The mathematics of light pulses in dispersive media

Guido Schneider and Hannes Uecker

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Abstract

In this survey article we discuss a number of mathematical questions related to the behavior of light pulses in dispersive media. Mathematically, we analyze the dynamics of modulating pulse solutions of a nonlinear wave equation. Such solutions consist of a pulse-like envelope advancing in the laboratory frame and modulating an underlying wave-train. We explain the role of the Nonlinear Schrödinger equation in the description of pulses with the same carrier wave. We show that there is almost no interaction of well prepared pulses with different carrier waves. Finally, we discuss the question: Do modulating pulse solutions exist for all times? We discuss the relevance of the presented results for fiber optics and photonic crystals.

1 Introduction

The transport of information over long distances through optical fibers is one of the key technologies of the post-industrial society. Information is encoded digitally by ones and zeroes, i.e., by sending a light pulse through the optical fiber or not. Physically such a light pulse is a complicated structure. It consists of an underlying electromagnetic carrier wave moving with phase velocity c_p and of a pulse-like envelope moving with group velocity c_g and modulating the underlying carrier wave, see Fig. 1.

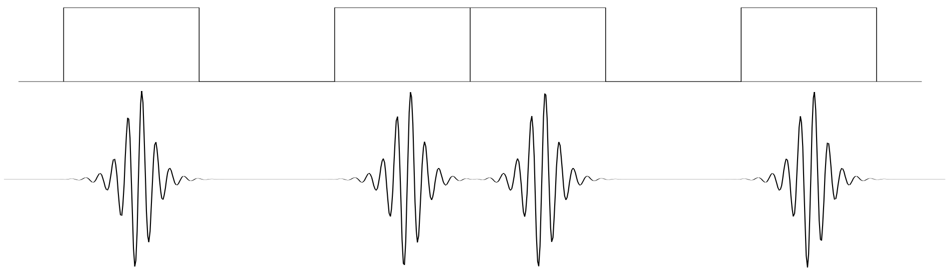

Figure 1: 0s and 1s are encoded physically by sending a light pulse or not; thus, for instance, the above electromagnetic wave encodes the sequence 101101.

The analysis of the evolution of such a light pulse is a nontrivial task. The system shows dispersion and (weak) dissipation, i.e., harmonic waves with different wave numbers travel at different speeds and energy is lost in a wave number dependent way. Moreover, there is a nonlinear response by the optical fiber. Thus, at a first glance it looks like a typical example for the application of numerical methods. However, a direct simulation of Maxwell's equations which describe these electromagnetic waves is beyond any present possibilities. This can be seen as follows: The wavelength of the underlying carrier wave is around 10^{-7} m. Resolving this structure in a fiber of 10 km =10^4 m gives in uniform one dimensional spatial discretization 10^{11} points, not to speak about the transverse directions and the temporal discretization. Therefore, before making any numerical investigations, the system has to be analyzed and simpler, numerically more suitable, models have to be derived. Interestingly, by using only a pencil and a sheet of paper a lot of things can be concluded without using any computer. This will be the subject of this survey article.

Using multiple scaling analysis we derive a formula for the optimal shape of the envelope of the pulse. Optimal means that it is more or less of a permanent form, i.e., in the ideal case the pulse is time periodic in a frame moving with the group velocity of the envelope. We will explain that the dynamics of pulses with the same carrier wave, i.e. with the same wave length, can be described by the dynamics of the envelope alone which is governed by a Nonlinear Schrödinger equation (NLS equation). The NLS equation is a universal nonlinear partial differential equation. Universal here means that additional to nonlinear optics it appears in the above sense in many contexts, for instance water waves, plasma physics, and lattice vibrations. Moreover, the NLS equation is a completely integrable Hamiltonian system. As a result, the NLS equation can (in principle) be solved explicitly. The method is called the inverse scattering scheme. In particular, the NLS equation has explicit so-called N-soliton solutions. These are special localized waves with N humps, $N \in \mathbb{N}$, where the humps interact asymptotically in a very unexpected way which is similar to the superposition principle in linear equations.

We will also explain that pulses with different carrier waves, i.e. different wave lengths, do not interact in lowest order. This fact allows to increase the information rate through the fiber by using different bands, i.e. a number of different carrier waves.

Photonic crystals play an important role in nanotechnological devices. One of the ultimate goals is to use them as optical storage. We will explain the possibility of standing light pulses in photonic crystals.

Finally, we will explain that the formula for the pulses of permanent form is correct to any polynomial order in the amplitude parameter, but that exponentially small terms will hinder the existence of a modulating pulse of permanent form with finite energy. However, it turns out that such modulating pulses of permanent form exist with infinite energy and exponentially small tails.

The paper starts with a short description of the physical background in order to motivate the description of nonlinear optics by nonlinear wave equations. We concentrate on rigorous mathematical results and skip in our presentation almost all purely formal results. We use ideas from finite and infinite dimensional dynamical systems theory, from perturbation theory and from a functional analytic treatment of partial differential equations over unbounded domains in Sobolev spaces.

The subsequent methods and results are not restricted to models from nonlinear optics. They essentially apply to all equations for which a NLS equation can be derived as an amplitude equation. For systems with (significant) dissipation the role of the NLS equation is taken by other but related amplitude equations, for instance of Ginzburg–Landau type. We refrain from any details in case of dissipation and refer to the literature, for instance [Sch99, Mie02] and the references therein.

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2 Physical background

Light pulses are electromagnetic waves and described by Maxwell's equations, namely

$$\nabla\cdot\vec{B}=0 \quad , \quad \nabla\times\vec{E}+\partial_t\vec{B}=0,$$
$$\nabla\cdot\vec{D}=\rho \quad , \quad \nabla\times\vec{H}-\partial_t\vec{D}=\vec{J},$$

with $\vec{D}=\varepsilon_0\vec{E}+\vec{P}$ and $\vec{H}=\vec{B}/\mu_0-\vec{M}$. Here $\vec{E}=\vec{E}(\vec{x},t)$ is the electric field, $\vec{x}=(x,y,z)\in\mathbb{R}^3$, $t\in\mathbb{R}$ is the time, ε_0 the permittivity of vacuum, $\vec{P}$ the material polarization, $\vec{B}$ the magnetic flux, μ_0 the magnetic permeability of vacuum, $\vec{M}$ the material magnetization, ρ the charge density and $\vec{J}$ the electric current. These equations have to be closed with constitutive laws $\vec{P}=\vec{P}(\vec{E},\vec{H})$ and $\vec{M}=\vec{M}(\vec{E},\vec{H})$ describing the behavior of the medium. Depending on this choice there are linear and nonlinear, instantaneous and history dependent, dispersive and dissipative models.

In typical optical fibers there is no magnetization $\vec{M}$, no charge density ρ, and no electric current $\vec{J}$, and therefore, using $\nabla\times\nabla\vec{E}=\Delta\vec{E}-\nabla(\nabla\cdot\vec{E})$, Maxwell's equations for light in nonlinear optical material are given by

$$\triangle\vec{E}-\nabla(\nabla\cdot\vec{E})-\partial_t^2\vec{E}=\partial_t^2\vec{P}, \tag{1}$$

where we scaled the speed of light in vacuum and the dielectric constant to 1.

The constitutive law for the polarization $\vec{P}=\vec{P}_{\mathrm{l}}+\vec{P}_{\mathrm{nl}}$ splits into a linear and a nonlinear part, which in general both depend on the history of the electric field. In centrosymmetric isotropic bulk material, the constitutive law for the linear response $\vec{P}_{\mathrm{l}}$ is given by an instantaneous part $\vec{P}_{\mathrm{l}}^{\mathrm{i}}(\vec{x},t)=\vec{P}_{\mathrm{l}}^{\mathrm{i}}(\vec{x},\vec{E}(\vec{x},t))$ and a history dependent term

$$\vec{P}_{\mathrm{l}}^{\mathrm{h}}(\vec{x},t)=(\chi_1 *_t \vec{E})(\vec{x},t)=\int_{-\infty}^{\infty}\chi_1(t-\tau)\vec{E}(\vec{x},\tau)\,\mathrm{d}\tau, \tag{2}$$

where χ_1 in (2) is a scalar function, independent of $\vec{x}$, with $\chi_1(t)=0$ for $t<0$ due to causality, and similar for the nonlinear polarization. In case of optical fibers χ_1 does also depend on the transverse directions y,z, and in case of photonic crystals also on the longitudinal direction x.

In the simplest case $\vec{E}$ is linearly polarized and only depends on x, i.e.,

$$\vec{E}(\vec{x},t)=u(x,t)\hat{k} \quad \text{with} \quad \|\hat{k}\|_{\mathbb{R}^3}=1, \quad (1,0,0)\cdot\hat{k}=0. \tag{3}$$

Then, (1) simplifies to

$$\partial_t^2 u(x,t)=\partial_x^2 u(x,t)-\partial_t^2 p_{\mathrm{l}}(x,t)-\partial_t^2 p_{\mathrm{nl}}(x,t), \tag{4}$$

with $u(x,t),p_{\mathrm{l}}(x,t),p_{\mathrm{nl}}(x,t)\in\mathbb{R}$ such that $\vec{P}_{\mathrm{l}}(t,\vec{x})=p_{\mathrm{l}}(x,t)\hat{k}$, $\vec{P}_{\mathrm{nl}}(t,\vec{x})=p_{\mathrm{nl}}(x,t)\hat{k}$. The symmetry $(y,z)\mapsto-(y,z)$, which is present in most optical materials, prevents the occurrence of even terms in p with respect to u, thus, in general p_{nl} starts with cubic terms.

