U\in L^2(\mathcal{T})$ is called a Bloch wave.

The Bloch decomposition [RS78, XIII.16] $D : L^2(\mathcal{T}^*, L^2(\mathcal{T}))\to L^2(\mathbb{R}^2)$,

$$u(x) = D(U)(x) = \int_{\sigma\in\mathcal{T}^*} e^{\mathrm{i}\sigma\cdot x}U(\sigma,x)\,\mathrm{d}\sigma$$

is an isomorphism, its inverse given by means of Fourier–transform as

$$U(\sigma,x) = D^{-1}(u)(\sigma,x) = \frac{1}{2\pi}\sum_{j\in\mathcal{G}^*} e^{\mathrm{i}j\cdot x}\hat{u}(\sigma+j).$$

Since in our problem $\mathcal{T}$ is one dimensional, we identify $\mathcal{T} = \mathcal{T}_{2\pi}\times\{0\}$ with $\mathcal{T}_{2\pi}$, and we also identify $\mathcal{G}^* = \mathbb{Z}\times\{0\}$ with $\mathbb{Z}$. Then $L^2(\mathcal{T}^*, L^2(\mathcal{T}_{2\pi}))$ is our Bloch space, and the above formulas become

$$u(x) = D(U)(x) = \int_{\sigma\in\mathcal{T}^*} e^{\mathrm{i}\sigma\cdot x}U(\sigma,\xi)\,\mathrm{d}\sigma, \tag{48}$$

$$U(\sigma,\xi) = D^{-1}(u)(\sigma,\xi) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} e^{\mathrm{i}j\xi}\hat{u}((\sigma_1+j,\sigma_2)). \tag{49}$$

In an obvious sense $\sigma_1 \in \mathcal{T}_1$ is a true Bloch wavenumber, while $\sigma_2 \in \mathbb{R}$ is a usual Fourier wavenumber. Due to the construction we have

$$\begin{aligned} &U((\sigma_1 + j, \sigma_2), \xi) = e^{-\mathrm{i}j\xi} U(\sigma, \xi) \text{ for } j \in \mathbb{Z}, \text{ and} \\ &U(-\sigma, \xi) = \overline{U(\sigma, \xi)} \text{ for real–valued } u. \end{aligned} \tag{50}$$

In the following these periodicities (50) will be stated implicitly by denoting the domains as tori in the bounded directions. The Bloch operators $B(\varepsilon, \kappa, \sigma) : H^4(\mathcal{T}_{2\pi}) \to L^2(\mathcal{T}_{2\pi})$ are defined by

$$\begin{aligned} B(\varepsilon, \kappa, \sigma) U(\sigma, \xi) :=& e^{-\mathrm{i}\sigma \cdot x} \mathcal{L}(\varepsilon, \kappa)[e^{\mathrm{i}\sigma \cdot x} U(\sigma, \xi)] \\ =& -(1 + k^2(\partial_\xi + \mathrm{i}\sigma_1)^2 - \sigma_2^2)^2 U(\sigma, \xi) + (\varepsilon^2 - 3U_{\varepsilon,\kappa}^2(\xi)) U(\sigma, \xi). \end{aligned}$$

The main point is the identity [Mie97a, Theorem 2.1]

$$L^2\text{–spec}(\mathcal{L}) = \text{closure}\Big(\bigcup_{\sigma \in \mathcal{T}^*} \text{spec}(B(\varepsilon, \kappa, \sigma)) \Big).$$

However, here we only state and use results on the spectra of the $B(\varepsilon, \kappa, \sigma)$, since our analysis will be completely in Bloch space. For every fixed $\sigma \in \mathcal{T}^*$ the eigenvalue problem

$$B(\varepsilon, \kappa, \sigma) U(\sigma, \cdot) = \lambda U(\sigma, \cdot), \quad U(\sigma) \in H^4(\mathcal{T}_{2\pi}), \tag{51}$$

is self–adjoint in $L^2(\mathcal{T}_{2\pi})$. Hence we obtain a discrete set of real eigenvalues

$$\{\lambda_j(\sigma) \in \mathbb{R} : j \in \mathbb{N}\} \text{ with } \lambda_j(\sigma) \geq \lambda_{j+1}(\sigma) \to -\infty \text{ for } j \to \infty,$$

with a corresponding set of eigenfunctions $\{f_j(\sigma, \cdot) : j \in \mathbb{N}\}$ which we normalize such that $\|f_j(\sigma, \cdot)\|_{L^2(\mathcal{T})} = 1$. Due to the translational invariance of the original problem, we always have the eigenvalue $\lambda_1(0) = 0$ with associated eigenfunction $f_1(0, x) = \partial_\xi U_{\varepsilon,\kappa}(\xi)/\|U_{\varepsilon,\kappa}\|_{L^2(\mathcal{T})}$.

The spectral stability problem for $U_{\varepsilon,\kappa}$ can be solved by again applying Liapunov–Schmidt reduction, now to the linear eigenvalue problem (51). As in the cSHE one uses the fact that $B(\varepsilon, \kappa, \sigma)$ is a small perturbation of $B^\kappa(\sigma) := B(\kappa, \kappa, \sigma)$. This operator has constant coefficients. The eigenfunctions ϕ_m, $m \in \mathbb{Z}$, $\phi_m(\xi) = e^{\mathrm{i}m\xi}$ of $B^\kappa(\sigma)$ form a basis of $L^2(\mathcal{T}_{2\pi})$, with $B^\kappa(\sigma)\phi_m = (\mu_m(\kappa, \sigma) + \kappa^2)\phi_m$, where $\mu_m(\kappa, \sigma) = -(1 - (1+\kappa)(m+\sigma_1)^2 - \sigma_2^2)^2$. Thus, for $\sigma \in \mathcal{T}^*$ we have

$$\int_{\mathcal{T}_{2\pi}} (B^\kappa(\sigma) V(\sigma, \xi)) \overline{V(\sigma, \xi)} \, \mathrm{d}\xi \leq -\gamma(\kappa, \sigma) \|V(\sigma, \cdot)\|_{L^2(\mathcal{T}_{2\pi})}, \tag{52}$$

where $\gamma(\kappa, \sigma) = \min\{(1 - (1+\kappa)\sigma_1^2 - \sigma_2^2)^2, (1 - (1+\kappa)(\sigma_1 \pm 1)^2 - \sigma_2^2)^2\} + \kappa^2$. We define the dangerous set $S_0 = \{\sigma \in \mathcal{T}^* : \sigma_1^2 + \sigma_2^2 = 1 \text{ or } (\sigma_1 \pm 1)^2 + \sigma_2^2 = 1\}$.

From (52) we see that for σ bounded away from S_0 we obtain an appropriate bound on the spectrum of $B^\kappa(\sigma)$. In fact, choosing a small $\delta > 0$ and defining the set $G_\delta = \{\sigma \in \mathcal{T}^* : \text{dist}(\sigma, S_0) \geq \delta\}$ of good wave vectors, we find $\gamma(\kappa, \sigma) \geq \delta^2/2$ for all $\sigma \in G_\delta$ and for all sufficiently small κ. Moreover, for $\varepsilon \leq \varepsilon_0$ with ε_0 sufficiently small we have $\|B(\varepsilon, \kappa, \sigma) - B^\kappa(\sigma)\|_{L^2\to L^2} = \|\varepsilon^2 - \kappa^2 - 3U_{\varepsilon,\kappa}^2\|_\infty \leq \widetilde{a}^2$. Thus, for $\sigma \in G_\delta$ we obtain

$$\int_{\mathcal{T}_{2\pi}} (B(\varepsilon, \kappa, \sigma)V)\overline{V}\, \mathrm{d}x \leq -(\delta^2/2 - \widetilde{a}^2)\|V\|_{L^2(\mathcal{T}_{2\pi})}.$$

In particular, choosing $\delta = 3\varepsilon$ we have $\delta^2/2 - \widetilde{a}^2 \geq \varepsilon^2$. So far we only needed classical perturbation theory.

