On the interaction of NLS-described modulating pulses with different carrier waves

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Abstract

We give a detailed analysis of the interaction of two modulating pulse solutions of a nonlinear wave equation. These solutions consist of pulse-like envelopes satisfying approximately a Nonlinear Schrödinger equation, advancing in the laboratory frame, and modulating underlying wave-trains. We improve the bound for the possible shift of the envelopes caused by the interaction of two well prepared pulses from order $\mathcal{O}(1)$ to order $\mathcal{O}(\varepsilon)$. Thus we manifest the statement that there is almost no interaction of pulses with different carrier waves.

1 Introduction

The transport of information over long distances through optical fibers is encoded digitally by sending a light pulse or not. Physically such a light pulse is a complicated structure. It consists of an underlying electromagnetic carrier wave moving with phase velocity c_p and of a pulse like envelope moving with group velocity c_g and modulating the underlying carrier wave. The fact that there is very few interaction of pulses with different carrier waves allows to increase the information rate through the fiber by using different bands, cf. [Ace00].

In most theoretical descriptions the dynamics of the envelope of the modulating pulses is approximately described by a Nonlinear Schrödinger (NLS) equation. In such a description the envelope has an amplitude of order $\mathcal{O}(\varepsilon)$ and a width of order $\mathcal{O}(1/\varepsilon)$, where $\varepsilon > 0$ is a small perturbation parameter. See Figure 1.

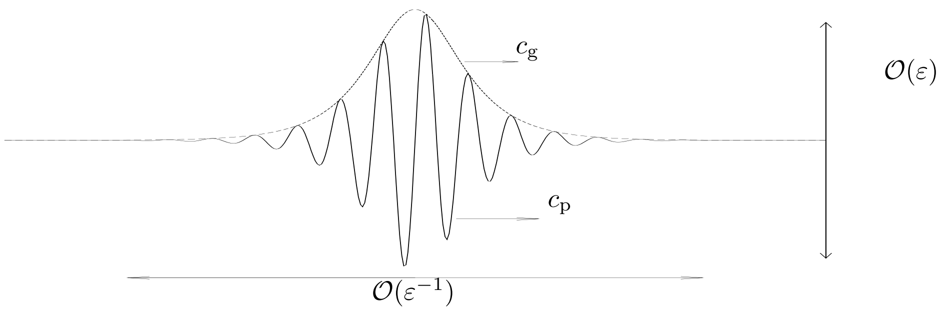

Figure 1: A modulating pulse described by the NLS-equation, see Lemma 3.1 below.

It is the purpose of this paper to show in detail that there is almost no interaction of two such NLS-described modulating pulses if they possess different carrier waves. It has been known for long time that pulses with different carrier waves do not interact in lowest order, see [PW96] for a rigorous proof and Remark 3.3 for the heuristic argument. Here we improve this statement by giving an $\mathcal{O}(\varepsilon)$-bound for the possible envelope shift resulting from the interaction. In order to do so we give a mathematical analysis of the interaction of two such modulating pulse solutions of a nonlinear wave equation. For well prepared NLS-described modulating pulse solutions we improve the bound for the physically relevant possible envelope shift caused by the interaction of the pulses from order $\mathcal{O}(1)$, cf. [PW96], to order $\mathcal{O}(\varepsilon)$ on an $\mathcal{O}(1/\varepsilon^2)$ time scale. The proof of the bound is based on an explicit description of the phases and on the consideration of pulses constructed with the help of higher order approximations.

On a $\mathcal{O}(1/\varepsilon^2)$ time scale, the natural time scale of the NLS–approximation, a modulating pulse of width $\mathcal{O}(1/\varepsilon)$ can pass at most $\mathcal{O}(1/\varepsilon)$ many modulating pulses of width $\mathcal{O}(1/\varepsilon)$. As a consequence of our result, the interaction of such a modulating pulse with $\mathcal{O}(1/\varepsilon)$ many modulating pulses with a different carrier wave can lead at most to an $\mathcal{O}(1)$-pulse shift. Thus, with respect to the question of the transport of information through glass fibers the influence of different frequencies to the dynamics in one band is negligible w.r.t. to the transport of digital information.

The plan of the paper is as follows. In Section 2 the relevance of the NLS-equation and the associated NLS-pulses is explained. The precise result is stated in Section 3. The proof is based on a number of Lemmas which are also stated in Section 3, but proved subsequently. In Section 4.1 we construct approximate modulating pulses with the help of the NLS-equation. A high order formal approximation of the interaction of two NLS-described modulating pulses with different carrier waves is constructed in Section 4.2, and the validity of this approximation on a time scale $\mathcal{O}(1/\varepsilon^2)$ is established in Section 4.3.

Although we restrict our analysis to a semilinear wave equation with cubic nonlinearity the statement can be transfered to all systems where the NLS-equation has been justified, i.e. semilinear wave equations with a quadratic nonlinearity in case of no resonances [Kal88, KSM92] and in case of resonances [Sch98b, Sch05], water wave models [Sch98a] and finally wave equations in periodic media [BSTU06].

Notation. Many possibly different constants that are independent of ε are denoted by C. The space $H^s(m)$ consists of s-times weakly differentiable functions for which $\|u\|_{H^s(m)} = \|u\rho^m\|_{H^s} = (\sum_{j=0}^{s} \int |\partial_x^j (u\rho^m)|^2 dx)^{1/2}$ with $\rho(x) = \sqrt{1+x^2}$ is finite, where we do not distinguish beween scalar and vector–valued functions or real- and complex-valued functions. The space C_b^s consists of s-times continuously differentiable functions for which $\|u\|_{C_b^s} = \sum_{j=0}^{s} \sup_{x\in\mathbb{R}} |\partial_x^j u|$ is finite. We sometimes write, e.g., $\|u(x)\|_{C_b^s}$ for the C_b^s norm of the function $x \mapsto u(x)$.