Due to the fact that we are mainly interested in the underlying mathematical structures, throughout the rest of the paper we choose

$$\partial_t^2 p(x,t) = u(x,t) - u^3(x,t)$$

as constitutive law, thus the toy problem for this paper is

$$\partial_t^2 u = \partial_x^2 u - u + u^3. \tag{5}$$

This choice is rather unphysical; however, it delivers a system with all properties in which we are interested, namely dispersive and nonlinear behavior. We refer to [SU03] for a mathematical discussion of a physically more realistic choice which includes dissipation and history dependence additional to dispersion and nonlinearity. Dissipation, i.e., wave number dependent damping, is usually very weak in the so-called transmission windows of optical fibers. However, it may become important over very long scales, while history dependence does not alter the analysis in an essential way.

3 Single pulses I

The description of light pulses, i.e. here of localized solutions of (5), is based on the derivation of a NLS equation by formal perturbation analysis. Therefore we introduce a small perturbation parameter

$$0 < \varepsilon \ll 1$$

which will relate the amplitude with the spatial and temporal scales. We seek $\mathcal{O}(\varepsilon)$-amplitude solutions which are slow spatial and temporal modulations of an underlying wave train $\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$, where k_0 and ω_0 are related by the dispersion relation $\omega_0^2 = k_0^2 + 1$ of the linearized problem $\partial_t^2 u = \partial_x^2 u - u$. Thus we substitute the ansatz

$$u_A(x,t) = \varepsilon(A(X,T)\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + \mathrm{c.c.}) + \mathcal{O}(\varepsilon^2), \tag{6}$$

into (5), where $X = \varepsilon(x - ct)$ with c to be determined, where $T = \varepsilon^2 t$, where c.c. means complex conjugate, and where $A(X,T)$ is a complex-valued amplitude. We sort the coefficients of the carrier wave $\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$ with respect to powers of ε and obtain

$$\begin{aligned} &\mathcal{O}(\varepsilon^1): \quad -\omega_0^2 A = -(k_0^2+1)A, \quad \text{dispersion relation}, \\ &\mathcal{O}(\varepsilon^2): \quad 2\mathrm{i}c\omega_0 A_X = 2\mathrm{i}k_0 A_X \;\Rightarrow\; c = k_0/\omega_0 = \omega'(k_0) =: c_\mathrm{g}', \quad \text{linear group velocity}, \end{aligned}$$

while at $\mathcal{O}(\varepsilon^3 \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)})$ we find that A should satisfy the NLS equation

$$2\mathrm{i}\omega_0 \partial_T A + (1 - (c_\mathrm{g}')^2)\partial_X^2 A + 3|A|^2 A = 0. \tag{7}$$

In fact, the Fourier transform of the ansatz (6) is strongly localized around k_0. Therefore, only the local shape of $\omega_{1,2} = \pm\sqrt{k^2+1}$ near k_0 is important to determine $c = c_\mathrm{g}'$ in (6) and the coefficients of the linear terms in (7), see Fig. 2.

Equation (7) has a four dimensional family of solutions of the form

$$A(X,T) = \tilde{A}(X - vT - X_0)\mathrm{e}^{\mathrm{i}(\tilde{v}X - \gamma_0 T + \phi_0)}, \quad \tilde{v} = (\omega_0 v)/(1 - (c_\mathrm{g}')^2),$$

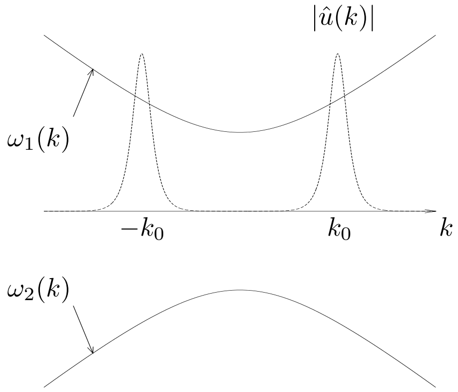

Figure 2: The two curves of eigenvalues $\omega_{1,2} = \pm i\sqrt{1+k^2}$. The Fourier transform of (6) is concentrated in an $\mathcal{O}(\varepsilon)$ neighborhood around $\pm k_0$, Therefore, the dynamics of (6) is determined by the expansion of ω_1 at k_0. $\omega_1'(k_0)$ gives the linear group velocity $c_{\rm g}'$, and the group velocity dispersion $\omega_1''(k_0)$ occurs as coefficient in the NLS equation. The concentration of Fourier modes $\hat{u}(k)$ is respected by the nonlinear interaction, i.e. convolution in Fourier space.

in which the real-valued function $\tilde{A}$ satisfies the second-order ordinary differential equation

$$\partial_X^2 \tilde{A} = C_1 \tilde{A} - C_2 \tilde{A}^3, \tag{8}$$

where

$$C_1 = \tilde{v}^2 - \frac{2\gamma_0\omega_0}{1-(c_{\rm g}')^2}, \quad C_2 = \frac{3}{(1-(c_{\rm g}')^2)}.$$

Since $c_{\rm g}' < 1$, we always have $C_2 > 0$, and for $C_1 > 0$ there exist the two explicit solutions

$$\tilde{A}_{\rm pulse}(X) = \pm\left(\frac{2C_1}{C_2}\right)^{1/2} \operatorname{sech}\,(C_1^{1/2}X) \tag{9}$$

to (8). These are called homoclinic since they connect the origin $(0,0)$ as a fixed point of the first order formulation of (8) with itself, see the left panel of Fig. 3, while solutions which connect different fixed points of a dynamical system are called heteroclinic.

The derivation of the NLS equation (7) was only formal in the sense that we simply ignored terms that are higher order w.r.t. ε or appeared at a different wave-number. They are contained in the residual, i.e.

$$\operatorname{Res}(u) := -\partial_t^2 u + \partial_x^2 u - u + u^3$$

contains the terms which do not cancel after inserting an approximation into (5). If $\operatorname{Res}(u)=0$, then u is an exact solution of (5). It is important to note that due to a possible 'accumulation of errors' the smallness of the residual alone does not imply the so-called validity of the approximation where validity means that there are solutions of (5) which behave as predicted by the NLS equation on the relevant $\mathcal{O}(1/\varepsilon^2)$ time–scale.

However, there are a number of mathematical validity results for (5), see [Kal88, KSM92, Sch98] and also §4. The above procedure thus identifies modulating pulse solutions of (5) which are described by the approximate formula

$$\begin{aligned} u_{\rm pulse}(x,t) &= \varepsilon\left(\tilde{A}_{\rm pulse}(X - vT - X_0)\mathrm{e}^{\mathrm{i}(\tilde{v}X-\gamma_0 T+\phi_0)}\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + \text{c.c.}\right) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon\left(\tilde{A}_{\rm pulse}(\varepsilon(x - c_{\rm g}'t - x_0 - \varepsilon vt))\mathrm{e}^{\mathrm{i}((k_0+\varepsilon\tilde{v})x-(\omega_0+\varepsilon^2\gamma_0)t+\phi_0)} + \text{c.c.}\right) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

accurately over time-scales of order $\mathcal{O}(1/\varepsilon^2)$. In particular, for w.l.o.g. $v = 0$, $x_0 = 0$ and $\phi_0 = 0$ we have

$$u_{\text{pulse}}(x,t) = \varepsilon \left(\tilde{A}_{\text{pulse}}(\varepsilon(x - c_{\text{g}}'t))\mathrm{e}^{\mathrm{i}k_0(x-(c_p'+\gamma_1\varepsilon^2)t)} + \text{c.c.} \right) + \mathcal{O}(\varepsilon^2),$$

where $c_{\text{p}}' = (1+k_0^2)^{1/2}/k_0$ is the linear phase velocity and $\gamma_1 = \gamma_0/k_0$.

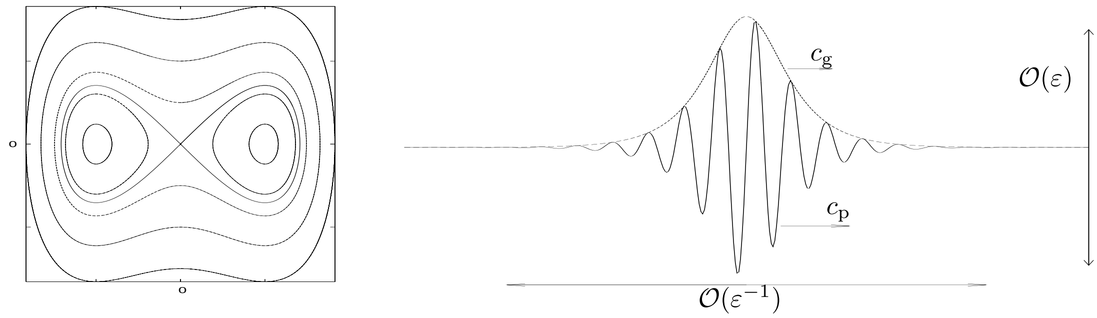

Figure 3: The $(\tilde{A}, \partial_X \tilde{A})$-phase portrait for (8), and a modulating pulse for (5) described by the NLS equation.