In order to decide on the stability of $U_{\varepsilon,\kappa}$ it remains to study the small (in modulus) eigenvalues of $B(\varepsilon, \kappa, \sigma)$ for σ close to S_0. This can be done by Liapunov–Schmidt reduction and gives similar but more involved algebraic problems as in the cSHE in section 3. Here we only report the results and refer to [Mie97a] for the details.

The sharpest conditions are obtained from the case σ close to 0. There, $B^\kappa(\sigma)$ has two small eigenvalues. For the corresponding eigenvalues $\lambda_2(\sigma) \leq \lambda_1(\sigma)$ of $B(\varepsilon, \kappa, \sigma)$ one obtains $\lambda_2(\sigma) \leq -2(\varepsilon^2 - \kappa^2)$ and

$$\lambda_1(\sigma) = -c_1(\varepsilon, \kappa)\sigma_1^2 - c_2(\varepsilon, \kappa)\sigma_2^2 + \mathcal{O}(|\sigma|^4), \qquad \text{where} \tag{53}$$

$$c_1 = 2(1+\kappa)\left(2 + 3\kappa - 4\frac{(1+\kappa)\kappa^2}{c(\varepsilon, \kappa)}\right) + \mathcal{O}((\varepsilon^2 - \kappa^2)^2), \quad c_2 = 2\kappa + \mathcal{O}(\varepsilon^2 - \kappa^2),$$

with $c(\varepsilon, \kappa) = -3\widetilde{a}^2/2 + \mathcal{O}(\widetilde{a}^4)$. These expansions follow from (4.4) in [Mie97a]. Clearly we obtain the two necessary conditions $c_1(\varepsilon, \kappa), c_2(\varepsilon, \kappa) \geq 0$ in order to have $\lambda_1(\varepsilon, \kappa, \widetilde{\sigma}) \leq 0$. Finally, the following necessary and sufficient conditions for $\lambda_1(\sigma) \leq 0$ and hence for the spectral stability of $U_{\varepsilon,\kappa}$ are calculated in [Mie97a]: $\varepsilon \leq \varepsilon_0$ for some $\varepsilon_0 > 0$, and

$$\varepsilon^2 \geq E_E(\kappa) = 3\kappa^2 - \kappa^3 + \mathcal{O}(\kappa^4), \text{ and } \kappa \geq E_Z(\varepsilon) = -\varepsilon^4/512 + \mathcal{O}(\varepsilon^6). \tag{54}$$

Compare to (31) and (32). The curves $\varepsilon^2 = E_E(\kappa)$ and $\kappa = E_Z(\varepsilon)$ are called Eckhaus respectively zigzag instability curves.

Like in the cSHE, typically c_2 is much smaller than c_1. Letting $\kappa = \beta\varepsilon$ with $\beta \in (0, 1/\sqrt{3})$ we find $c_2(\varepsilon, \beta\varepsilon) = 2\beta\varepsilon + \mathcal{O}(\varepsilon^2)$, but

$$c_1(\varepsilon, \beta\varepsilon) = 2(2 - 4\frac{\beta^2}{1-\beta^2}(1+\beta\varepsilon)^2) + \mathcal{O}(\varepsilon) = c_1^* + \mathcal{O}(\varepsilon),$$

with $c_1^* > 0$ independent of ε.

We summarize our results in the following lemma, where again we stick to the situation that $\kappa = \beta\varepsilon$ with $0 < \beta < 1/\sqrt{3}$. By this convention we now do not cover two small parts of the parameter region of spectrally stable rolls, namely the sets $\{(\varepsilon,\kappa) : 0 \le \varepsilon \le \varepsilon_0, E_Z(\varepsilon) \le \kappa \le 0\}$ and $\{(\varepsilon,\kappa) : 0 \le \varepsilon \le \varepsilon_0, E_E(\kappa) \le \varepsilon^2 \le 3\kappa^2\}$.

Lemma 16 *There exist continuous functions $a_j : (0, 1/\sqrt{3}) \to \mathbb{R}_+$, $j = 0,1,2$ and an $\varepsilon_0 > 0$ such that for all $\beta \in (0, 1/\sqrt{3})$ and for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. For $\kappa = \beta\varepsilon$ the largest eigenvalue $\lambda_1(\sigma)$ of $B(\varepsilon,\kappa,\sigma)$ fulfills*

$$\begin{aligned} \lambda_1(\sigma) &= -c_1\sigma_1^2 - c_2\sigma_2^2 + \mathcal{O}(|\sigma|^4) \\ &\le -a_0(\beta)\sigma_1^2 - \varepsilon a_0(\beta)\sigma_2^2 \quad \text{for } \sigma_1^2 + \varepsilon\sigma_2^2 \le a_2(\beta)\varepsilon^2/2, \\ \lambda_1(\sigma) &\le -2a_1(\beta)\varepsilon^2 \quad \text{for } \sigma_1^2 + \varepsilon\sigma_2^2 \ge a_2(\beta)\varepsilon^2/4. \end{aligned}$$

All other eigenvalues $\lambda_j(\varepsilon,\kappa,\sigma)$, $j \ge 2$, fulfill

$$\lambda_j(\sigma) \le -2a_1(\beta)\varepsilon^2 \text{ for all } \sigma \in \mathcal{T}^* = \mathcal{T}_1 \times \mathbb{R}.$$

Remark 17 As in the cSHE we call $\{\sigma \in \mathcal{T}_1 \times \mathbb{R} : \sigma_1^2 + \varepsilon\sigma_2^2 \le a_2(\beta)\varepsilon^2\}$ the set of center wave vectors, and its complement the set of stable wave vectors. We again have the asymptotics $a_0(\beta), a_1(\beta) \to 0$ for $\beta \to 0$ or $\beta \to 1/\sqrt{3}$. Therefore our result will be worse for β close to 0 or β close to $1/\sqrt{3}$, i.e. for (ε,κ) close to the zigzag or to the Eckhaus instability curve. Finally we remark, that there exists an $a_3(\beta) > 0$ such that $\lambda_1(\sigma) - \lambda_2(\sigma) \ge a_3(\beta)\varepsilon^2$ for $\sigma_1^2 + \varepsilon\sigma_2^2 \le a_2(\beta)\varepsilon^2$. This will be used to define projections $\widetilde{P}_1(\sigma) : L^2(\mathcal{T}_{2\pi}) \to \text{span}\{f_1(\sigma)\}$, depending smoothly on σ. For $\sigma_1^2 + \varepsilon\sigma_2^2 > a_2(\beta)\varepsilon^2$ these projections may no longer be well defined, since λ_1 may intersect with λ_2.

6 Nonlinear Stability of rolls for the SHE

To prove (9) with respect to small perturbations in $H^2(3)$ we want to apply Theorem 1. Since the inverse Bloch transform maps $H^2(3)$ into a Bloch space with regularity and weights, and since moreover the wave vector domain in the renormalization process will depend on n, we first set up notations and the functional analytic frame, and then state and prove our result Theorem 20.

6.1 Bloch spaces with regularity and weights

In section 4 we heavily relied on the fact that Fourier transform is an isomorphism between $H^m(k)$ and $H^k(m)$. Bloch transform is an isomorphism between $H^m(k)$ and a Bloch space with regularity and weights: we define

$$\mathcal{B}(k,m,b) := \{V \in H^k(\mathcal{T}^*, H^m(\mathcal{T}_{2\pi})) : \|V\|_{\mathcal{B}(k,m,b)} < \infty\}, \text{ where}$$
$$\|V\|^2_{\mathcal{B}(k,m,b)} := \sum_{|\alpha|\le k}\sum_{\beta \le m} \|(\partial_\sigma^\alpha \partial_\xi^\beta V)\rho^b\|^2_{L^2(\mathcal{T}^*, L^2(\mathcal{T}_{2\pi}))}, \quad \rho(\sigma) = (1+|\sigma|^2)^{1/2}. \quad (55)$$

Note that by definition $V \in H^k(\mathcal{T}^*, H^m(\mathcal{T}_{2\pi}))$ has the periodicities (50). Clearly the weight does not affect the properties of the Bloch space with respect to the bounded wave vector direction $\sigma_1 \in \mathcal{T}_1$. Identically to Lemma 5.4 in [Sch97] one can now prove the following fundamental Lemma.