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2 The NLS-pulses

As a prototype of a nonlinear wave equation possessing approximate modulating pulse solutions we consider throughout this paper the semilinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u + u^3, \tag{1}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$. It is well known that on time-scales of order $\mathcal{O}(1/\varepsilon^2)$ equation (1) has $\mathcal{O}(\varepsilon)$-amplitude solutions which are slow spatial and temporal modulations of an underlying carrier wave $e^{i(k_0x-\omega_0 t)}$, where $\varepsilon > 0$ is a small perturbation parameter and where k_0 and ω_0 are related by the linear dispersion relation $\omega_0^2 = k_0^2 + 1$. Such solutions are described by the formula

$$u_A(x,t) = \varepsilon(A(X,T)e^{i(k_0x-\omega_0 t)} + \text{c.c.}) + \mathcal{O}(\varepsilon^2).$$

Here $X = \varepsilon(x - c_{\rm g}' t)$ and $T = \varepsilon^2 t$ are the long spatial and temporal scales, respectively, $c_{\rm g}' = k_0/(1+k_0^2)^{1/2}$ is the linear group velocity, and the complex envelope A satisfies

$$2i\omega_0 \partial_T A + (1 - (c_{\rm g}')^2)\partial_X^2 A + 3|A|^2 A = 0. \tag{2}$$

The nonlinear Schrödinger equation (2) has a three-parameter family of time-periodic solutions of the form

$$A(X,T) = \tilde{A}(X - X_0)e^{-i\gamma_0 T}e^{i\phi_0},$$

in which the real-valued function $\tilde{A}$ satisfies the second-order ordinary differential equation

$$\partial_X^2 \tilde{A} = C_1 \tilde{A} - C_2 \tilde{A}^3, \tag{3}$$

where $C_1 = -2\gamma_0\omega_0/(1-(c_{\rm g}')^2)$, $C_2 = 3/(1-(c_{\rm g}')^2)$. For $\gamma_0 < 0$ and $\omega_0 > 0$ this equation has two homoclinic solutions

$$\tilde{A}_{\rm pulse}(X) = \pm\left(\frac{2C_1}{C_2}\right)^{1/2} \text{sech}\,(C_1^{1/2}X) \tag{4}$$

which connect the origin of the $(\tilde{A}, \partial_X \tilde{A})$-phase plane with itself and which fulfill

$$|\tilde{A}_{\rm pulse}(X)| \leq Ce^{-r|X|}, \quad r = \sqrt{\frac{-2\gamma_0\omega_0}{1-(c_{\rm g}')^2}}. \tag{5}$$

This procedure therefore gives modulating pulse solutions of the nonlinear wave equation which are described by the approximate formula

$$\begin{aligned} u_{\rm pulse} =& \varepsilon(\tilde{A}_{\rm pulse}(X - X_0)e^{-i\gamma_0 T}e^{i(k_0x-\omega_0 t)} + \text{c.c.}) \\ =& \varepsilon(\tilde{A}_{\rm pulse}(\varepsilon(x - c_{\rm g}' t) - \varepsilon x_0)e^{i(k_0 x - (\omega_0 + \gamma_0\varepsilon^2)t)} + \text{c.c.}) \end{aligned} \tag{6}$$

over time-scales of order $\mathcal{O}(1/\varepsilon^2)$.

3 The result

In contrast to the formal analysis of Section 2 the well known 'non-existence of breathers' result [Den93, BMW94] does not allow the global existence of modulating pulse solutions, i.e. there are no solutions to (1) of the form

$$u(x,t) = v(x - c_{\mathrm{g}}t, k_0 x - \omega_0 t),$$

where v is 2π-periodic in its second argument and $\lim_{\xi\to\pm\infty} v(\xi, y) = 0$. However, to any polynomial order such solutions can be computed. This means that there are approximate modulating pulse solutions for which the residual

$$\mathrm{Res}(u) = -\,\partial_t^2 u + \partial_x^2 u - u + u^3$$

can be made small to any power of ε. The residual contains the terms which do not cancel after inserting an approximation into (1). If $\mathrm{Res}(u) = 0$ then u is an exact solution of (1). For our purposes we need the following approximate modulating pulses.

Lemma 3.1 *Let $s \geq 2$, $k_0 > 0$ and $\gamma_0 < 0$. For sufficiently small $\varepsilon > 0$ there exists a two-dimensional family of approximate modulating pulse solutions to (1) of the form*

$$u(x,t) = \varepsilon v_{k_0}(x - c_{\mathrm{g}}t + x_0, k_0 x - \omega t + \phi), \tag{7}$$

parametrized by envelope shift $x_0 \in \mathbb{R}$ and phase shift $\phi \in [0, 2\pi)$, where v_k is 2π-periodic in its second argument, $\omega = \omega_0 + \gamma_0\varepsilon^2 + \mathcal{O}(\varepsilon^4) = k_0 c_{\mathrm{p}}$ with phase velocity $c_{\mathrm{p}} = c_{\mathrm{p}}' + \gamma_1\varepsilon^2 + \mathcal{O}(\varepsilon^4)$, $\gamma_1 = \gamma_0/k_0$, and group velocity $c_{\mathrm{g}} = k_0/\omega = 1/c_{\mathrm{p}}$. Moreover,

$$\varepsilon v_{k_0}(\xi, y) = \varepsilon \tilde{A}_{\mathrm{pulse}}(\varepsilon\xi)e^{iy} + \mathrm{c.c.} + \mathcal{O}(\varepsilon^3 e^{-r\varepsilon|\xi|}) \tag{8}$$

with $\tilde{A}_{\mathrm{pulse}}$ and $r > 0$ given by the homoclinic solution of (4) and (5). The residual fulfills

$$\left\|\mathrm{Res}(\varepsilon v_{k_0})\right\|_{H^s} \leq C\varepsilon^{11/2}. \tag{9}$$