For the transport of information the global existence of modulating pulse solutions would be an important goal, i.e., we investigate if there are exact solutions to equation (5) of the form

$$u(x,t) = v(x - c_{\text{g}}t, k_0(x - c_{\text{p}}t)),$$

where v is 2π-periodic in its second argument with

$$\lim_{\xi\to\pm\infty} v(\xi, y) = 0.$$

This question will be discussed in detail in §7. As a first result we note [GS01] that such solutions can be computed approximately to any polynomial order in ε by extending the ansatz (6) by higher order terms and applying a small correction to the linear group speed, i.e., using $c_g = c_g' + \mathcal{O}(\varepsilon^2)$, see the right panel of Fig. 3. In other words, the following Lemma allows to find modulating pulse solutions which make the residual arbitrarily small.

To measure the residual we use Sobolev spaces [Ada75] $H^s = H^s(\mathbb{R}, \mathbb{C})$. For simplicity we restrict to $s \in \mathbb{N}$. Then H^s consists of all functions $f : \mathbb{R} \to \mathbb{C}$ which together with their distributional derivatives up to order s are square integrable, equipped with the norm

$$\|u\|_{H^s} = \sum_{j=0}^{s} \|\partial_x^j u\|_{L^2}.$$

In our spatially one-dimensional setting, H^s is a subset of the space of uniformly bounded and m times continously differentiable functions $C_b^m(\mathbb{R}, \mathbb{C})$ if $s > m + 1/2$, $m \in \mathbb{N}$, and the embedding is continuous, i.e., $\|u\|_{C_b^m} \leq C\|u\|_{H^s}$. These so–called Sobolev embeddings can be used to show that nonlinear terms such as u^3 are well-defined and continuous mappings from $H^s \to H^s$ if $s > 1/2$.

Lemma 3.1 *Let $s \geq 2$, $k_0 > 0$, $n \in \mathbb{N}$, and $\gamma_1 < 0$. For sufficiently small $\varepsilon > 0$ there exists a two-dimensional family, parameterized by envelope shift $x_0 \in \mathbb{R}$ and phase shift $\phi \in [0, 2\pi)$, of approximate modulating pulse solutions to (5) of the form*

$$u(x,t) = \varepsilon v_{k_0}(x - c_{\text{g}}t - x_0, k_0(x - c_{\text{p}}t) + \phi),$$

where v is 2π-periodic in its second argument, $c_{\rm p} = c'_{\rm p} + \gamma_1 \varepsilon^2$, $c_{\rm g} = 1/c_{\rm p}$, and where, for some $r > 0$, $\varepsilon v_{k_0}(\xi, y) = \varepsilon \tilde{A}_{\rm pulse}(\varepsilon\xi) {\rm e}^{{\rm i}y} + \mathcal{O}(\varepsilon^3 {\rm e}^{-r\varepsilon|\xi|}) + c.c.$, and

$$\|{\rm Res}(\varepsilon v_{k_0})\|_{H^s} \leq C\varepsilon^{n+1/2}.$$

4 Interaction of pulses with the same frequency

By the derivation of the NLS equation for the nonlinear wave equation (5) not only modulating pulse solutions of the nonlinear wave equation are identified. The complete dynamics known for the NLS equation can also be expected to be found approximately in the nonlinear wave equation.

We refer to the excellent textbooks [AS81, DJ89, SS99] about the various dynamics known for the NLS equation. For our purposes the fact is essential that the NLS equation is a completely integrable Hamiltonian system. Hamiltonian means that (7) can be written as $\partial_T A = J\delta H(A)$ where $J = -{\rm i}/(2\omega_0)$ is a skew symmetric operator and δ denotes the variational derivative of the Hamiltonian

$$H(A) = \int \left[\frac{1 - c_g'^2}{2} |\partial_X A|^2 - \frac{3}{4}|A|^4 \right] dX.$$

An immediate consequence is that the Hamiltonian $H(A)$ is conserved by the flow of (7), but in fact various further properties follow. Completely integrable means here that there are infinitely many independent conserved quantities for (7), and that there exists a transformation which is called inverse scattering scheme and which uses these conserved quantities to map (7) to a linear system which can (in principle) be solved explicitly. As a result, there are explicit though somewhat lengthy formulae (similar to (9)) for so-called N-soliton solutions of the NLS equation. In general, N-solitons are localized solutions which consist of N humps and which for $t \to \pm\infty$ asymptote to N solitons with different speeds. In particular, the individual humps interact in a very special way which is rather unexpected in a nonlinear equation: asymptotically for $t \to \pm\infty$ the interaction preserves the shapes and speeds of the individual humps, and only alters the relative positions. Thus, the humps are similar to elastic particles, and this motivates the name soliton. The change of position after interaction is $\mathcal{O}(1)$ in the NLS equation and is called a pulse shift. Formally, the N-solitons yield modulating N-pulse solutions for the nonlinear wave equation (5) with $\mathcal{O}(1/\varepsilon)$ pulse-shifts after interaction, see Fig. 4.

However, as already said in §3, the formal derivation of the NLS equation for the nonlinear wave equation (5) alone does not imply that the dynamics found in the NLS equation can also be found in the nonlinear wave equation (5): There are amplitude equations derived in a formally correct way by multiscale analysis which do not reflect the dynamics of the original system, see, e.g., [Sch95]. We now discuss the validity of the approximation, that is, how well solutions of the nonlinear wave equation (5) can be approximated via the solutions of the NLS equation.

Let $A \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$ be a solution of the NLS equation (7) with $s_A \geq 1$ defined below. Then

$$\varepsilon\psi_c(x,t) = \varepsilon A(\varepsilon(x - c'_{\rm g}t), \varepsilon^2 t) {\rm e}^{{\rm i}(k_0 x - \omega_0 t)} + {\rm c.c.} \tag{10}$$

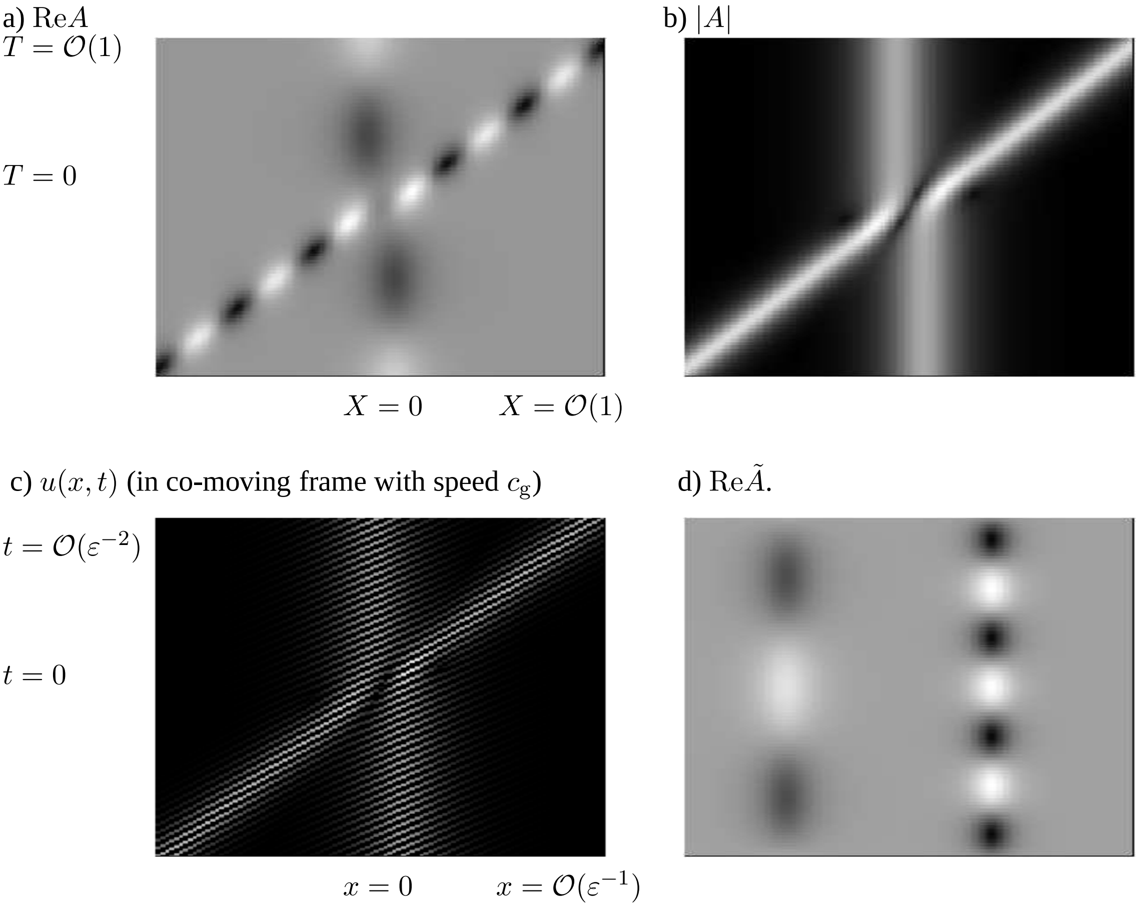

Figure 4: A 2-soliton A with interaction in the NLS equation and the associated modulating 2-pulse solution in the nonlinear wave equation (a)–(c), with interaction at $(X,T) = (0,0)$, and a time periodic 2-soliton $\tilde{A}$ in the NLS equation (d). For graphical reasons, black has been assigned to $u = 0$ in c).