Lemma 18 *The mapping $D : L^2(\mathcal{T}^*, L^2(\mathcal{T}_{2\pi})) \to L^2(\mathbb{R}^2)$ defined by (48) is an isomorphism between $\mathcal{B}(k,m,m)$ and $H^m(k)$.*

Corresponding to $H^2(3)$ in x–space, our basic space to work in will be $\mathcal{B}(3,2,2)$ and to abbreviate we set $\mathcal{B}(b) = \mathcal{B}(3,2,b)$. Due to Lemma 18 we have a well defined operation $*$ in $\mathcal{B}(3,2,2)$, associated to pointwise multiplication in x–space. Also note that $\mathcal{B}(3,2,2) \hookrightarrow C^1(\mathcal{T}^*, H^2(\mathcal{T}_{2\pi}))$.

6.2 Scaling and nonlinear interaction in Bloch space

For the cSHE the renormalization process was based on rescaling in the Fourier wave vector, corresponding to rescaling in x–space by the formula (34). Using the same idea for the SHE the wave vector domain $\mathcal{T}^* = \mathcal{T}_1 \times \mathbb{R}$ changes on rescaling in σ. Thus, for $L > 0$ we define the spaces

$$\mathcal{B}_L(k,m,b) := H^k(\mathcal{T}_L \times \mathbb{R}, H^m(\mathcal{T}_{2\pi})), \quad \mathcal{B}_L(b) := \mathcal{B}_L(3,2,b),$$

equipped with the norm (55) with $\|\cdot\|_{L^2(\mathcal{T}_L\times\mathbb{R},L^2(\mathcal{T}_{2\pi}))}$ instead of $\|\cdot\|_{L^2(\mathcal{T}_1\times\mathbb{R},L^2(\mathcal{T}_{2\pi}))}$. For $L_1, L_2 > 0$ we define the scaling operators

$$\mathcal{R}_{(\frac{1}{L_1},\frac{1}{L_2})} : \mathcal{B}(k,m,m) \to \mathcal{B}_{L_1}(k,m,m), \; (\mathcal{R}_{(\frac{1}{L_1},\frac{1}{L_2})}U)(\sigma,\xi) = U((\frac{\sigma_1}{L_1},\frac{\sigma_2}{L_2}),\xi).$$

Remark 19 Note that these operators do not correspond to scaling operators in x–space. This is the case if we define $\widetilde{\mathcal{B}}_L(k,m,b) = H^k(\mathcal{T}_L \times \mathbb{R}, H^m(\mathcal{T}_{2\pi/L}))$ and $\widetilde{\mathcal{R}}_{1/L}U(\sigma,\xi) = U(\sigma/L, L\xi)$. Then we have $L^d D^{-1}\mathcal{R}_L u = \widetilde{\mathcal{R}}_{1/L} D^{-1} u$, and $D_L^{-1} := D^{-1}\mathcal{R}_L$ defines an isomorphism between $H^m(k)$ and $\widetilde{\mathcal{B}}_L(k,m,m)$, suggesting that the renormalization process which follows below could be suitably controlled in x–space. However, the norms of D_L^{-1} and D_L depend on L. Therefore this approach fails. For this reason we use the more simple definition above. This gives the necessity to formulate Theorem 1 in sequences of Banach spaces.

We set $\mathcal{R}_{1/L} = \mathcal{R}_{(1/L,1/L)}$ and $\mathcal{R}^{1/L} = \mathcal{R}_{(1/L,1/\sqrt{L})}$. Analogously to (35) we obtain

$$\|\mathcal{R}_{(1/L_1,1/L_2)}U\|_{\mathcal{B}_{L_1}(k,m,b)} \le C\frac{\max\{L_1,L_2\}^b}{\min\{L_1,L_2\}^k}\sqrt{L_1L_2}\|U\|_{\mathcal{B}(k,m,b)}. \tag{56}$$

We denote by $*$ the operation “convolution” in $\mathcal{B}(k,m,b)$ that is associated to pointwise multiplication in x–space, i.e. for $U, V \in \mathcal{B}(k,m,b)$ we define

$$(U * V)(\sigma,\xi) = D^{-1}(DU(x)DV(x))(\sigma,\xi).$$

By Lemma 18 this definition is reasonable for $b>3/2$. We have the formula

$$
\begin{aligned}
(U*V)(\sigma,\xi) &= \sum_{j\in\mathbb{Z}} e^{\mathrm{i}j\xi}\int_{m\in\mathbb{R}^2}\hat{u}((\sigma_1+j-m_1,\sigma_2-m_2))\hat{v}(m)\,\mathrm{d}m\\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_{m\in\mathcal{T}_1\times\mathbb{R}}\hat{u}((\sigma_1+j-m_1-k,\sigma_2-m_2))e^{\mathrm{i}(j+k)\xi}\hat{v}((m_1+k,m_2))e^{\mathrm{i}k\xi}\,\mathrm{d}m\\
&= \int_{m\in\mathcal{T}_1\times\mathbb{R}}\Big(\sum_{j\in\mathbb{Z}} e^{\mathrm{i}j\xi}\hat{u}((\sigma_1-m_1+j,\sigma_2-m_2))\Big)\Big(\sum_{k\in\mathbb{Z}} e^{\mathrm{i}k\xi}\hat{v}((m_1+k,m_2))\Big)\,\mathrm{d}m\\
&= \int_{m\in\mathcal{T}_1\times\mathbb{R}} U(\sigma-m,\xi)V(m,\xi)\,\mathrm{d}m
\end{aligned}
$$

where we used (50). Scaling of σ gives

$$
\begin{aligned}
\mathcal{R}_{(1/L_1,1/L_2)}(U*V)(\sigma,\xi) &= \int_{m\in\mathcal{T}_1\times\mathbb{R}} U((\frac{\sigma_1}{L_1}-m_1,\frac{\sigma_2}{L_2}-m_2),\xi)V(m,\xi)\,\mathrm{d}m\\
&= L_1^{-1}L_2^{-1}\int_{\widetilde{m}\in\mathcal{T}_{L_1}\times\mathbb{R}} U((\frac{\sigma_1-\widetilde{m}_1}{L_1},\frac{\sigma_2-\widetilde{m}_2}{L_2}),\xi)V((\frac{\widetilde{m}_1}{L_1},\frac{\widetilde{m}_2}{L_2}),\xi)\,\mathrm{d}\widetilde{m}\\
&=: L_1^{-1}L_2^{-1}(\mathcal{R}_{(1/L_1,1/L_2)}U *_{L_1}\mathcal{R}_{(1/L_1,1/L_2)}V)(\sigma,\xi). \qquad (57)
\end{aligned}
$$

In the following we drop the subscript and write $*$ for $*_{L_1}$. As before we write V^{*p} for the p–times convolution. Introducing $V(t)=D^{-1}v(t)$, equation (47) turns into

$$
\begin{aligned}
\partial_t V(t,\sigma,\xi) &= B(\varepsilon,\kappa,\sigma)V(t,\sigma,\xi)+N(V)(t,\sigma,\xi), \quad \text{where} \qquad (58)\\
N(V)(t,\sigma,\xi) &= -3U_{\varepsilon,\kappa}(\xi)V(t)^{*2}(\sigma,\xi)-V(t)^{*3}(\sigma,\xi).
\end{aligned}
$$

Next introducing $\widetilde{V}(t)=\mathcal{R}^{\varepsilon}V(t)\in B_{1/\varepsilon}(2)$, equation (58) turns into

$$
\partial_t\widetilde{V} = (\mathcal{R}^{\varepsilon}B(\varepsilon,\kappa,\sigma))\widetilde{V}+\mathcal{R}^{\varepsilon}N(\mathcal{R}^{1/\varepsilon}\widetilde{V}). \qquad (59)
$$

Note that the spectrum of $\mathcal{R}^{\varepsilon}B(\varepsilon,\kappa,\cdot)$ is given by the surfaces $\mathcal{T}_{1/\varepsilon}\times\mathbb{R}\ni\sigma\mapsto\mathcal{R}^{\varepsilon}\lambda_j(\sigma)$, $j\in\mathbb{N}$, in particular $\mathcal{R}^{\varepsilon}\lambda_1(\sigma)=c_1\varepsilon^2\sigma_1^2+(c_2/\varepsilon)\varepsilon^2\sigma_2^2+\mathcal{O}(|(\varepsilon\sigma_1,\sqrt{\varepsilon}\sigma_2)|^4)$. Equation (59) is the analogue of (40). In fact, the analysis of section 4 for (40) now translates in an almost automatic way to (59).