Proof. See Section 4.1. ■

Remark 3.2 For u_{pulse} defined in (6) we have $\|\mathrm{Res}(u_{\mathrm{pulse}})\|_{H^s} = \mathcal{O}(\varepsilon^{5/2})$. In particular, to achieve (9) we need the $\mathcal{O}(\varepsilon^2)$ correction to the linear group velocity $c_g' = k_0/\omega_0$ from Section 2, i.e., $c_g = c_g' + \mathcal{O}(\varepsilon^2)$ in (7) and in Fig.1.

To analyze the interaction of two approximate modulating pulses from Lemma 3.1 with different carrier waves we introduce subscripts A and B to indicate the wave numbers $k_A \neq k_B$ of each pulse, the associated group velocities $c_{g,A}$ and $c_{g,B}$, the envelope shifts x_A and x_B and so on. Note that $k_A \neq k_B$ implies $c_{g,A} \neq c_{g,B}$. If the two pulses are separated initially, and, say, $x_A > x_B$ and $k_A < k_B$ such that $c_{g,A} < c_{g,B}$ and the faster pulse is in front, then, since the pulses are exponentially localized, it is natural to expect that the dynamics of the two pulses can be described by the sum of the two individual pulses, at least on an $\mathcal{O}(1/\varepsilon^2)$ time–scale, which is the natural time scale to approximate solutions of (1) by solutions of the NLS, cf. [KSM92]. However, if the two pulses are, say, $\mathcal{O}(1/\varepsilon)$ separated initially, with

$x_A > x_B$ and $k_A > k_B$, then, since the group velocities differ by $\mathcal{O}(1)$, the two pulses must interact on an $\mathcal{O}(1/\varepsilon^2)$ time–scale. Clearly this is the mathematically more interesting case.

For notational simplicity we assume that $\phi_A = \phi_B = 0$ and thus study the interaction of

$$\varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A x - \omega_A t) \quad \text{and} \quad \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B x - \omega_B t), \quad k_A \neq k_B.$$

We prove that the form of the pulses is almost preserved and that the interaction mainly leads to phase-shifts $\varepsilon\Omega_j$ and to envelope shifts $\varepsilon\delta_A$, i.e. after interaction the solution looks like

$$\varepsilon v_{k_A}(x - c_{g,A}t + x_A + \varepsilon\delta_A, k_A x - \omega_A t + \varepsilon\Omega_A) + \varepsilon v_{k_B}(x - c_{g,B}t + x_B + \varepsilon\delta_B, k_B x - \omega_B t + \varepsilon\Omega_B),$$

for some $\delta_A, \delta_B, \Omega_A, \Omega_B \in \mathbb{R}$.

Remark 3.3 There is a simple argument why the evolution equations for v_{k_A} and v_{k_B} decouple in lowest order in terms of ε. Going into the scaling of the envelope, v_{k_A} and v_{k_B} have an amplitude and a width of order $\mathcal{O}(1)$. But since the group velocities differ by an order $\mathcal{O}(1/\varepsilon)$ in this scaling the interaction time of v_{k_A} and v_{k_B} is only of order $\mathcal{O}(\varepsilon)$. Therefore, the influence of a term $v_{k_A}v_{k_B}$ on the dynamics of v_{k_A} and v_{k_B} is $\mathcal{O}(\varepsilon)$ and so in lowest order the evolution equations for v_{k_A} and v_{k_B} decouple. This property can be observed in a number of problems, cf. [PW96].

However, transfering as in the subsequent Remark 3.7 the estimates from [PW96] gives an $\mathcal{O}(1)$-bound for the possible shift of the envelope for $\varepsilon \to 0$. Here we improve the bound for the physically relevant envelope shift from order $\mathcal{O}(1)$ to order $\mathcal{O}(\varepsilon)$ in case of well prepared approximate modulating pulses from Lemma 3.1. Thus we quantify the statement that there is almost no interaction of pulses with different carrier waves. We do this by extracting explicity the shift of the phase of the underlying carrier wave.

The idea is to construct an approximation

$$\varepsilon\tilde{\Psi}(x,t) = \varepsilon\Psi(x,t) + \varepsilon^3 h(x,t), \tag{10}$$

of the pulse interaction, where

$$\begin{aligned}\varepsilon\Psi(x,t) :=& \varepsilon v_{k_A}(x - c_{g,A}t + x_A, k_A x - \omega_A t + \varepsilon\Omega_A(\eta_B)) \\ &+ \varepsilon v_{k_B}(x - c_{g,B}t + x_B, k_B x - \omega_B t + \varepsilon\Omega_B(\eta_A)),\end{aligned} \tag{11}$$

with explicit functions Ω_A, Ω_B, given by

$$\Omega_A = \int_{-\infty}^{\eta_B} \frac{3|B_1|^2}{\omega_A(c_A - c_B)}\, d\tilde{\eta}_B + \Omega_A^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_B = \varepsilon(x + x_B - c_{g,B}t), \tag{12}$$