defines a formal approximation of the solutions u of the nonlinear wave equation (5). For our purposes it turns out to be advantageous to consider the extended approximation

$$\varepsilon\psi(x,t) = \varepsilon A(\varepsilon(x - c_{\rm g}'t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + \varepsilon^3 A_3(\varepsilon(x - c_{\rm g}'t), \varepsilon^2 t)\mathrm{e}^{3\mathrm{i}(k_0x-\omega_0t)} + \text{c.c.} \quad (11)$$

where $A_3 = A^3/(9\omega_0^2 - 9k_0^2 - 1)$ is also in $C([0,T_0], H^{s_A}(\mathbb{R},\mathbb{C}))$, if $s_A \geq 1$. In summary, if $s_A > m + 1/2$, then there exist C, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned}\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\psi(\cdot,t) - \varepsilon\psi_c(\cdot,t)\|_{C_b^m} &\leq C \sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon^3 A_3(\varepsilon\cdot, \varepsilon^2 t)\mathrm{e}^{3\mathrm{i}(k_0x-\omega_0t)} + \text{c.c.}\|_{C_b^m} \\ &\leq C\varepsilon^3 \sup_{t\in[0,T_0/\varepsilon^2]} \|A_3(\varepsilon\cdot, \varepsilon^2 t)\|_{C_b^m} \leq C\varepsilon^3 \sup_{T\in[0,T_0]} \|A_3(\cdot,T)\|_{C_b^m} \\ &\leq C\varepsilon^3 \sup_{T\in[0,T_0]} \|A_3(\cdot,T)\|_{H^{s_A}}\end{aligned}$$

due to Sobolev's embedding theorem. As a consequence, if u can be approximated by $\varepsilon\psi$ up to an error of order $\mathcal{O}(\varepsilon^\beta)$ then it can also be approximated up to an error of order $\mathcal{O}(\varepsilon^{\min(3,\beta)})$ by ψ_c. In detail this means that

$$\|u - \varepsilon\psi_c\|_{C_b^m} \leq \|u - \varepsilon\psi\|_{C_b^m} + \|\varepsilon\psi - \varepsilon\psi_c\|_{C_b^m} \leq C\varepsilon^\beta + C\varepsilon^3 \leq 2C\varepsilon^{\min(3,\beta)}.$$

In order to estimate the difference $u - \varepsilon\psi =: \varepsilon^{3/2} R$ we derive an equation for R and estimate R. In order to do so we need estimates for the residual $\mathrm{Res}(\varepsilon\psi)$. By the choice of A and A_3 all terms up to formal order $\mathcal{O}(\varepsilon^4)$ are eliminated in the residual. Therefore there exist C, $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|\mathrm{Res}(\varepsilon\psi(t))\|_{H^s} \leq C\varepsilon^{7/2}.$$

The loss of $\varepsilon^{1/2}$ comes from the scaling properties of the L^2-norm.

With $u = \varepsilon\psi + \varepsilon^{3/2} R$ we find

$$\partial_t^2 (\varepsilon\psi + \varepsilon^{3/2} R) \;=\; \partial_x^2 (\varepsilon\psi + \varepsilon^{3/2} R) - (\varepsilon\psi + \varepsilon^{3/2} R) - (\varepsilon\psi + \varepsilon^{3/2} R)^3$$

such that R satisfies

$$\partial_t^2 R = \partial_x^2 R - R + f \tag{12}$$

with

$$f = -3\varepsilon^2 \psi^2 R - 3\varepsilon^{5/2} \psi R^2 - \varepsilon^3 R^3 + \varepsilon^{-3/2} \mathrm{Res}(\varepsilon\psi).$$

Thus,

$$\|f\|_{H^s} \leq C_1 \varepsilon^2 \|R\|_{H^s} + C_2(C_R) \varepsilon^{5/2} \|R\|_{H^s}^2 + C_3 \varepsilon^2 \tag{13}$$

as long as $\|R(t)\|_{H^s} \leq C_R$ with a constant C_R determined below, constants C_1, C_3 independent of C_R and $\varepsilon \in (0,1)$ and a constant C_2 depending on C_R but independent of $\varepsilon \in (0,1)$.

The equation for R is solved here for simplicity with zero initial conditions. We use energy estimates and define the energy

$$E(R) = \sum_{j=0}^{s} \int_{-\infty}^{\infty} (\partial_t \partial_x^j R)^2 + (\partial_x^{j+1} R)^2 + (\partial_x^j R)^2 dx.$$

For $j = 0$ and $\int = \int_{-\infty}^{\infty}$ we obtain

$$\begin{aligned} \frac{1}{2}\partial_t \int (\partial_t R)^2 + (\partial_x R)^2 + R^2 dx &= \int (\partial_t R)(\partial_t^2 R) + (\partial_x R)(\partial_t \partial_x R) + R(\partial_t R) dx \\ &= \int [(\partial_t R)(\partial_x^2 R) - (\partial_t R) R + (\partial_t R) f + (\partial_x R)(\partial_t \partial_x R) + R(\partial_t R)] dx \\ &= \int (\partial_t R) f dx \end{aligned}$$

which can be estimated with the Cauchy Schwarz inequality by

$$\begin{aligned} \left| \int (\partial_t R) f dx \right| &\leq \|\partial_t R\|_{L^2} \|f\|_{L^2} \leq \|\partial_t R\|_{L^2} (C_1 \varepsilon^2 \|R\|_{H^s} + C_2(C_R) \varepsilon^{5/2} \|R\|_{H^s}^2 + C_3 \varepsilon^2) \\ &\leq C_1 \varepsilon^2 E(R) + C_2(C_R) \varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2 E(R)^{1/2} \\ &\leq (C_1 + C_3) \varepsilon^2 E(R) + C_2(C_R) \varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2. \end{aligned}$$

Since exactly the same estimates hold for $j = 1, \ldots, s$ we finally find

$$\partial_t E(R) \leq (C_1 + C_3) \varepsilon^2 E(R) + C_2(C_R) \varepsilon^{5/2} E(R)^{3/2} + C_3 \varepsilon^2. \tag{14}$$

Now assume that $\varepsilon^{1/2}C_2(C_R)E^{1/2}(R) \leq 1$. Then, for $0 \leq t \leq T_0/\varepsilon^{-2}$,

$$E(R(t)) \leq C_3 \mathrm{e}^{(C_1+C_3+1)T_0} =: C_R^2 \tag{15}$$

by Gronwall's lemma which translates differential inequalities like (14) into pointwise estimates like (15), see, e.g., [Hen81, Lemma 7.1.1] for a very general version.

Choosing $\varepsilon_0 > 0$ so small that

$$\varepsilon_0^{1/2} C_2(C_R) C_R^{1/2} \leq 1 \tag{16}$$

we are done. In detail, to a given $C_R = C_R(T_0, C_1, C_3)$ defined in (15) we have a $C_2(C_R)$ by (13) and to this C_2 we have an $\varepsilon_0 > 0$ by (16). Hence, there are solutions u of (5) which behave for all $t \in [0, T_0/\varepsilon^2]$ as predicted by the NLS equation (7).

Theorem 4.1 *Fix $s_A \geq s+3 \geq 4$. Let $A \in C([0,T_0], H^{s_A})$ be a solution of the NLS equation (7). There exist C, $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (5) such that $\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\cdot,t) - \varepsilon\psi(\cdot,t)\|_{H^s} \leq C\varepsilon^{3/2}$.*

Remark 4.2 The time scale $\mathcal{O}(T_0/\varepsilon^2)$ is necessary to describe non-trivial dynamics. The error of order $\mathcal{O}(\varepsilon^{3/2})$ is much smaller than the approximation which is of order $\mathcal{O}(\varepsilon)$. Adding higher order terms, like A_3, to the approximation $\varepsilon\psi$ allows to decrease the magnitude of the residual further, in particular we can obtain $\mathcal{O}(\varepsilon^{11/2})$. This results in an error of order $\mathcal{O}(\varepsilon^{7/2})$ instead of $\mathcal{O}(\varepsilon^{3/2})$. However, the time scale $\mathcal{O}(1/\varepsilon^2)$ of validity in general can not be extended.

As a consequence of Theorem 4.1 modulating pulse solutions for the nonlinear wave equation (5) with the same carrier wave interact as predicted by the NLS equation, i.e., we have approximately the persistence of the modulating pulse solutions after the nonlinear interaction and $\mathcal{O}(1/\varepsilon)$ pulse-shifts in the nonlinear wave equation (5). For the transport of information through optical fibers the interaction of pulses is in general undesirable. However, even if the envelopes are in a very general form, like in real world technical devices, the NLS equation can now be used to compute numerically how far the modulating pulse solutions have to be separated such that there is no nonlinear interaction during the journey through the fiber.

5 Interaction of pulses with different frequencies

The information rate through the fiber can be increased by using different bands, i.e., different basic wave numbers, cf. [Ace00]. As explained in Remark 5.1 below (after fixing some notation), there is a simple argument why wave packets with different wave numbers do not interact in lowest order w.r.t. ε. Moreover, for pulses from Lemma 3.1 the argument can be refined, and in this section we explain that there is almost no interaction of such pulses associated to different carrier waves by giving an $\mathcal{O}(\varepsilon)$-bound for the possible shift of the envelope resulting from the interaction. For general wave packets the shift of the envelope will be in general $\mathcal{O}(1)$. Thus, it is advantageous to use well-prepared pulses for the transport of information.