6.3 The result

We assume the initial conditions for (47) to be given at time $1/\varepsilon^2$. In order to prove the nonlinear diffusive stability of a spectrally stable roll $u_{\varepsilon,\beta\varepsilon}$, $(\beta,\varepsilon)\in\mathcal{P}:=(0,1/\sqrt{3})\times(0,\varepsilon_0)$, and in particular to estimate the size of the domain of attraction, we will consider the rescaled problem (59). Thus we obtain the condition (60) on the size of $\widetilde{V}(1/\varepsilon^2)=\mathcal{R}^{\varepsilon}D^{-1}v(1/\varepsilon^2)$ in $B_{1/\varepsilon}(2)$. This condition is interpretated in Lemma 21.

Theorem 20 *There exist continuous functions $\delta, C : (0, 1/\sqrt{3}) \to \mathbb{R}_+$ and a continuous function $\mathcal{A} : \mathcal{P} \times \mathcal{B}_{1/\varepsilon}(2) \to \mathbb{R}$ such that for all $(\beta, \varepsilon) \in \mathcal{P}$ the following holds. Let $\kappa = \beta\varepsilon$ and let $v = v(t, x)$ be the solution to (47) with the initial conditions $v(1/\varepsilon^2, \cdot) = v_0(\cdot)$ satisfying*

$$\|\mathcal{R}^\varepsilon D^{-1} v_0\|_{\mathcal{B}_{1/\varepsilon}(2)} \le \delta(\beta). \tag{60}$$

Then

$$\|v(t, x) - \frac{\alpha^*}{\sqrt{c_1 c_2}\, t} e^{-\frac{\xi^2}{4c_1 t} - \frac{x_2^2}{4c_2 t}} f_1(0, \xi)\|_{L^\infty(\mathbb{R}^d)} \le C(\beta)\varepsilon^{-3/2} t^{-3/2}, \tag{61}$$

as $t \to \infty$, where $\alpha^* = \mathcal{A}(\beta, \varepsilon, v_0)$ and $f_1(0, \xi) = \partial_\xi U_{\varepsilon,\kappa}(\xi)/\|U_{\varepsilon,\kappa}\|_{L^2(\mathcal{T}_{2\pi})}$.

Lemma 21 *The condition (60) holds for initial conditions $v_0 \in H^2(3)$ given in the form $v_0(x) = \varepsilon^{3/2}(\mathcal{R}^\varepsilon A_n(x))e^{in\xi} + c.c.$ with $n^2\|A_n\|_{H^2(3)} \le C\delta(\beta)$.*

Proof. By (34) we obtain $\hat{v}_0(\sigma) = \hat{A}_n((\frac{\sigma_1 - n}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}})) + \hat{A}_n^c((\frac{\sigma_1 - n}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}}))$, where $\hat{A}^c(\sigma) = \overline{\hat{A}(-\sigma)}$. This shows that $\hat{v}_0$ is concentrated over two ellipses, centered at $(\pm n, 0)$ with the lengths of the σ_1 and σ_2– axis given by $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\sqrt{\varepsilon})$, respectively. Without loss of generality we assume supp $\hat{A}_n \subset \{\sigma : |\sigma| \le 1/2\}$. Then

$$\begin{aligned} V_0(\sigma, \xi) &= (D^{-1} v_0)(\sigma, \xi) \\ &= \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} e^{ij\xi} (\hat{A}_n(\frac{\sigma_1 - n + j}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}}) + \hat{A}_n^c(\frac{\sigma_1 - n + j}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}})) \\ &= \frac{1}{2\pi} \left(e^{in\xi} \hat{A}_n(\frac{\sigma_1}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}}) + e^{-in\xi} \hat{A}_n^c(\frac{\sigma_1}{\varepsilon}, \frac{\sigma_2}{\sqrt{\varepsilon}}) \right). \end{aligned}$$

Hence we obtain $\widetilde{V}_0(\sigma, \xi) = \mathcal{R}^\varepsilon V_0(\sigma, \xi) = \frac{1}{2\pi}(e^{in\xi}\hat{A}_n(\sigma) + e^{-in\xi}\hat{A}_n^c(\sigma))$. This shows that $\|\partial_\sigma^\alpha \widetilde{V}_0(\sigma, \cdot)\|^2_{H^2(\mathcal{T}_{2\pi})} = (1 + n^2 + n^4)(\partial_\sigma^\alpha \hat{A}_n(\sigma))^2$. Thus we have $\|\widetilde{V}\|_{\mathcal{B}_{1/\varepsilon}(2)} \le Cn^2\|\hat{A}_n(\sigma)\|_{H^3(2)}$, and hence $\|\widetilde{V}\|_{\mathcal{B}_{1/\varepsilon}(2)} \le \delta(\beta)$ if $n^2\|\hat{A}_n(\sigma)\|_{H^3(2)} \le C\delta(\beta)$. □

Remark 22 From Lemma 21 we in particular see, that the attracted neighborhood of $u_{\varepsilon,\beta\varepsilon}$ is of diameter $\mathcal{O}(\varepsilon^{3/2})$ in $L^\infty(\mathbb{R}^2)$. It is clear, that (60) also holds for v_0 given in the form

$$v_0(x) = \sum_n \varepsilon^{3/2} A_n(\varepsilon\xi, \sqrt{\varepsilon}x_2)e^{in\xi} + c.c., \text{ where } \|\sum_n n^2 A_n\|_{H^2(3)} \le C\delta(\beta).$$

In this case the Fourier modes of v_0 are concentrated in ellipses centered at the wave vectors $(n, 0), n \in \mathbb{Z}$, with the σ_1 and σ_2 semi–axis of order ε and $\sqrt{\varepsilon}$. For the functions $\mathcal{A}, \delta, C$ we have

$$|\mathcal{A}(\beta, \varepsilon, v_0| \le C(\beta)\|\mathcal{R}^\varepsilon D^{-1} v_0\|_{\mathcal{B}_{1/\varepsilon}(2)} \tag{62}$$

with $C(\beta)$ independent of ε, and $\delta(\beta) \to 0$ and $\overline{C}(\beta) \to \infty$ for $\beta \to 0, 1/\sqrt{3}$.

6.4 Proof of Theorem 20

We need to separate the eigenfunctions $V(\sigma,\cdot) \in L^2(\mathcal{T}_{2\pi})$ of $B(\varepsilon,\kappa,\sigma)$ belonging to the surface $\{\sigma_1^2 + \varepsilon\sigma_2^2 \le a_2\varepsilon^2\} \ni \sigma \mapsto \lambda_1(\sigma)$ from the linearly exponentially damped ones. As in the cSHE this is done via mode–filters.