$$\Omega_B = \int_{-\infty}^{\eta_A} \frac{3|A_1|^2}{\omega_B(c_B - c_A)}\, d\tilde{\eta}_A + \Omega_B^0 + \mathcal{O}(\varepsilon^2 e^{-r|\eta_B|}), \quad \eta_A = \varepsilon(x + x_A - c_{g,A}t), \tag{13}$$

where B_1 and A_1 are given by (4) with constants $C_{1,B}, C_{2,B}$ and $C_{1,A}, C_{2,A}$, respectively, and where Ω_A^0 and Ω_B^0 are constants which normalize the initial phases. Note that Ω_A depends on $x - c_{g,B}t$ and Ω_B on $x - c_{g,A}t$ as the phase shift accounts for so called cross phase modulation. In (10), $h(x,t)$ are higher order terms, and the ansatz leads to estimates for the residual similar to the ones of Lemma 3.1, i.e., $\|\mathrm{Res}(\varepsilon\tilde{\Psi})\|_{H^s} = \mathcal{O}(\varepsilon^{11/2})$, and $h(x,t)$ is$\mathcal{O}(1)$–bounded on the natural $\mathcal{O}(1/\varepsilon^2)$ time scale. We remark that $\|\mathrm{Res}(\varepsilon\Psi)\|_{H^s} = \mathcal{O}(\varepsilon^{5/2})$ would not allow to prove estimates for the approximation of solutions of (1) by $\varepsilon\Psi$ on the $\mathcal{O}(1/\varepsilon^2)$ time scale.

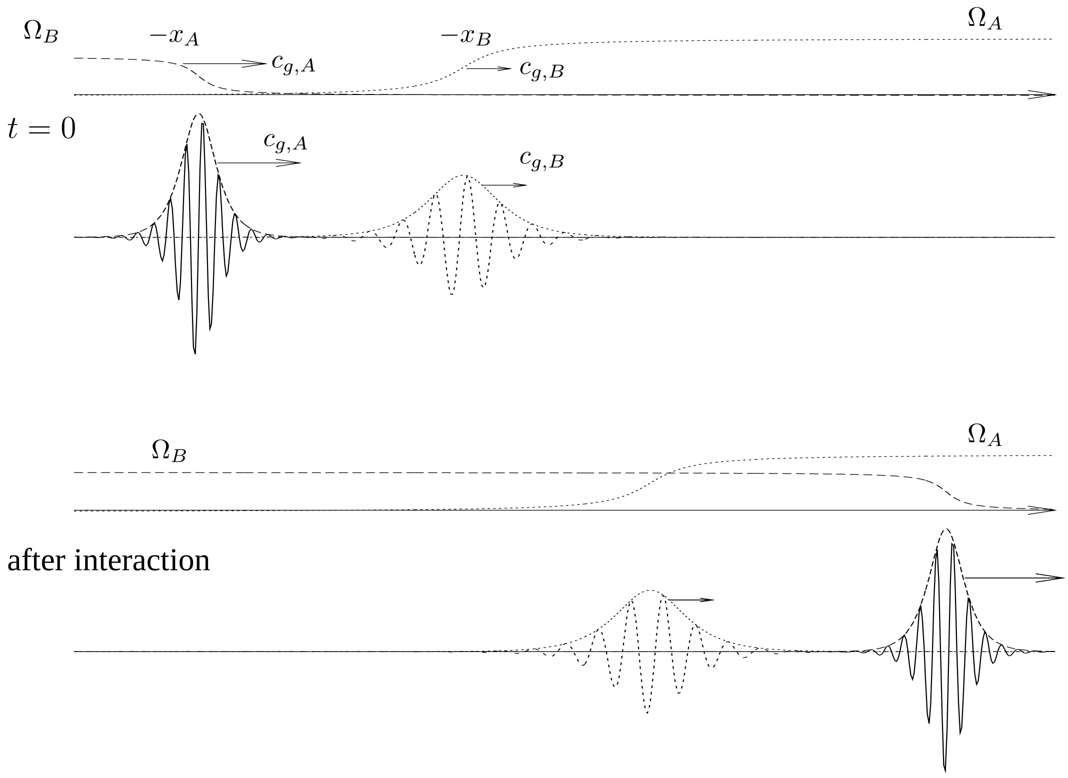

Figure 2: Illustration of our main result, as detailed in Lemma 3.4 and Theorem 3.6 below. Here $k_A > k_B$ and the slower pulse is in front. Thus, $c_B - c_A < 0$ in (13), and Ω_B is a decaying function of x. The constants Ω_A^0 and Ω_B^0 have been chosen in such a way that at $t = 0$ (upper two pictures) there are no phase-shifts for the pulses, i.e., Ω_B is exponentially small near the position $-x_B$ of v_{k_B}, while Ω_A is exponentially small near the position $-x_A$ of v_{k_A}. Note that Ω_A moves with the B–pulse, while Ω_B moves with the A–pulse.