We introduce subscripts A and B to indicate the wave numbers $k_A \neq k_B$ of each pulse, the associated group velocities $c_{g,A}$ and $c_{g,B}$, the envelope shifts x_A and x_B and so on. If the two pulses are separated initially, and, say, $x_A > x_B$ and $k_A < k_B$ such that $c_{g,A} < c_{g,B}$ and the faster pulse is in front, then, since the pulses are exponentially localized, it is natural to expect that the dynamics of the two pulses can be described by the sum of the two individual pulses, at least on the natural $\mathcal{O}(1/\varepsilon^2)$ time–scale. However, if the two pulses are, say, $\mathcal{O}(1/\varepsilon)$ separated initially, with $x_A > x_B$ and $k_A > k_B$, then, since the group velocities differ by $\mathcal{O}(1)$, the two pulses must interact on an $\mathcal{O}(1/\varepsilon^2)$ time–scale. Clearly this is the mathematically more interesting case.

For notational simplicity we assume that $\phi_A = \phi_B = 0$ and thus study the interaction of

$$\varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A(x - c_{p,A}t)) \quad \text{and} \quad \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B(x - c_{p,B}t)), \quad k_A \neq k_B.$$

We prove that the form of the pulses is almost preserved and that the interaction mainly leads to phase-shifts $\varepsilon\Omega_A$ and $\varepsilon\Omega_B$ with $\Omega_A, \Omega_B \in \mathbb{R}$ bounded independent of ε.

Remark 5.1 That the amplitude equations for εv_{k_A} and εv_{k_B} decouple in lowest order can be seen as follows. Going into the scaling of the envelope, εv_{k_A} and εv_{k_B} have an amplitude and a width of order $\mathcal{O}(1)$. But since the group velocities differ by an order $1/\varepsilon$ in this scaling the interaction time of εv_{k_A} and εv_{k_B} is only $\mathcal{O}(\varepsilon)$. Thus, the influence of a term $v_{k_A} v_{k_B}$ on the dynamics of v_{k_A} and v_{k_B} is $\mathcal{O}(\varepsilon)$ in the NLS scaling and therefore in lowest order the evolution equations for v_{k_A} and v_{k_B} decouple. This argument is not restricted to v_{k_A} and v_{k_B}. It holds for all wave-packets. Moreover, this property can be observed in a number of problems. For modulating pulse solutions such a statement can be found for instance in [PW96] where it has been shown that the two NLS equations for counter-propagating waves decouple.

The estimates from [PW96] still only transfer into $\mathcal{O}(1)$-bounds for the possible envelope shifts of the pulses for $\varepsilon \to 0$. However, for well-prepared pulses, i.e., $n \geq 5$ in Lemma 3.1, by extracting explicitly the phase shift of the underlying carrier wave we can refine the bound on the envelope shifts to $\mathcal{O}(\varepsilon)$. In detail, we show that after interaction the solution is close to

$$\begin{aligned} &\varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A(x - c_{p,A}t) + \varepsilon\Omega_A(\eta_B)) \\ &\quad + \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B(x - c_{p,B}t) + \varepsilon\Omega_B(\eta_A)), \end{aligned} \tag{17}$$

with explicit functions Ω_A, Ω_B, given by

$$\Omega_A = \int_{-\infty}^{\eta_B} \frac{3|B_1|^2}{\omega_A(c_A - c_B)} \, d\tilde{\eta}_B + \Omega_A^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_B = \varepsilon(x + x_B - c_{g,B}t), \tag{18}$$

$$\Omega_B = \int_{-\infty}^{\eta_A} \frac{3|A_1|^2}{\omega_B(c_B - c_A)} \, d\tilde{\eta}_A + \Omega_B^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_A = \varepsilon(x + x_A - c_{g,A}t), \tag{19}$$

where B_1 and A_1 are given by (9) with constants $C_{1,B}, C_{2,B}$ and $C_{1,A}, C_{2,A}$, respectively, and where Ω_A^0 and Ω_B^0 are constants which normalize the initial phases, see Fig. 5. Note that Ω_A depends on $x - c_{g,B}t$ and Ω_B on $x - c_{g,A}t$ as the phase shift accounts for so-called cross phase modulation.

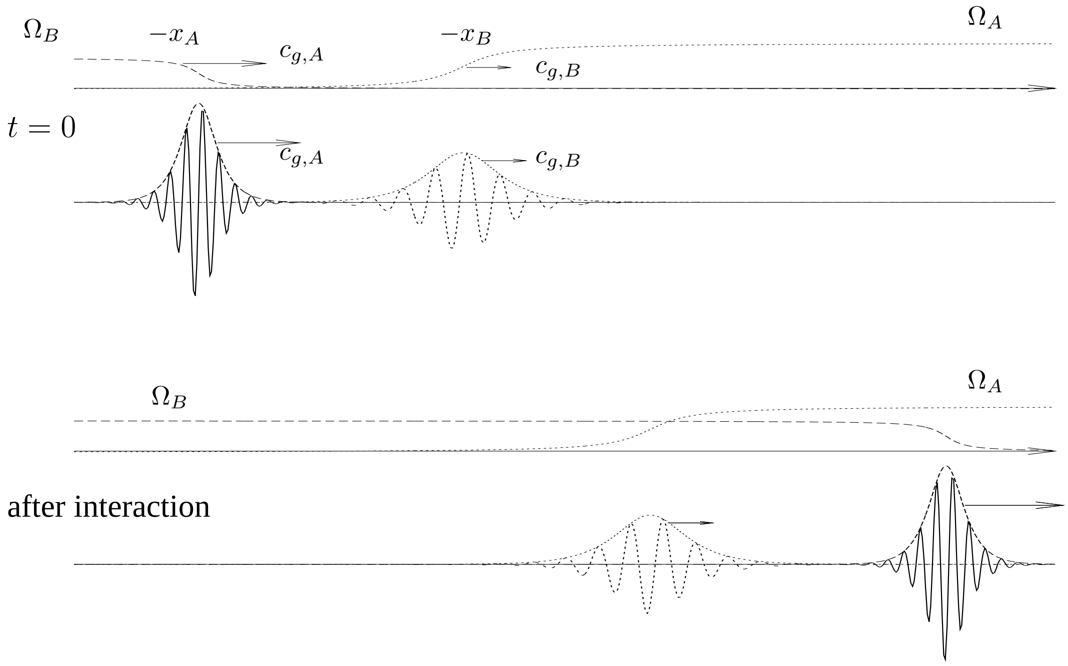

Figure 5: Illustration of the interaction of two pulses εv_{k_A} and εv_{k_B} with the associated cross–phase modulations Ω_A and Ω_B. Here $k_A > k_B$ and the slower pulse is in front. Thus, $c_B - c_A < 0$ in (19), and Ω_B is a decaying function of x. The constants Ω_A^0 and Ω_B^0 have been chosen in such a way that at $t = 0$ (upper two pictures) there are no phase-shift for the pulses, i.e., Ω_B is exponentially small near the position $-x_B$ of εv_{k_B}, while Ω_A is exponentially small near the position $-x_A$ of εv_{k_A}. Note that Ω_B travels with εv_{k_A} and Ω_A with εv_{k_B}.

Theorem 5.2 *Let $s \geq 2$, $k_A, k_B > 0$, $k_A \neq k_B$, $\gamma_{1,A}, \gamma_{1,B} < 0$, $x_A, x_B \in \mathbb{R}$ in Lemma 3.1, and $T_0 > 0$. There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist solutions u of (5) such that*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(x,t) - \varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A(x - c_{p,A}t) + \varepsilon\Omega_A(\eta_B)) \\ - \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B(x - c_{p,B}t + \varepsilon\Omega_B(\eta_A))\|_{C_b^{s-1}} \leq C_2\varepsilon^3 \qquad (20)$$

with v_{k_A}, v_{k_B} from Lemma 3.1 and Ω_A, Ω_B given by (18),(19).

Remark 5.3 To obtain an estimate for the physically relevant shift of the envelope, suppose that the error comes from a shift of the envelope. Then, due to the long wave form of the envelope, "vertical" estimates of order $\mathcal{O}(\varepsilon^3)$ in L^∞ can lead on a pulse of amplitude $\mathcal{O}(\varepsilon)$ only to a possible envelope shift εa of order $\mathcal{O}(\varepsilon)$, due to

$$\varepsilon g(\varepsilon(x + \varepsilon a)) - \varepsilon g(\varepsilon x) = \varepsilon g'(\varepsilon x)\varepsilon^2 a + \mathcal{O}(\varepsilon(\varepsilon^2 a)^2) = \mathcal{O}(\varepsilon^3).$$

Idea of the proof of Theorem 5.2 (See [CBSU06] for more details.) We make the ansatz

$$u(x,t) = \varepsilon\psi(x,t) := \varepsilon A_1(\eta_A)E + \varepsilon B_1(\eta_B)F \\ + \varepsilon^3 A_2(\eta_A, T)E + \varepsilon^3 B_2(\eta_B, T)F + \text{c.c.} + \text{h.o.t.} \qquad (21)$$

where $T=\varepsilon^2 t$, and where

$$E=\mathrm{e}^{\mathrm{i}(k_A x-\omega_A t+\varepsilon\Omega_A(\eta_B))}, \quad F=\mathrm{e}^{\mathrm{i}(k_B x-\omega_B t+\varepsilon\Omega_B(\eta_A))}, \quad \eta_A=\varepsilon(x-c_{\mathrm{g},A}t),\ \eta_B=\varepsilon(x-c_{\mathrm{g},B}t).$$