Lemma 23 *Let $\widetilde{M} \in C_b^n(\mathcal{T}\times\mathbb{R}, \mathrm{Lin}(H^{m_1}(\mathcal{T}_{2\pi}), H^{m_2}(\mathcal{T}_{2\pi})))$ and let $L_1, L_2 > 1$. Then $(\mathcal{R}_{(\frac{1}{L_1},\frac{1}{L_2})}M)V(\sigma) = (\mathcal{R}_{(1/L_1,1/L_2)}\widetilde{M}(\sigma))V(\sigma)$ defines a linear operator $(\mathcal{R}_{(\frac{1}{L_1},\frac{1}{L_2})}M) : \mathcal{B}_{L_1}(n,m_1,b) \to \mathcal{B}_{L_1}(n,m_2,b)$. There exists a $C > 0$ such that*

$$\|(\mathcal{R}_{(1/L_1,1/L_2)}M)V\|_{\mathcal{B}_{L_1}(n,m_2,b)} \le C\|V\|_{\mathcal{B}_{L_1}(n,m_1,b)}.$$

Proof. This directly follows from the definition of $\mathcal{B}(n,m,b)$ since $(\mathcal{R}_{(\frac{1}{L_1},\frac{1}{L_2})}\widetilde{M})$ is uniformly bounded for all $L_1, L_2 \ge 1$. □

From Lemma 16 we know, that at least for $\sigma_1^2 + \varepsilon\sigma_2^2 \le a_2(\beta)\varepsilon^2$ the eigenvalue $\lambda_1(\sigma)$ of $B(\varepsilon,\kappa,\sigma)$ is bounded away from the rest of the spectrum. Thus for fixed σ with $\sigma_1^2 + \varepsilon\sigma_2^2 \le a_2(\beta)\varepsilon^2$, through the Dunford–integral

$$P_1(\sigma) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (B(\varepsilon,\kappa,\sigma) - s\mathrm{Id})^{-1}\,\mathrm{d}s,$$

a $B(\varepsilon,\kappa,\sigma)$–invariant projection from $L^2(\mathcal{T}_{2\pi})$ onto span$f_1(\sigma)$ is defined, where Γ is a curve in $\mathbb{C}$ surrounding $\lambda_1(\sigma)$ in the resolvent set of $B(\varepsilon,\kappa,\sigma)$. Since the eigenvalue problem (51) is self–adjoint in $L^2(\mathcal{T}_{2\pi})$ the projection is orthogonal in $L^2(\mathcal{T}_{2\pi})$ and we have the explicit formula

$$P_1(\sigma)V(\cdot) = \left(\int_{\mathcal{T}_{2\pi}} \overline{f_1(\sigma,\xi)}V(\xi)\,\mathrm{d}\xi\right) f_1(\sigma,\cdot).$$

For $\sigma_1^2 + \varepsilon\sigma_2^2 \ge a_2(\beta)$ (where no smoothness of $\sigma \mapsto P_1(\sigma)$ may be guaranteed) we continue P_1 in an arbitrary but smooth way. Next fix a smooth cut–off–function

$$\chi(\sigma) = \begin{cases} 1 & \text{for} \quad 0 \le |\sigma| \le a_2/4 \\ \in [0,1] & \text{for} \quad a_2/4 \le |\sigma| \le a_2/2 \\ 0 & \text{for} \quad a_2/1 \le |\sigma| \end{cases}.$$

Then the function $\widetilde{E}^c = \chi^{1/\varepsilon}P_1 \in C^3(\mathcal{T}_1\times\mathbb{R}, \mathrm{Lin}(H^2(\mathcal{T}_{2\pi}), H^2(\mathcal{T}_{2\pi})))$ defines the center mode filter $E^c : B(n,m,b) \to B(n,m,b)$, where we used $\chi^{1/\varepsilon} = \mathcal{R}^{1/\varepsilon}\chi$. We denote the stable mode filter by $E^s = Id - E^c$ and introduce auxiliary mode filters E^{ch} and E^{sh} defined via the multipliers $\widetilde{E}^{ch} : \sigma \mapsto \chi^\varepsilon(\sigma/2)P_1(\sigma)$ and $\widetilde{E}^{sh} : \sigma \mapsto \mathrm{Id} - \chi^\varepsilon(2\sigma)P_1(\sigma)$. We then have $E^{ch}E^c = E^c$ and $E^{sh}E^s = E^s$. By Lemma 23 the following holds:

Lemma 24 *There exists a constant $C > 0$ such that for all $L > 1/\varepsilon$ we have $\|(\mathcal{R}_{1/L}E^*)V\|_{\mathcal{B}_L(b)} \leq C\|V\|_{\mathcal{B}_L(b)}$ for E^* equal to E^c, E^{ch}, E^s or E^{sh}.*

Now we define the center part $\widetilde{V}_c$ and the stable part $\widetilde{V}_s$ to be the solutions of

$$\begin{aligned} \partial_t \widetilde{V}_c &= \Lambda^c \widetilde{V}_c + N^c(\widetilde{V}_c, \widetilde{V}_s), \quad \widetilde{V}_c(1/\varepsilon^2) = (\mathcal{R}^\varepsilon E^c)\widetilde{V}(1/\varepsilon^2), \\ \partial_t \widetilde{V}_s &= \Lambda^s \widetilde{V}_s + N^s(\widetilde{V}_c, \widetilde{V}_s), \quad \widetilde{V}_c(1/\varepsilon^2) = (\mathcal{R}^\varepsilon E^s)\widetilde{V}(1/\varepsilon^2), \end{aligned} \tag{63}$$

where $N^*(\widetilde{V}_c, \widetilde{V}_s) = \mathcal{R}^\varepsilon E^c N(E^{ch}\mathcal{R}^{1/\varepsilon}\widetilde{V}_c + E^{sh}\mathcal{R}^{1/\varepsilon}\widetilde{V}_s))$ for $*$ equal to c or s, and Λ^c and Λ^s are defined by

$$(\Lambda^c \widetilde{V})(\sigma) = \begin{cases} (\mathcal{R}^\varepsilon \lambda_1 P_1)(\sigma)\widetilde{V}(\sigma) & , |\sigma| \leq a_2/2 \\ (\mathcal{R}^\varepsilon \lambda_1^c P_1)(\sigma)\widetilde{V}(\sigma) & , |\sigma| \geq a_2/2 \end{cases},$$

$$(\Lambda^s V)(\sigma) = \begin{cases} (\mathcal{R}^\varepsilon (B(\varepsilon,\kappa,\cdot) - \lambda_1 P_1 + \lambda_1^s P_1)(\sigma)\widetilde{V}(\sigma) & , |\sigma| \leq a_2/4 \\ (\mathcal{R}^\varepsilon B(\varepsilon,\kappa,\sigma))\widetilde{V}(\sigma) & , |\sigma| \geq a_2/4 \end{cases}.$$

Here, λ_1^c is a smooth continuation of λ_1 for $\sigma_1^2 + \varepsilon\sigma_2^2 \geq \varepsilon^2 a_2/2$ such that $\lambda_1^c(\sigma) \leq -a_0\sigma_1^2 - \varepsilon a_0 \sigma_2^2$, and λ_1^s is a smooth continuation of λ_1 for $\sigma_1^2 + \varepsilon\sigma_2^2 \leq \varepsilon^2 a_2/4$ such that $\lambda_1^s(\sigma) \leq -a_1\varepsilon^2$. Due to the construction we have $(\mathcal{R}^\varepsilon E^{ch})\widetilde{V}_c(t) = \widetilde{V}_c(t)$ and $(\mathcal{R}^\varepsilon E^{sh})\widetilde{V}_s(t) = \widetilde{V}_s(t)$ for all $t \geq 1/\varepsilon^2$. Therefore, if $(\widetilde{V}_c, \widetilde{V}_s)$ solves (63), then $\widetilde{V} = \widetilde{V}_c + \widetilde{V}_s$ solves (59). Using the same scalings like in (43), i.e. defining

$$U_n(T,\Sigma) = \widetilde{V}_c(L^{2n}T/\varepsilon^2, \Sigma/L^n), \quad V_n(T,\Sigma) = L^n \widetilde{V}_s(L^{2n}T/\varepsilon, \Sigma/L^n)$$