Lemma 3.4 *Let $s \geq 2$, $k_A, k_B > 0$, $k_A \neq k_B$, $\gamma_{0,A}, \gamma_{0,B} < 0$, $x_A, x_B \in \mathbb{R}$ in Lemma 3.1, and $T_0 > 0$. Then there exist $\varepsilon_0 > 0$ and $C, C_{\text{res}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists an approximation $\varepsilon\Psi$ of the pulse interaction in the form (10), where*

$$\|h(\cdot, t)\|_{C_b^{s-1}} \leq C, \tag{14}$$

and

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|\text{Res}(\varepsilon\tilde{\Psi})\|_{H^s} \leq C_{\text{res}}\varepsilon^{11/2}. \tag{15}$$

Proof. See Section 4.2. ■

Using (15) it is easy to show that given initial data close to $\varepsilon\tilde{\Psi}(x, 0)$ the solution u to (1) stays close to the approximation $\varepsilon\tilde{\Psi}$ of the pulse interaction.

Lemma 3.5 *Under the assumptions of Lemma 3.4 there exist $\varepsilon_0 > 0$ and $C_1, C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds: if*

$$\left\|u(\cdot, 0) - \varepsilon\tilde{\Psi}(\cdot, 0)\right\|_{H^s} + \left\|\partial_t u(\cdot, 0) - \varepsilon\frac{d}{dt}\tilde{\Psi}(\cdot, 0)\right\|_{H^{s-1}} \leq C_1\varepsilon^{7/2} \tag{16}$$

with $\varepsilon\tilde{\Psi}$ from (10), then

$$\sup_{t\in[0,T_0/\varepsilon^2]}\left(\left\|u(\cdot,t)-\varepsilon\tilde{\Psi}(\cdot,t)\right\|_{H^s}+\left\|\partial_t u(\cdot,t)-\varepsilon\frac{d}{dt}\tilde{\Psi}(\cdot,t)\right\|_{H^{s-1}}\right)\leq C_2\varepsilon^{7/2}\,, \tag{17}$$

Proof. See Section 4.3. ■

The triangle inequality with (14) and (17) and Sobolev's embedding theorem, $H^s\subset C_b^{s-1}$, immediately gives the main result, see fig. 2 for illustration.

Theorem 3.6 *Let $s\geq 2$, $k_A,k_B>0$, $k_A\neq k_B$, $\gamma_{0,A},\gamma_{0,B}<0$, $x_A,x_B\in\mathbb{R}$ in Lemma 3.1, and $T_0>0$. Then there exist $\varepsilon_0>0$ and $C_1,C_2>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ the following holds: if*

$$\begin{aligned}&\left\|u(x,0)-v_{k_A}(x{+}x_A,k_Ax{+}\varepsilon\Omega_A(\eta_B|_{t=0}))-v_{k_B}(x{+}x_A,k_Bx{+}\Omega_B(\eta_A|_{t=0}))\right\|_{H^s}\\&+\left\|\partial_t u(x,0)-\frac{d}{dt}\left[v_{k_A}(x{+}x_A,k_Ax{+}\varepsilon\Omega_A(\eta_B|_{t=0}))+v_{k_B}(x{+}x_A,k_Bx{+}\Omega_B(\eta_A|_{t=0}))\right]\right\|_{H^{s-1}}\\&\leq C_1\varepsilon^{7/2}.\end{aligned} \tag{18}$$

where Ω_A,Ω_B are given by (12),(13), then

$$\begin{aligned}\sup_{t\in[0,T_0/\varepsilon^2]}&\big\|u(x,t){-}v_{k_A}(x{-}c_{g,A}t{+}x_A,k_Ax{-}\omega_At{+}\varepsilon\Omega_A(\eta_B))\\&-v_{k_B}(x{-}c_{g,B}t+x_B,k_Bx{-}\omega_Bt{+}\varepsilon\Omega_B(\eta_A))\big\|_{C_b^{s-1}}\leq C_2\varepsilon^3.\end{aligned} \tag{19}$$

Remark 3.7 If $x_A-x_B\geq C\varepsilon^{-(1+\delta)}$ for a $\delta>0$, then initially, i.e., at $t=0$, Ω_A is exponentially small near $-x_A$ and Ω_B is exponentially small near $-x_B$, see fig. 2 for illustration, and we may replace (18) by the more readable condition

$$\begin{aligned}&\left\|u(x,0)-v_{k_A}(x{+}x_A,k_Ax)-v_{k_B}(x{+}x_A,k_Bx)\right\|_{H^s}\\&+\left\|\partial_t u(x,0)-\frac{d}{dt}\left[v_{k_A}(x{+}x_A,k_Ax)+v_{k_B}(x{+}x_A,k_Bx)\right]\right\|_{H^{s-1}}\leq C_1\varepsilon^{7/2},\end{aligned} \tag{20}$$

which means that u is really close to the sum of two pulses. It finally remains to transfer the result (19) into an estimate for a possible shift of the envelope. Suppose that the error comes from a shift of the envelope. Then due to the long wave form of the envelope "vertical" estimates of order $\mathcal{O}(\varepsilon^3)$ in L^∞ can lead on a pulse of amplitude $\mathcal{O}(\varepsilon)$ only to a possible envelope shift εa of order $\mathcal{O}(\varepsilon)$, due to

$$\varepsilon g(\varepsilon(x+\varepsilon a))-\varepsilon g(\varepsilon x)=\varepsilon g'(\varepsilon x)\varepsilon^2a+\mathcal{O}(\varepsilon(\varepsilon^2a)^2)=\mathcal{O}(\varepsilon^3).$$

This, together with Theorem 3.6 means that there is almost no interaction of well prepared modulating pulse solutions to carrier waves with different wave numbers, i.e., the physically relevant envelopes are almost not affected by the interaction.