In (21), h.o.t. stands for terms of higher order in ε, which are algebraically determined similar to A_3 in (11), and which do not lead to new aspects compared to Section 3. We choose A_1 and B_1 as given by Lemma 3.1. If we choose Ω_A, Ω_B to satisfy

$$\partial_{\eta_B}\Omega_A=\frac{3|B_1|^2}{\omega_A(c_{\mathrm{g},A}-c_{\mathrm{g},B})} \quad \text{and} \quad \partial_{\eta_A}\Omega_B=\frac{3|A_1|^2}{\omega_B(c_{\mathrm{g},B}-c_{\mathrm{g},A})} \tag{22}$$

which yields (18) and (19), then the coefficients at $\varepsilon^3 E$ and $\varepsilon^3 F$ vanish. At $\varepsilon^5 E$ and $\varepsilon^5 F$ we find that A_2, B_2 satisfy the linear equations

$$2i\omega_A\partial_T A_2+(1-c_{\mathrm{g},A}^2)\partial_{\eta_A}^2 A_2+G_A = 0\,, \tag{23}$$
$$2i\omega_B\partial_T B_2+(1-c_{\mathrm{g},B}^2)\partial_{\eta_B}^2 B_2+G_B = 0\,, \tag{24}$$

with, by construction, zero initial data, and where

$$\begin{aligned} G_A &= 6|A_1|^2A_2+3A_1^2A_{-2}+6(B_1B_{-2}+B_2B_{-1})A_1 \\ &\quad +\varepsilon^{-1}[\mathrm{i}(1-c_{\mathrm{g},A}^2)\partial_{\eta_B}^2\Omega_A A_1+2\mathrm{i}(c_{g,A}c_{g,B}-1)(\partial_{\eta_B}\Omega_A)(\partial_{\eta_A}A_1))\,], \\ G_B &= 6|B_1|^2B_2+3B_1^2B_{-2}+6(A_1A_{-2}+A_2A_{-1})B_1 \\ &\quad +\varepsilon^{-1}[\mathrm{i}(1-c_{\mathrm{g},B}^2)\partial_{\eta_A}^2\Omega_B B_1+2\mathrm{i}(c_{g,A}c_{g,B}-1)(\partial_{\eta_A}\Omega_B)(\partial_{\eta_B}B_1)\,]. \end{aligned}$$

The argument given in Remark 5.1 applied to the terms multiplied by ε^{-1} shows

Lemma 5.4 *There exists a $C>0$ such that for all $\varepsilon\in(0,1]$ there exists a unique solution $(A_2,B_2)\in C([0,T_0],H^s\times H^s)$ to (23)-(24) with zero initial data. It satisfies*

$$\sup_{T\in[0,T_0]}\|(A_2,B_2)(T)\|_{H^s\times H^s}\leq C.$$

This shows that $\|\mathrm{Res}(\varepsilon\psi)\|_{H^s}=\mathcal{O}(\varepsilon^{11/2})$. Similar to the proof of Theorem 4.1 we write $u(x,t)=\varepsilon\psi(x,t)+\varepsilon^{7/2}R$, where we can employ the higher weight of R due to the smaller residual. The equation for R looks exactly as (12). Thus, $\sup_{t\in[0,T_0/\varepsilon^2]}\|R\|_{H^s}\leq C$ as above, which concludes the proof of Theorem 5.2. ■

Theorem 5.2 can be extended in at least two directions. On a time scale $\mathcal{O}(1/\varepsilon^2)$ a modulating pulse can pass at most $\mathcal{O}(1/\varepsilon)$ many modulating pulses of width $\mathcal{O}(1/\varepsilon)$. The interaction of such a modulating pulse with $\mathcal{O}(1/\varepsilon)$ many modulating pulses with a different carrier wave can lead at most to an $\mathcal{O}(1)$-pulse shift. Thus, with respect to the question of the transport of information through optical fibers the influence of different frequencies to the dynamics at some frequency is negligible. Finally, a possibility to increase the rate of information through the fibers is to decrease the gap between the wave numbers. Formally we find for $k_A-k_B=\mathcal{O}(\varepsilon^\mu)$ with $0\leq\mu\leq 1$ a pulse shift of order $\mathcal{O}(\varepsilon^{1-2\mu})$. Thus we must expect a certain payoff between the number of different carrier frequencies $1/(k_A-k_B)$ and the spacing of bits.

6 Pulses in photonic crystals: Standing light

One of the major goals of nanotechnology is photonics, i.e. the construction of 'electronic' devices where the electrons are completely replaced by photons. In this context, the question of optical storage plays a major role. One theoretical possibility are photonic crystals. These are optical materials with a periodic structure with a wave length comparable to the wave length of light. Due to the periodic structure the linearized problem is no longer solved by Fourier modes, but by so-called Bloch modes. The curves of eigenvalues plotted as a function over the Bloch wave numbers can now possess horizontal tangencies, i.e. vanishing group velocities. Thus, in principle, standing light pulses are possible. This will be explained in detail in the following, see also [BSTU06] for more details.

Again we consider a semilinear wave equation

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) - \chi_3(x)u^3(x,t) \tag{25}$$

with $x \in \mathbb{R}$ and $t \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, but now in a spatially periodic medium. This means that the coefficient functions $\chi_j = \chi_j(x)$ satisfy $\chi_j(x) = \chi_j(x+L)$ for $j = 1,2,3$. We assume here that the χ_j are smooth functions, that $\chi_1(x) > 0$ and that $\chi_2(x) > 0$ for all $x \in [0,L)$, and, without loss of generality, $L = 2\pi$ throughout this section. The linearized problem

$$\partial_t^2 v(x,t) = \chi_1(x)\partial_x^2 v(x,t) - \chi_2(x)v(x,t)$$

is solved by the Bloch waves

$$v(x,t) = \tilde{v}_n(\ell,x)\mathrm{e}^{\mathrm{i}\ell x}\mathrm{e}^{\mathrm{i}\omega_n(\ell)t}$$

where $n \in \mathbb{Z} \setminus \{0\}$, $\ell \in (-1/2,1/2]$, with $\tilde{v}_n$ and ω_n determined by (27) below. Here, $\omega_n(\ell) \in \mathbb{R}$ satisfies $\omega_{n+1}(\ell) \geq \omega_n(\ell)$, $\omega_{-n}(\ell) = -\omega_n(\ell)$, and $\tilde{v}_n(x,\ell)$ satisfies

$$\tilde{v}_n(\ell,x) = \tilde{v}_n(\ell,x+2\pi) \quad \text{and} \quad \tilde{v}_n(\ell,x) = \tilde{v}_n(\ell+1,x)\mathrm{e}^{\mathrm{i}x}. \tag{26}$$

The Bloch wave transform of a function $u : \mathbb{R} \to \mathbb{C}$ is a generalization of Fourier transform and formally given by

$$\tilde{u}(\ell,x) = \sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jx}\hat{u}(\ell+j).$$

By construction, $\tilde{u}$ satisfies (26), and $\ell \in (-1/2,1/2]$ is called a Bloch or pseudo wave number. From Parseval's identity $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ it follows that Bloch transform is an isomorphism from $H^m(\mathbb{R},\mathbb{C})$ to the Bloch space $L^2((-1/2,1/2],H^m_{\text{per}}((0,2\pi)))$, and its inverse is given by

$$u(x) = \int_{-1/2}^{1/2} \mathrm{e}^{\mathrm{i}\ell x}\tilde{u}(\ell,x)\,d\ell.$$

See [RS78, Sca99] for further properties and applications of Bloch transform.

For fixed Bloch wave number ℓ the Bloch modes $\tilde{v}_n(\ell,x)$ satisfy the spatially periodic eigenvalue problem

$$-\tilde{\Lambda}(\ell,\partial_x)\tilde{v}_n(\ell,\cdot) = \chi_1(\cdot)(\partial_x + \mathrm{i}\ell)^2\tilde{v}_n(\ell,\cdot) - \chi_2(\cdot)\tilde{v}_n(\ell,\cdot) = -(\omega_n(\ell))^2\tilde{v}_n(\ell,\cdot). \tag{27}$$

Since the operator $\tilde{\Lambda}(\ell, \partial_x)$ is elliptic in the bounded domain $[0, 2\pi)$ with periodic boundary conditions we have for fixed ℓ countable many eigenvalues $\lambda_n = \omega_n^2$, $n \in \mathbb{N}$. In the space $L^2_{\chi_1}(0, 2\pi)$ where

$$\langle \tilde{u}(\ell, \cdot), \tilde{v}(\ell, \cdot) \rangle_{\chi_1} = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\ell, x) \overline{\tilde{v}(\ell, x)} \, \frac{1}{\chi_1(x)} \, dx \,, \tag{28}$$

the operator $\tilde{\Lambda}(\ell, \partial_x)$ is positive definite and self adjoint such that the eigenvalues $\lambda_n(\ell)$ are real and positive. They are ordered by $\lambda_n(\ell) \leq \lambda_{n+1}(\ell)$.

We now explain the possibility of horizontal tangencies for the curves $\ell \mapsto \omega_n(\ell)$ by discussing periodic coefficients as perturbation of the spatially homogeneous case.

Example 6.1 The solutions of the constant coefficient case

$$\partial_t^2 v(x,t) = \partial_x^2 v(x,t) - v(x,t) \tag{29}$$

are given by the Fourier modes $v(x,t) = \mathrm{e}^{\mathrm{i}(kx \pm \mu(k)t)}$, where $(\mu(k))^2 = k^2 + 1$. We consider artificially the problem in a spatially periodic set-up. In a Bloch wave representation we have

$$v(x,t) = \mathrm{e}^{\mathrm{i}nx} \mathrm{e}^{\mathrm{i}\ell x} \mathrm{e}^{\mathrm{i}\omega_n^{\pm}(\ell) t},$$

where $k = n + \ell$, with $n \in \mathbb{Z}$ here and $\ell \in (-\frac{1}{2}, \frac{1}{2}]$. The eigenvalues are related by $\omega_n^{\pm}(\ell) = \pm\mu(n+\ell)$, i.e., they are obtained from wrapping $\pm\mu(\cdot)$ around a cylinder, see the left panel of Fig. 6.