we write the variation of constant formula for U_n, V_n as

$$\begin{aligned} U_n(T) &= e^{\Lambda_n^c(T-1/L^2)}\mathcal{R}_{1/L}U_n(1/L^2) + \int_{1/L^2}^T e^{\Lambda_n^c(T-\tau)} N_n^c(U_n(\tau), V_n(\tau))\,\mathrm{d}\tau, \\ V_n(T) &= e^{\Lambda_n^s(T-1/L^{2n})}\mathcal{R}_{1/L}V_n(1/L^2) + \int_{1/L^2}^T e^{\Lambda_n^s(T-\tau)} N_n^s(U_n(\tau), V_n(\tau))\,\mathrm{d}\tau, \end{aligned} \tag{64}$$

where $T \in [1/L^2, 1]$, $n \in \mathbb{N}$, and

$$\begin{aligned} \Lambda_n^c &= L^{2n}\varepsilon^{-2}(\mathcal{R}_{1/L^n}\Lambda^c), \quad N_n^c(U_n, V_n) = L^{2n}\varepsilon^{-2}\mathcal{R}_{1/L^n}N^c(\mathcal{R}_{L^n}U_n, L^{-n}\mathcal{R}_{L^n}V_n), \\ \Lambda_n^s &= L^{2n}\varepsilon^{-2}(\mathcal{R}_{1/L^n}\Lambda^s), \quad N_n^s(U_n, V_n) = L^{3n}\varepsilon^{-2}\mathcal{R}_{1/L^n}N^s(\mathcal{R}_{L^n}U_n, L^{-n}\mathcal{R}_{L^n}V_n). \end{aligned}$$

Defining $X_n = Y_n = \mathcal{B}_{L^n/\varepsilon}(2)$, $\widetilde{X}_n = \mathcal{B}_{L^n/\varepsilon}(1)$, and $H_n^c = H_n^s = \mathcal{R}_{1/L^n}$, the system (64) is in the form (21). Using (56), assumption A3) in Theorem 1 is fulfilled with $m_c = m_s = 3$. We define

$$\begin{aligned} &\Pi_n \in \mathrm{Lin}(X_n, \mathbb{R}) \text{ by } \Pi_n U = \langle U(0,\cdot), f_1(0,\cdot)\rangle_{L^2(\mathcal{T}_{2\pi})}, \text{ and} \\ &\Psi_n \in X_n \text{ by } \Psi_n(\Sigma,\cdot) = e^{-c_1\Sigma_1^2 - (c_2/\varepsilon)\Sigma_2^2}\chi(\Sigma/L^n)f_1(\Sigma/L^n,\cdot), \end{aligned} \tag{65}$$

and show the assumptions A1), A2) and A4). Here, $\Pi_n \in \mathrm{Lin}(X_n, \mathbb{R})$, as well as $\|\Pi_n\|_{\mathrm{Lin}(\mathcal{B}_{L^n/\varepsilon}(2),\mathbb{R})} \le C$, follows from $\mathcal{B}_{L^n/\varepsilon}(2) \hookrightarrow C^0(\mathcal{T}_{L^n/\varepsilon} \times \mathbb{R}, H^2(\mathcal{T}_{2\pi}))$ and from $U(-\sigma,\xi) = \overline{U(\sigma,\xi)}$ for real–valued u. The estimate $\|\psi_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \le C = C(\beta)$ is clear.

In order to show A2), the crucial point for the cSHE in section 4 was the vanishing of the projection of the quadratic terms on the critical eigenspace at wave vector 0. The key estimate was the estimate on s_1 in Lemma 13, which relied on Lemma 12. In $\mathcal{B}_L(b) = \mathcal{B}_L(3,2,b)$ we have the following analogue of Lemma 12. Using the respective spaces also the proof is completely the same.

Lemma 25 *Let $\widetilde{K} \in C_b^3((\mathcal{T}_1 \times \mathbb{R})^2, H^2(\mathcal{T}_{2\pi}, \mathbb{C}))$ with $\|\widetilde{K}(\sigma, \sigma - m)(\cdot)\|_{H^2(\mathcal{T}_{2\pi})} \le C(|\sigma| + |\sigma - m|)^\gamma$. For $A, B \in \mathcal{B}_L(2)$ define*

$$(\mathcal{R}_{1/L}K)(A,B)(\sigma,\xi) = \int_{m\in\mathcal{T}_L\times\mathbb{R}} (\mathcal{R}_{1/L}\widetilde{K}(\sigma,\sigma-m,\xi))A(\sigma,\xi)B(\sigma-m,\xi)\,\mathrm{d}m.$$

Then $(\mathcal{R}_{1/L}K)$ is a bilinear mapping from $[\mathcal{B}_L(2)]^2$ into $\mathcal{B}_L(2)$. There exists a $C > 0$ such that for all $L > 1$ we have

$$\|(\mathcal{R}_{1/L}K)(A,B)\|_{\mathcal{B}_L(2-\gamma)} \le CL^{-\min\{\gamma,1\}}\|A\|_{\mathcal{B}_L(2)}\|B\|_{\mathcal{B}_L(2)}.$$

Lemma 26 *For all $\beta \in (0, 1/\sqrt{3})$ there exists a constant $C(\beta) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the assumptions A1) and A2) hold with $C = C(\beta)$.*

Proof. Using Lemma 16, assumption A1) follows from $\|e^{\Lambda_n^c T}U_n(\Sigma)\|_{H^2(\mathcal{T}_{2\pi})} \le e^{-a_0|\Sigma|^2 T}\|U_n(\Sigma)\|_{H^2(\mathcal{T}_{2\pi})}$ and $\|e^{\Lambda_n^s T}V_n(\Sigma)\|_{H^2(\mathcal{T}_{2\pi})} \le e^{-a_1 L^{2n}T}\|V_n(\Sigma)\|_{H^2(\mathcal{T}_{2\pi})}$. The well definedness and continuity statements in A2) follow from Sobolev imbeddings. We estimate $\|N_n^s(U_n,V_n)\|_{\mathcal{B}_{L^n/\varepsilon}(2)}$ and $\|N_n^c(U_n,V_n)\|_{\mathcal{B}_{L^n/\varepsilon}(1)}$, the Lipschitz estimates in A2) then follow analogously. For notational simplicity we assume $(\mathcal{R}^\varepsilon_{1/L^n}E^{ch})U_n = U_n$ and $(\mathcal{R}^\varepsilon_{1/L^n}E^{sh})V_n = V_n$. Using (56), (57) and $\|U_{\varepsilon,\kappa}\|_{L^\infty} = \mathcal{O}(\varepsilon)$ we obtain $\|N_n^s(U_n,V_n)\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \le C\varepsilon^{1/2}L^n\ \|(U_n,V_n)\|^2_{\mathcal{B}_{L^n/\varepsilon}(2)}$ and $\|N_n^c(U_n,V_n)\|_{\mathcal{B}_{L^n/\varepsilon}(1)} = L^{2n}\varepsilon^{-2}\|s_1\|_{\mathcal{B}_{L^n/\varepsilon}(1)} + \mathcal{O}(\varepsilon^{1/2}L^{-n})$, where the last term comes from estimating the mixed convolution of $\mathcal{R}^{1/\varepsilon}_{L^n}U_n$ with $L^{-n}\mathcal{R}^{1/\varepsilon}_{L^n}V_n$ in $\mathcal{B}_{L^n/\varepsilon}(2)$. The term s_1 contains the quadratic terms in U_n, given as follows. For $\sigma, m \in \mathcal{T}^* = \mathcal{T}_1 \times \mathbb{R}$ we define

$$\widetilde{K}_1(\sigma,\sigma-m)(\xi) = \int_{\mathcal{T}_{2\pi}} \widetilde{U}_{\varepsilon,k}(\widetilde{\xi})\overline{f_1(\sigma-m,\widetilde{\xi})}f_1(m,\widetilde{\xi})f_1(\sigma,\widetilde{\xi})\,\mathrm{d}\widetilde{\xi}f_1(\sigma,\xi),$$