4 The proofs

4.1 Construction of well-prepared pulses

Lemma 3.1 can be proved as in [GS01] with the help of spatial dynamics and invariant manifold theory. In order to keep the paper as self-contained and as simple as possible, here we give a proof only using simple perturbation analysis. We make the ansatz

$$\varepsilon v_{k_0} = \varepsilon A_1 E + \varepsilon^3 A_3 E^3 + \varepsilon^5 A_5 E^5 + \text{c.c.}$$

where the A_j depend on the variable $\xi = \varepsilon(x - c_g t)$ and where $E = e^{i(k_0 x - \omega t)}$. With $A_{-1} := \bar{A}_1$ this yields

$$\begin{aligned}\operatorname{Res}(\varepsilon v_{k_0}) =& \varepsilon(\omega^2 - k_0^2 - 1)A_1 E + 2\varepsilon^2(-i\omega c_g + ik_0)\partial_X A_1 E \\ &+ \varepsilon^3((1-c_g^2)\partial_X^2 A_1 + 3A_1|A_1|^2)E + \varepsilon^5(3A_3A_{-1}^2)E \\ &+ \varepsilon^3((9\omega^2 - 9k_0^2 - 1)A_3 + A_1^3)E^3 + 2\varepsilon^4(-3i\omega c_g + 3ik_0)\partial_X A_3 E^3 \\ &+ \varepsilon^5((1-c_g^2)\partial_X^2 A_3 + 6A_3|A_1|^2)E^3 \\ &+ \varepsilon^5((25\omega^2 - 25k_0^2 - 1)A_5 + 3A_3A_1^2)E^5 + \mathcal{O}(\varepsilon^6) + \text{c.c.}\end{aligned} \qquad (21)$$

We choose $\omega^2 - k_0^2 - 1 = \gamma_2\varepsilon^2$, $\gamma_2 = 2\omega_0\gamma_0 + \varepsilon^2\gamma_0^2$, which cancels the $\mathcal{O}(\varepsilon)E$ term in $\operatorname{Res}(v_{k_0})$ and adds $\gamma_2 A_1 E$ to the $\mathcal{O}(\varepsilon^3)E$ term. Next we choose $c_g = k_0/\omega$ which cancels the $\mathcal{O}(\varepsilon^2)E$ and $\mathcal{O}(\varepsilon^4)E^3$ terms. Now we proceed in a somewhat non standard way which however will simplify the estimates of the pulse–pulse interactions, cf. Remark 4.1 below. Define A_5 by

$$(25\omega^2 - 25k_0^2 - 1)A_5 + 3A_3A_1^2 = 0,$$

which cancels the $\mathcal{O}(\varepsilon^5)E^5$ term. A_3 can be defined by

$$A_3 \;=\; -\alpha A_1^3 - \varepsilon^2\alpha((1-c_g^2)\partial_X^2 A_3 + 6A_3|A_1|^2), \quad \alpha = (9\omega^2 - 9k_0^2 - 1)^{-1}$$

which means that $A_3 = \tilde{A}_3 + \mathcal{O}(\varepsilon^2)$, $\tilde{A}_3 = -\alpha A_1^3$. So in order to cancel terms up to $\mathcal{O}(\varepsilon^5)$ it suffices to set

$$A_3 \;=\; -\alpha A_1^3 + \varepsilon^2\alpha^2((1-c_g^2)\partial_X^2 A_1^3 + 6A_1^3|A_1|^2). \qquad (22)$$

Now this is used to define A_1 as the solution of

$$\begin{aligned} 0 &= (1-c_g^2)\partial_X^2 A_1 + \gamma_2 A_1 + 3A_1|A_1|^2 + 3\varepsilon^2\tilde{A}_3 A_{-1}^2 \\ &= (1-c_g^2)\partial_X^2 A_1 + \gamma_2 A_1 + 3A_1|A_1|^2 - 3\varepsilon^2\alpha|A_1|^4 A_1. \end{aligned}$$

For all values $0 < \varepsilon \ll 1$ and $\gamma_2 < 0$ this equation has two solutions homoclinic to the origin in the $(A_1, \partial_X A_1)$-plane which yield the approximate pulse solutions.

Thus, all terms up to $\mathcal{O}(\varepsilon^5)$ in the residual cancel such that formally $\operatorname{Res}(v_{k_0}) = \mathcal{O}(\varepsilon^6)$. We obtain $\|\operatorname{Res}(\varepsilon\Psi)\|_{H^s} \leq C\varepsilon^{11/2}$ due to $\|A(\varepsilon\cdot)\|_{L^2} = \varepsilon^{-1/2}\|A\|_{L^2}$ while $\|\partial_x A(\varepsilon\cdot)\|_{L^2} = \varepsilon^{1/2}\|\partial_X A\|_{L^2}$ and similar for the higher order derivatives, i.e., the loss of $\varepsilon^{1/2}$ comes from the way the L^2-norm scales in terms of ε. The exponential bound in (8) follows directly from (5) and the definition of εv_{k_0}. ■