For all $\ell \in (-1/2, 1/2]$ except for $\ell = 0, 1/2$ all eigenvalues of $\tilde{\Lambda}(\ell, \partial_x)$ in Example 6.1 are simple. By classical perturbation arguments [Kat66], for periodic $\chi_j = 1 + \mathcal{O}(\delta)$ the eigenvalues are smooth functions of δ and stay separated for $\delta > 0$ sufficiently small. However, for $\ell = 0, 1/2$ all eigenvalues are double and generically for small $\delta > 0$ the eigenvalues will split. This is exactly what happens in the spatially periodic case.

Example 6.2 Let $\chi_2(x) = 1 + 2\delta \cos(2nx) = 1 + \delta(\mathrm{e}^{\mathrm{i}2nx} + \mathrm{e}^{-\mathrm{i}2nx})$ with $\delta > 0$ small and a fixed $n \in \mathbb{N}$. Setting

$$\tilde{v}_n(\ell, x) = \sum_{k \in \mathbb{Z}} \hat{v}_k^n(\ell) \mathrm{e}^{\mathrm{i}kx},$$

the eigenvalue problem (27) is given by the infinitely many equations

$$(1 + (k+\ell)^2)\hat{v}_k^n(\ell) + \delta(\hat{v}_{k+2n}^n(\ell) + \hat{v}_{k-2n}^n(\ell)) - \lambda_n(\ell)\hat{v}_k^n(\ell) = 0, \qquad (k \in \mathbb{Z}). \tag{30}$$

For $\delta = 0$ we have (with some abuse of notation) $\lambda_n(0) = \lambda_{-n}(0)$, i.e. a crossing of the curves of eigenvalues at $\ell = 0$. Due to the continuity of single eigenvalues or subspaces to eigenvalues separated from the rest, for small $\delta > 0$ and $\ell = 0$, the infinite dimensional eigenvalue problem in lowest order can be reduced to the two-dimensional problem

$$\det \begin{pmatrix} 1 + (-n)^2 - \lambda_n(0) & \delta \\ \delta & 1 + (n)^2 - \lambda_n(0) \end{pmatrix} = 0,$$

for $\hat{v}_n^n$ and $\hat{v}_{-n}^n$. Hence $\lambda_{\pm n}(0) = 1 + n^2 \pm \delta$. Thus, $\lambda_n(\ell)$ and $\lambda_{-n}(\ell)$ split at the crossings, i.e. at $\ell = 0$, and recombine in a different way. These new curves are also denoted with $\lambda_n(\ell)$ now ordered such that $\lambda_{n+1}(\ell) \geq \lambda_n(\ell)$ but now and in the following indexed with $n \in \mathbb{N}$. As before we let $\lambda_n(\ell) = \omega_n^2(\ell)$ and $\omega_n(\ell) = -\omega_{-n}(\ell) > 0$, see the right panel of Fig. 6.

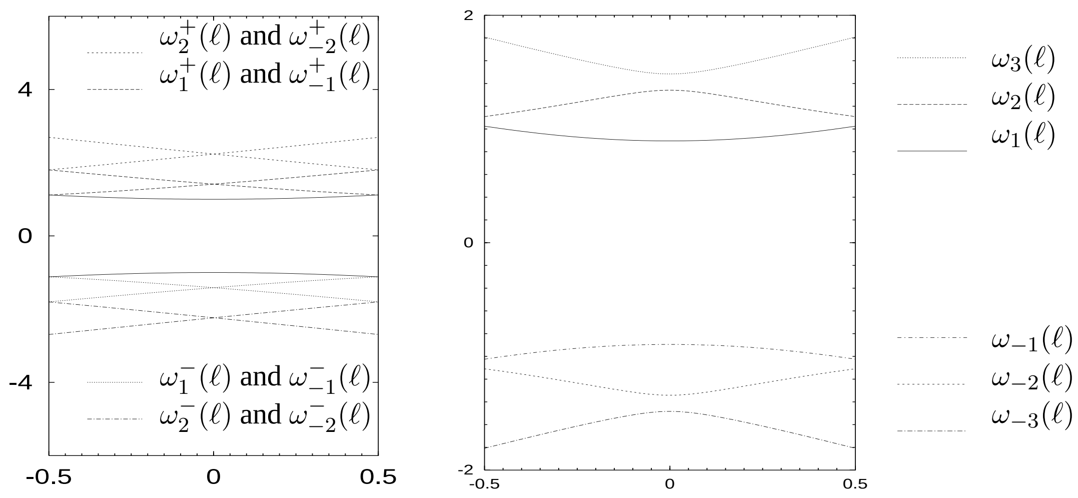

Figure 6: The curves of eigenvalues for the homogeneous case (29) in Bloch-representation, and the splitting of eigenvalues for (30). The Bloch modes of the ansatz (31) are strongly concentrated on an $\mathcal{O}(\varepsilon)$ neighborhood of the basic Bloch wave numbers $\pm\ell_0$ and the evolution of the wave packet will be strongly determined by the associated curves $\omega_{\pm n_0}$ at $\pm\ell_0$. Thus, the occurrence of horizontal tangencies as explained in Example 6.2 corresponds to vanishing group velocity c_g', i.e. to standing light pulses.

Thus, on a linear level we have a situation as in the spatially homogeneous case: we have curves of eigenvalues over wave numbers except that associated eigenfunctions are no longer Fourier modes, but Bloch modes. Then, similar to the spatially homogeneous case, slow modulations in time and space of such a Bloch mode (indexed with n_0) may be described by the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x - c_g' t), \varepsilon^2 t)\tilde{v}_{n_0}(\ell_0, x)e^{i\ell_0 x}e^{i\omega_{n_0}(\ell_0)t} + \text{c.c.}, \tag{31}$$

where c.c. means complex conjugate, where $0 < \varepsilon \ll 1$ is a small parameter, and where $c_g' = \partial_\ell \omega_{n_0}(\ell_0)$ is the linear group velocity. The complex valued amplitude $A(X,T) \in \mathbb{C}$ describes slow modulations in time $T = \varepsilon^2 t$, and space $X = \varepsilon(x - c_g' t)$, of the underlying wave $\tilde{v}_{n_0}(\ell_0, x)e^{i\ell_0 x}e^{i\omega_{n_0}(\ell_0)t}$. The Bloch modes of the ansatz are strongly concentrated in an $\mathcal{O}(\varepsilon)$ neighborhood of the basic Bloch wave numbers $\pm\ell_0$ and the evolution of the wave packet will be strongly determined by the associated curves $\omega_{\pm n_0}$ at $\pm\ell_0$. Plugging the ansatz into (25) one finds that A has to satisfy a NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2 \tag{32}$$

with coefficients $\nu_1 = -\frac{1}{2}\partial_\ell^2 \omega_{n_0}(\ell_0) \in \mathbb{R}$ and

$$\nu_2 = \frac{3}{2\omega_{n_0}(\ell_0)} \int_0^{2\pi} \frac{\chi_3(x)}{\chi_1(x)} |v_{n_0}(\ell_0, x)|^4 \, dx \in \mathbb{R}.$$

The occurrence of the nonlinear term $i\nu_2 A|A|^2$ is a priori not clear at all. However, the nonlinear interaction corresponds in Bloch space to a convolution with respect to the Bloch wave numbers. Thus, the concentration of modes is respected by the nonlinear interaction which can be described in lowest order by $i\nu_2 A|A|^2$.

In general, the dispersion relation $\ell \mapsto \omega_n(\ell)$ and hence the coefficient ν_1 as well as ν_2 have to be calculated numerically. On the other hand, for a given material, these coefficients can be tailored by adjusting the grating, i.e. the periodic functions χ_j. This is a highly nontrivial optimization problem [HFBW01].

The justification of (32) for (25) in the sense of error estimates proceeds similar to the proof of Theorem 4.1, but the functional analysis becomes somewhat more complicated [BSTU06]. The physical detection of the pulses predicted by (32) is a nontrivial task, since they are localized in the photonic crystal and cannot be 'seen'. One possibility would be the interaction with other modulating pulses. However, similar to the analysis in §5, only pulses with carrier waves close to the carrier wave of the standing pulse will have any relevant, in terms of ε, effect on the standing pulse, and vice versa. Nevertheless, due to the higher dispersion, the influence is in general much larger than in homogeneous optical fibers, cf. [TPB04].

7 Single pulses II

We found approximate modulating pulse solutions with the help of the NLS equation up to a time-scale of order $\mathcal{O}(1/\varepsilon^2)$. Since these solutions are essential for the transport of information the following question occurs: do these solutions exist for all $t \in \mathbb{R}$? More precisely, are there 'breather solutions', which are time-periodic solutions in a moving frame and which are spatially localized, i.e., which decay to zero for $|x| \to \infty$? Such solutions are known explicitly for the sine-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - \sin(u),$$

which first appeared in differential geometry in the description of surfaces with constant negative curvature [Enn70], but which also appears in crystallography and in particle physics. In fact, like the NLS equation, the sine-Gordon equation is a completely integrable Hamiltonian system. See [DJ89] for more background and references.