where $\varepsilon\widetilde{U}_{\varepsilon,\kappa} = U_{\varepsilon,\kappa}$. Then $\widetilde{K}_1 \in C_b^3(\mathcal{T}^* \times \mathcal{T}^*, H^2(\mathcal{T}_{2\pi}))$ with $\widetilde{K}_1(0,0) = 0$. This holds because $U_{\varepsilon,\kappa}$ is even, thus $f_1(0) = \partial_\xi U_{\varepsilon,\kappa}/\|U_{\varepsilon,\kappa}\|_{L^2(\mathcal{T}_{2\pi})}$ is odd, and hence the integral goes over an odd function for $\sigma = m = 0$. See also the following remark. We obtain $\|\widetilde{K}_1(\sigma,\sigma-m)\|_{H^2(\mathcal{T}_{2\pi})} = \mathcal{O}(|\sigma|+|m|)$, and s_1 is given as

$$\begin{aligned} s_1(\Sigma,\xi) &= -\frac{3\varepsilon^{3/2+1}}{L^{2n}}(\mathcal{R}^{\varepsilon}_{1/L^n}E^c)\widetilde{U}_{\varepsilon,\kappa}(U_n * U_n)(\Sigma,\xi) \\ &= -\frac{3\varepsilon^{5/2}}{L^{2n}}\chi(\frac{\Sigma}{L^n})\Bigg(\int\limits_{\widetilde{\xi}\in\mathcal{T}_{2\pi}}\Big\{\widetilde{U}_{\varepsilon,\kappa}(\widetilde{\xi})\int\limits_{M\in\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R}}\Big\{U_n(\Sigma-M,\widetilde{\xi})U_n(M,\widetilde{\xi})\Big\}\,\mathrm{d}M \\ &\qquad\qquad \overline{(\mathcal{R}^{\varepsilon}_{1/L^n}f_1(\Sigma,\widetilde{\xi}))}\Big\}\,\mathrm{d}\widetilde{\xi}\Bigg)(\mathcal{R}^{\varepsilon}_{1/L^n}f_1(\Sigma,\xi)) \\ &= -\frac{3\varepsilon^{5/2}}{L^{2n}}\chi(\frac{\Sigma}{L^n})\int_{\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R}}(\mathcal{R}^{\varepsilon}_{1/L^n}\widetilde{K}_1(\Sigma,\Sigma-M)(\xi))a_n(\Sigma-M)a_n(M)\,\mathrm{d}M, \end{aligned}$$

where $U_n(\Sigma,\cdot) = a_n(\Sigma)(\mathcal{R}^{\varepsilon}_{1/L^n}f_1(\Sigma,\cdot))$ and we have one factor ε from the modulus of $U_{\varepsilon,\kappa}$. Using Lemma 25 with $\gamma=1$ we gain a factor $\varepsilon^{1/2}L^{-n}$ in $\mathcal{B}_{L^n/\varepsilon}(1)$, i.e. $\|s_1\|_{\mathcal{B}_{L^n/\varepsilon}(1)} \le C\varepsilon^{5/2+1/2}L^{-3n}\|U_n\|^2_{\mathcal{B}_{L^n/\varepsilon}(2)}$. □

Remark 27 As pointed out in the Introduction, the fact that $\widetilde{K}_1(0,0)=0$ comes from the translational invariance of the SHE. To see this, consider (47), i.e. $v_t = \mathcal{L}v + F(v)$, over the domain $\mathcal{T}_{2\pi}$, which corresponds to $\sigma=0$ in (58). Then there exist a smooth one dimensional center manifold $\mathcal{M}^c$, given in the form $\mathcal{M}^c = \{\gamma f_1 + h(\gamma) : \gamma\in(-\delta,\delta)\}$ for some $\delta>0$. Here $f_1 = f_1(0) = \partial_\xi u_{\varepsilon,\kappa}/\|\partial_\xi u_{\varepsilon,\kappa}\|_{L^2(\mathcal{T}_{2\pi})}$ as before, and $h(\gamma)\in H^2(\mathcal{T}_{2\pi})\setminus\mathrm{span}\{f_1\}$ with $h(\gamma)=\mathcal{O}(\gamma^2)$. The function h is determined by the invariance condition. This means that $v(t)=\gamma(t)f_1 + h(\gamma(t))$ solves (47) if and only if $\gamma(t)$ solves the reduced equation $\dot\gamma f_1 = P^cF(\gamma f_1 + h(\gamma))$, where the projection P^c is precisely $P_1(0)$. Thus the right hand side of the reduced equation is given by

$$P^cF(\gamma f_1+h(\gamma)) = \int_{\mathcal{T}_{2\pi}}\Big\{\big(-3u_{\varepsilon,\kappa}(\xi)(\gamma^2 f_1(\xi))^2 + \mathcal{O}(\gamma^3)\big)+\mathcal{O}(\gamma^3)\big)\overline{f_1(\xi)}\Big\}\,\mathrm{d}\xi f_1.$$

Since $\mathcal{M}^c$ consists of fixed points we already know that $P^cF(\gamma f_1 + h(\gamma))=0$ and since the quadratic term in $P^cF(\gamma f_1+h(\gamma))=0$ is given by $\widetilde{K}_1(0,0)\gamma^2$ we can conclude $\widetilde{K}_1(0,0)=0$ without any calculation.

Lemma 28 *Let Π_n and Ψ_n be defined by (65). For all $\beta\in(0,1/\sqrt{3})$ there exist a $C(\beta)>0$ such that A4) holds with $C=C(\beta)$ for all $\varepsilon\in(0,\varepsilon_0)$.*

Proof. The estimate $\|e^{\Lambda^c_n(1-1/L^2)}\mathcal{R}_{1/L}\Psi_{n-1}-\Psi_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \le C(\beta)L^{-n}$ works the same way as in the complex case, see a) in Lemma 14. To show

$$\|e^{\Lambda^c_n(1-1/L^2)}\mathcal{R}_{1/L}\Theta\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \le C(\beta)L^{-1}\|\Theta\|_{\mathcal{B}_{L^{n-1}/\varepsilon}(2)} \text{ if } \Pi_{n-1}\Theta=0 \qquad (66)$$

we apply the mean value theorem in $\mathcal{B}_{L^n/\varepsilon}(2)$ and obtain $\|\Theta(\Sigma/L,\cdot)\|_{H^2(\mathcal{T}_{2\pi})} \leq |\Sigma| L^{-1} \, \|\Theta\|_{C^1(\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R},H^2(\mathcal{T}_{2\pi}))}$. Using $\mathcal{B}_{L^n/\varepsilon}(2) \hookrightarrow C^1(\mathcal{T}_{L^n/\varepsilon}, H^2(\mathcal{T}_{2\pi}))$ the rest of the proof of (66) follows analogously to the proof of b) in Lemma 14. □