4.2 Construction of a formal approximation for the pulse interaction

To prove Lemma 3.4 we make the ansatz

$$\begin{aligned}\varepsilon\tilde{\Psi}(x,t) =& \varepsilon A_1 E + \varepsilon^3 A_3 E^3 + \varepsilon^5 A_5 E^5 + \varepsilon B_1 F + \varepsilon^3 B_3 F^3 + \varepsilon^5 B_5 F^5 \\ &+ \varepsilon^3 Y_A E + \varepsilon^3 Y_B F + \varepsilon^3 M_{\text{mixed}} + \text{c.c.}\end{aligned} \tag{23}$$

where $T = \varepsilon^2 t$,

$$E = e^{i(k_A x - \omega_A t + \varepsilon \Omega_A(\eta_B))}, \quad F = e^{i(k_B x - \omega_B t + \varepsilon \Omega_B(\eta_A))},$$
$$\eta_A = \varepsilon(x + x_A - c_A t), \quad \eta_B = \varepsilon(x + x_B - c_B t),$$

where A_1 and B_1 depend on η_A resp. η_B, where $Y_A = Y_A(\eta_A, T)$ and $Y_B = Y_B(\eta_B, T)$, and where A_3, A_5, B_3, B_5 depend on η_A, η_B and T. Although only two of η_A, η_B and T are independent, for notational clarity we write $A_3 = A_3(\eta_A, \eta_B, T)$ and so on. In (23), A_1 and B_1 are chosen as in Section 4.1, while A_3, B_3, A_5 and B_5 will be small corrections compared to (22). Thus, the first line in (23) essentially corresponds to $\varepsilon\Psi$ from (11).

The term $M_{\text{mixed}} = M_{\text{mixed}}(A_1, A_3, A_5, B_1, B_3, B_5, Y_A, Y_B, E, F)$ accounts for terms involving both E and F, i.e., for the mixed frequencies, which are generated by the nonlinearity according to the formula

$$\begin{aligned}(\varepsilon A_1 E + \varepsilon^3 A_3 E^3 + \varepsilon^5 A_5 E^5 + \varepsilon B_1 F + \varepsilon^3 B_3 F^3 + \varepsilon^5 B_5 F^5 + \varepsilon^3 Y_A E + \varepsilon^3 Y_B F + \text{c.c.})^3 \\ = \sum_{k_1+\ldots+k_{16}=3,\ k_j \geq 0} \frac{3!}{k_1! \cdots k_{16}!} (\varepsilon A_1 E)^{k_1} \cdots (\varepsilon^3 \overline{Y_B F})^{k_{16}}.\end{aligned}$$

At $\varepsilon^3 E^2 F$ for example the term $A_1^2 B_1$ appears. To cancel this we extend the ansatz by $\alpha_{21}\varepsilon^3 A_1^2 B_1 E^2 F$ and get an algebraic equation for α_{21} of the form

$$(1 + (2i\omega_A + i\omega_B)^2 + (2ik_A + ik_B)^2)\alpha_{21} = 3.$$

The procedure is essentially the same for each such term yielding

$$(1 + (l\omega_A + j\omega_B)^2 + (lk_A + jk_B)^2)\alpha_{lj} = \beta_{lj}.$$

Now M_{mixed} contains all these extensions, which means that we can concentrate on the remaining terms of the residual.

Exactly as in Sec.4.1 we choose $\omega_A^2 - k_A^2 - 1 = \gamma_2^A \varepsilon^2$ and $\omega_B^2 - k_B^2 - 1 = \gamma_2^B \varepsilon^2$. The group velocities can also be set analogously to $c_A = k_A/\omega_A$ and $c_B = k_B/\omega_B$.

At $\varepsilon^3 E$ and $\varepsilon^3 F$ we obtain

$$(2(k_A - \omega_A c_B)\partial_{\eta_B}\Omega_A - 6|B_1|^2)A_1 + (1 - c_A^2)\partial_{\eta_A}^2 A_1 + \gamma_2^A A_1 - 3A_1|A_1|^2 + 3\varepsilon^2 A_3 A_{-1}^2 = 0,$$
$$(2(k_B - \omega_B c_A)\partial_{\eta_A}\Omega_B - 6|A_1|^2)B_1 + (1 - c_B^2)\partial_{\eta_B}^2 B_1 + \gamma_2^B B_1 - 3B_1|B_1|^2 + 3\varepsilon^2 B_3 B_{-1}^2 = 0.$$

By associating the coupling terms $|B_1|^2 A_1$ resp. $|A_1|^2 B_1$ with $\partial_{\eta_B}\Omega_A$ resp. $\partial_{\eta_A}\Omega_B$ this coupled system splits into a decoupled set of equations. Moreover, by replacing A_3, B_3 by $\tilde{A}_3 = -\alpha A_1^3$ and $\tilde{B}_3 = -\beta B_1^3$, $\alpha = (9\omega_A^2 - 9k_A^2 - 1)^{-1}, \beta = (9\omega_B^2 - 9k_B^2 - 1)^{-1}$, we choose A_1 and B_1 as the solutions of

$$(1 - c_A^2)\partial_{\eta_A}^2 A_1 + \gamma_2^A A_1 - 3A_1|A_1|^2 + 3\varepsilon^2 \tilde{A}_3 A_{-1}^2 = 0,$$
$$(1 - c_B^2)\partial_{\eta_A}^2 B_1 + \gamma_2^B B_1 - 3B_1|B_1|^2 + 3\varepsilon^2 \tilde{B}_3 B_{-1}^2 = 0.$$

Thus, A_1, B_1 only depend on η_A, η_B, respectively. Finally we choose Ω_A and Ω_B to satisfy, respectively,

$$\partial_{\eta_B}\Omega_A = \frac{3|B_1|^2}{\omega_A(c_A - c_B)} \quad \text{and} \quad \partial_{\eta_A}\Omega_B = \frac{3|A_1|^2}{\omega_B(c_B - c_A)} \,. \tag{24}$$

Thus

$$\Omega_A = \int_{-\infty}^{\eta_B} \frac{3|B_1|^2}{\omega_A(c_A - c_B)}\, d\tilde{\eta}_B + \Omega_A^0 \quad \text{and} \quad \Omega_B = \int_{-\infty}^{\eta_A} \frac{3|A_1|^2}{\omega_B(c_B - c_A)}\, d\tilde{\eta}_A + \Omega_B^0 \,,$$

with suitable constants of integration Ω_A^0 and Ω_B^0, cf. Fig. 2.