Thus, the question is whether 'breathers' can also exist in other nonlinear wave equations, for instance of the type

$$\partial_t^2 u = \partial_x^2 u - u + g(u),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth, odd function which satisfies $g(u) = \mathcal{O}(u^3)$ and $g'''(0) > 0$. It turns out that for $g(u)$ close to $u - \sin(u)$ the sine-Gordon equation is the only such equation. For a precise statement see [BMW94, Den93]. In the following we explain why this 'non existence of breathers' result holds. Moreover, we will explain positive results for generalized breather solutions.

The solutions we are interested in are obtained from the ansatz

$$u(x,t) = v(x - c_{\mathrm{g}}t, x - c_{\mathrm{p}}t) = v(\xi, y),$$

where v is periodic in y with period $2\pi/k_0$ for some $k_0 > 0$. They are homoclinic solutions of the evolutionary system

$$(1 - c_{\mathrm{g}}^2)\partial_\xi^2 v + (1 - c_{\mathrm{p}}^2)\partial_y^2 v - v + g(v) = 0, \qquad (33)$$

which generalizes the spatial dynamics approach of Kirchgässner [Kir82], i.e., we look for v with

$$\lim_{\xi \to \pm\infty} v(\xi, y) = 0.$$

In order to obtain (33) we have chosen $c_g = 1/c_p$ according to the linear relation $c_g' = 1/c_p'$.

Hence, v has to be in the intersection of the stable and unstable manifold of the origin. The stable and unstable manifolds are the nonlinear counterparts to the stable and unstable subspaces in case of linear equations and are tangential to these subspaces. Therefore, we look at the linearization around the fixed point $(v, \partial_\xi v) \equiv (0,0)$ in order to compute the dimensions of these manifolds. The linearization of (33) is given by

$$(1 - c_\mathrm{g}^2)\partial_\xi^2 v + (1 - c_\mathrm{p}^2)\partial_y^2 v - v = 0. \tag{34}$$

Since we are interested in periodic solutions w.r.t. y we use Fourier series

$$v(\xi, y) = \sum_{m \in \mathbb{Z}} v_m(\xi) \mathrm{e}^{\mathrm{i}mk_0 y}$$

and find $\partial_\xi^2 v_m = -\lambda_m^2 v_m$ which is solved by $u_m(x) = \mathrm{e}^{\mathrm{i}\lambda_m x} u_m(0)$ where $\lambda_m^2 = \frac{m^2 k_0^2 (1 - c_\mathrm{p}^2) + 1}{(1 - c_\mathrm{g}^2)}$. Due to the cubic nonlinearity we can restrict to odd $m \in \mathbb{Z}$. Therefore, for c_p close to c_p' the eigenvalues λ_m are on the imaginary axis for $|m| > 3$. The eigenvalues $\lambda_{\pm 1}$ are on the real axis for $c_\mathrm{p} < c_\mathrm{p}'$. Hence we have a two-dimensional stable and a two-dimensional unstable manifold. These manifolds intersect for the sine-Gordon equation, but in general two two-dimensional manifolds will not intersect in an infinite-dimensional phase space. This makes the sine-Gordon equation exceptional in this class of equations.

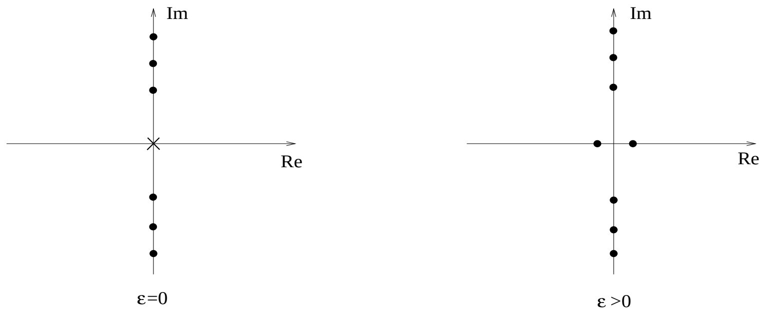

Figure 7: The spectrum of the linearization (34), where ε is defined in Theorem 7.1.

A time-periodic solution in a moving frame is called generalized moving breather or generalized modulating pulse solution if not necessarily

$$\lim_{\xi \to \pm\infty} v(\xi, y) = 0,$$

but $v(\xi, y)$ is small for $|\xi| \to \infty$. In [GS01], the existence of generalized modulating pulse solutions with $\mathcal{O}(\varepsilon^n)$-tails has been established. For simplicity we restrict to $g(u) = u^3$.

Theorem 7.1 *Fix a positive integer n and a positive real number k_0. For sufficiently small $\varepsilon > 0$ (depending upon n and k_0) there exists an infinite-dimensional, continuous family of modulating pulse solutions to equation (5) of the form*

$$u(x, t) = v(x - c_\mathrm{g} t, x - c_\mathrm{p} t),$$

where v is $2\pi/k_0$-periodic in its second argument and $c_\mathrm{p} = c_\mathrm{p}' + \gamma_1 \varepsilon^2$, $c_\mathrm{g} = 1/c_\mathrm{p}$. These solutions satisfy

$$v(\xi, y) = v(-\xi, y), \quad |v(\xi, y) - 2h(\xi, y, \varepsilon)| \le \varepsilon^{n+1}, \qquad \xi, y \in \mathbb{R},$$

where $h(\xi, y, \varepsilon) = \varepsilon B_\mathrm{pulse}(\varepsilon\xi) \sin k_0 y + \mathrm{c.c.} + \mathcal{O}(\varepsilon^2)$ and $\lim_{\xi \to \pm\infty} h(\xi, y, \varepsilon) = 0$.

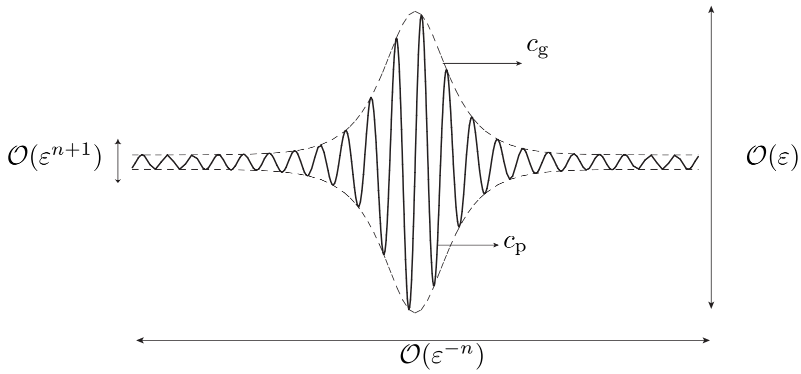

Figure 8: A generalized modulating pulse solution.

The modulating pulse solutions of Theorem 7.1 are found in the intersection of the infinite-dimensional center stable and infinite-dimensional center unstable manifold. For $|\xi| \to \infty$ the solutions converge with some exponential rate towards the center manifold. Thus, a secular growth of the solutions is possible. However, for this special equation the boundedness for $|\xi| \to \infty$ follows with the help of the Hamiltonian structure due to the fact that the Hamiltonian restricted to the center manifold is positive definite.

For general, especially quasilinear, systems the norm induced by the Hamiltonian is too weak compared with the norm used for the construction of the invariant manifolds. Thus, in general, generalized modulating pulse solutions can only be constructed for $|\xi| \leq 1/\varepsilon^n$, cf. [GS05]. This result has been improved in [GS06] to exponentially small tails and exponentially large intervals, i.e., $|\xi| \leq \exp(-1/\varepsilon)$.

8 Outlook and related fields

The above analysis can be extended into a number of directions. First we may consider different constitutive laws for the polarization, as for instance

$$\partial_t^2 p(x,t) = u(x,t) + \partial_t^2 (u(x,t)^3)$$

leading to quasilinear systems, cf. [GS05].

Recently so-called ultra-short pulses have attracted a lot of interest, cf. [SW04]. They play an important role in spectroscopy. For such pulses the length of the envelope and the wavelength of the underlying carrier wave have a comparable size.

In materials with broken up–down symmetry also quadratic terms are present. Then, from a mathematical point of view, the proof of the above approximation results is a much more challenging task. The idea is to use normal form transforms or averaging methods to eliminate the quadratic terms and to reduce the proof to the cubic case, cf. [Sch98, BSTU06]. The case of quadratic resonant media has been treated recently in [Sch05].

There is another famous system with dispersive behavior for which the NLS equation can be derived, namely the water wave problem, cf. [Zak68]. Estimates for the residual can be found in [CSS92]. Here, quadratic terms are present. The elimination of these terms is complicated due to some resonance at the wavenumber $k = 0$ and other resonances present in case of small positive surface tension. Estimates for model problems can be found in [DS05]. A first attempt for the water wave problem as been made in [SW06] where the validity of the approximation over at least the right time scale has been shown.

More generally, as already pointed out in the introduction, the methods reviewed here can be applied to all dispersive nonlinear equations for which the NLS equation can be derived.

There are still many open questions. A serious difficulty in the description of photonic crystals comes from the fact that the coefficient functions χ_j very often are step functions, i.e., they are not smooth. Another challenging problem is the justification of the NLS equation when the original equation possesses quasilinear quadratic terms. The elimination of these terms by normal form transforms gives a loss of regularity complicating the local existence and uniqueness theory of solutions substantially.

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Addresses of the authors:

Institut für Analysis, Dynamik und Modellierung
Universität Stuttgart
Pfaffenwaldring 57
D - 70569 Stuttgart
firstname.lastname@mathematik.uni-stuttgart.de