Thus, by Theorem 1, we conclude that there exist $\delta(\beta), C(\beta) > 0$ and $L_0 > 1$ such that for $L \in [L_0, L_0]^2$ and $Z_0(1) = (U_0(1), V_0(1)) \in [\mathcal{B}_{1/\varepsilon}(2)]^2$ with $\|Z_0(1)\|_{\mathcal{B}_{1/\varepsilon}(2)} \leq \delta$ we have

$$\|U_n(1) - \alpha^*(\varepsilon,\kappa,Z_0(1))\Psi_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \leq CL^{-n}, \quad \|V_n(1)\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \leq CL^{-n}, \quad (67)$$

where $\alpha^* : \mathcal{P}\times[\mathcal{B}_{1/\varepsilon}(2)]^2 \to \mathbb{R}$ is continuous with $|\alpha^*(\varepsilon,\kappa,Z| \leq C(\beta)\|Z\|_{\mathcal{B}_{1/\varepsilon}(2)}$. Then $\mathcal{A}(\beta,\varepsilon,v_0) = \alpha^*(\varepsilon,\kappa,(\mathcal{R}^\varepsilon E^c D^{-1}v_0, \mathcal{R}^\varepsilon E^s D^{-1} v_0))$ is continuous and fulfills (62). To conclude (61), let $\widetilde{\Psi}_n(\Sigma,\cdot) = e^{-c_1\Sigma_1^2-(c_2/\varepsilon)\Sigma_2^2}\chi(\Sigma/L^n)f_1(0,\cdot)$ and $R_n = U_n(1) + L^{-n}V_n(1) - \alpha^*\widetilde{\Psi}_n$. Then $\|\widetilde{\Psi}_n - \Psi_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \leq CL^{-n}$ and hence $\|R_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \leq CL^{-n}$. The solution v of (47) fulfills

$$\begin{aligned} v(L^{2n}/\varepsilon^2) &= D(V^c(L^{2n}/\varepsilon^2) + V^s(L^{2n}/\varepsilon^2)) = D(\mathcal{R}^{1/\varepsilon}_{L^n}(U_n(1) + L^{-n}V_n(1))) \\ &= \alpha^* D\mathcal{R}^{1/\varepsilon}_{L^n}\widetilde{\Psi}_n + D\mathcal{R}^{1/\varepsilon}_{L^n}R_n, \end{aligned}$$

and since

$$\begin{aligned} (D\mathcal{R}^{1/\varepsilon}_{L^n}\widetilde{\Psi}_n)(x) &= \int_{T_1\times\mathbb{R}} e^{\mathrm{i}\sigma\cdot x}\mathcal{R}^{1/\varepsilon}_{L^n}\chi e^{-c_1\sigma_1^2-(c_2/\varepsilon)\sigma_2^2}\,\mathrm{d}\sigma f_1(0,\xi) \\ &= \varepsilon^{3/2}L^{-2n}\int_{\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R}} e^{\mathrm{i}\sigma\cdot(\varepsilon x_1/L^n,\sqrt{\varepsilon}\xi/L^n)}\chi(\sigma)e^{-c_1\sigma_1^2-(c_2/\varepsilon)\sigma_2^2}\,\mathrm{d}\sigma f_1(0,\xi) \\ &= \varepsilon^{3/2}L^{-2n}(\mathcal{R}^\varepsilon_{1/L^n}\frac{(4\pi)^{-1}}{\sqrt{c_1(c_2/\varepsilon)}}e^{-\frac{\xi^2}{4c_1}-\frac{x_2^2}{4(c_2/\varepsilon)}})f_1(0,\xi) + \mathcal{O}(e^{-L^n/\varepsilon}), \end{aligned}$$

we obtain in $L^\infty(\mathbb{R}^2)$, letting $t = L^{2n}/\varepsilon^2$,

$$\begin{aligned} &\|v(t,x) - \frac{(4\pi)^{-1}\alpha^*}{\sqrt{c_1c_2}\;t}e^{-\frac{\xi^2}{4c_1t}-\frac{x_2^2}{4c_2t}}f_1(0,\xi)\|_{L^\infty} = \|(D\mathcal{R}^{1/\varepsilon}_{L^n}R_n)(x)\|_{L^\infty} + \mathcal{O}(e^{-L^n/\varepsilon}) \\ &= \|\varepsilon^{3/2}L^{-2n}\int_{\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R}} e^{\mathrm{i}\sigma\cdot(\varepsilon x_1/L^n,\sqrt{\varepsilon}\xi/L^n)}R_n(\sigma,\xi)\,\mathrm{d}\sigma\|_{L^\infty(\mathbb{R}^2)} + \mathcal{O}(e^{-L^n/\varepsilon}) \\ &\leq \varepsilon^{-1/2}t^{-1}\sup_{\xi\in\mathcal{T}_{2\pi}}\int_{\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R}}|R_n(\sigma,\xi)|\,\mathrm{d}\sigma \\ &\leq C\varepsilon^{-1/2}t^{-1}\sup_{\xi\in\mathcal{T}_{2\pi}}\|R_n(\cdot,\xi)(1+|\cdot|^2)\|_{L^2(\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R})}\|(1+|\cdot|^2)^{-1}\|_{L^2(\mathcal{T}_{L^n/\varepsilon}\times\mathbb{R})} \\ &\leq C\varepsilon^{-1/2}t^{-1}\|R_n\|_{\mathcal{B}_{L^n/\varepsilon}(2)} \leq C\varepsilon^{-3/2}t^{-3/2}. \end{aligned}$$

This completes the proof of Theorem 20. □

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft DFG under Grant Mi 459/2-2. The author thanks Alexander Mielke and Guido Schneider for stimulating comments and discussions.

REFERENCES

[BK92] J. Bricmont and A. Kupianen. Renormalization group and the Ginzburg–Landau equation. *Comm. Math. Phys.*, 150:287–318, 1992.

[BKL94] J. Bricmont, A. Kupianen, and G. Lin. Renormalization group and Asymptotics of Solutions of Nonlinear Parabolic Equations. *Comm. Pure Appl. Math.*, 6:893–922, 1994.

[CE90] P. Collet and J.-P. Eckmann. *Instabilities and Fronts in Extended Systems.* Princeton University Press, 1990.

[EWW97] J.-P. Eckmann, C.E. Wayne, and P. Wittwer. Geometric stability analysis of periodic solutions of the Swift–Hohenberg equations. *Comm. Math. Phys.*, 190:173–211, 1997.

[Gal94] Th. Gallay. Local stability of critical fronts in parabolic partial differential equations. *Nonlinearity*, 7:741–764, 1994.

[Man92] Paul Manneville. *Dissipative Structures and Weak Turbulence.* Academic Press, 1992.

[Mie95] Alexander Mielke. A new approach to sideband instabilities using the principle of reduced instability. In A. Doelman and A. van Harten, editors, *Nonlinear dynamics and pattern formation in the natural environment*, pages 206–222. Longman UK, 1995.

[Mie97a] Alexander Mielke. Instability and stability of rolls in the Swift–Hohenberg equation. *Comm. Math. Phys.*, 189:829–853, 1997.

[Mie97b] Alexander Mielke. Mathematical analysis of sideband instabilities with application to Rayleigh–Bénard convection. *J. Nonlinear Science*, 7:57–99, 1997.

[RS78] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV.* Academic Press, 1978.

[Sch94] Guido Schneider. Error estimates for the Ginzburg–Landau approximation. *J. Appl. Math. Phys. (ZAMP)*, 45:433–457, 1994.

[Sch96] Guido Schneider. Diffusive Stability of Spatial Periodic Solutions of the Swift–Hohenberg equation. *Comm. Math. Phys.*, 178:679–702, 1996.

[Sch97] Guido Schneider. Nonlinear Stability of Taylor–vortices in infinite cylinders, Habilitation thesis, Universität Hannover, 1997. Submitted to Arch. Rat. Mech. Anal. 1997.

[Sch98] Guido Schneider. Nonlinear diffusive stability of spatially periodic solutions – Abstract theorem and higher space dimensions. In E. Yanagida Y. Nishiura, I. Takagi, editor, *Proceedings International Conference on Asymptotics of Nonlinear Diffusive Systems, Sendai, Japan, 28.7-1.8*, pages 159–167. Tohoku Mathematical Publications, Nr. 8, 1998.

[vH91] A. van Harten. On the validity of Ginzburg–Landau's equation. *J. Nonlinear Science*, 1:397–422, 1991.

[Way97] E.C. Wayne. Invariant manifolds for parabolic partial differential equations on unbounded domains. *Arch. Rat. Mech. Anal.*, 138:279–306, 1997.

[Wei81] Fred B. Weissler. Existence and Nonexistence of global solutions for a semilinear heat equation. *Israel Journal of Mathematics*, 38:29–40, 1981.