At $\varepsilon^5 E^3$ and $\varepsilon^5 F^3$ we choose A_3 and B_3 as

$$\begin{aligned} A_3 &= -\alpha A_1^3 + \alpha^2\varepsilon^2(6|A_1|^2A_1^3 + 6|B_1|^2A_1^3 + (c_A^2 - 1)\partial_{\eta_A}^2 A_1^3 + 3A_1^2 Y_A), \\ B_3 &= -\beta B_1^3 + \beta^2\varepsilon^2(6|B_1|^2B_1^3 + 6|A_1|^2B_1^3 + (c_B^2 - 1)\partial_{\eta_B}^2 B_1^3 + 3B_1^2 Y_B). \end{aligned}$$

Compared to Section 4.1, in the equation for A_3 there are new coupling terms $\varepsilon^2(6|B_1|^2A_1^3 + 3A_1^2Y_A)$, and similar for B_3. As long as Y_A and Y_B are $\mathcal{O}(1)$ bounded, which we will show below, these terms do not make any diffuculties as they only appear in the $\mathcal{O}(\varepsilon^2)$ part and the defining equations for A_1 and B_1 only use $\tilde{A}_3$ and $\tilde{B}_3$. The $\mathcal{O}(1)$–boundedness of Y_A and Y_B also yields $A_3 - \tilde{A}_3, B_3 - \tilde{B}_3 = \mathcal{O}(\varepsilon^3)$.

At $\varepsilon^5 E$ and $\varepsilon^5 F$ we get, respectively,

$$2i\omega_A\partial_T Y_A + (1 - c_A^2)\partial_{\eta_A}^2 Y_A + \gamma_2^A Y_A + G_A = 0, \tag{25}$$
$$2i\omega_B\partial_T Y_B + (1 - c_B^2)\partial_{\eta_B}^2 Y_B + \gamma_2^B Y_B + G_B = 0, \tag{26}$$

with

$$\begin{aligned} G_A &= 6|A_1|^2Y_A + ((1 - c_B^2)(\partial_{\eta_B}\Omega_A)^2 + 6Y_B B_{-1})A_1 \\ &\quad +\varepsilon^{-1}(i(1 - c_B^2)(\partial_{\eta_B}^2\Omega_A)A_1 + 2i(c_Ac_B - 1)(\partial_{\eta_B}\Omega_A)(\partial_{\eta_A}A_1))\,, \\ G_B &= 6|B_1|^2Y_B + ((1 - c_A^2)(\partial_{\eta_B}\Omega_B)^2 + 6Y_A A_{-1})B_1 \\ &\quad +\varepsilon^{-1}(i(1 - c_A^2)(\partial_{\eta_A}^2\Omega_B)B_1 + 2i(c_Ac_B - 1)(\partial_{\eta_A}\Omega_B)(\partial_{\eta_B}B_1))\,. \end{aligned}$$

Finally, at $\varepsilon^5 E^5$ and $\varepsilon^5 F^5$ we choose A_5 and B_5 to satisfy

$$(25\omega_A^2 - 25k_A^2 - 1)A_5 + 3A_3A_1^2 = 0 \quad \text{and} \quad (25\omega_B^2 - 25k_B^2 - 1)B_5 + 3B_3B_1^2 = 0.$$

Hence formally all terms up to order $\mathcal{O}(\varepsilon^5)$ in the residual cancel. Therefore, to prove (15) and the second estimate in (14) it remains to show that Y_A and Y_B are $\mathcal{O}(1)$ bounded for $T \leq T_0$, where, by construction, we may choose $Y_A|_{T=0} = Y_B|_{T=0} = 0$. This is done in Lemma 4.2 below. Thus, the proof of Lemma 3.4 is complete. The loss of $\varepsilon^{1/2}$ in (15) again comes from the way the L^2-norm scales under $X = \varepsilon x$. ■

Remark 4.1 Here we need well prepared pulses. As already said, cf. Remark 3.3, in lowest order the NLS-equations for A_1 and B_1 decouple. However, if A_1 resp. B_1 would have been chosen to be time–dependent solutions of (2), then G_A resp. G_B would have contained the terms $\varepsilon^{-1}2c_{g,A}\partial_T\partial_{\eta_A}A_1$ resp. $\varepsilon^{-1}2c_{g,B}\partial_T\partial_{\eta_B}B_1$, which can not be handled by the subsequent analysis, i.e. the estimates would become worse.

Lemma 4.2 *For all $s \geq 2$ there exists a $C > 0$ such that for all $\varepsilon \in (0, 1]$ the following holds. System (25)-(26) with zero initial data has a unique solution $Y_A, Y_B \in C([0, T_0], H^s)$. It satisfies*

$$\sup_{T\in[0,T_0]} \|(Y_A, Y_B)(T)\|_{H^s} \leq C.$$

Proof. We rewrite (25)-(26) as

$$\partial_T \begin{pmatrix} Y_A \\ Y_B \end{pmatrix} = M \begin{pmatrix} Y_A \\ Y_B \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \omega_A^{-1} G_A \\ \omega_B^{-1} G_B \end{pmatrix},$$

where M is the linear part of (25),(26). The operator $M : D(M) \to H^s$ with $D(M) = H^{s+2}$ generates a uniformly bounded C_0 semigroup in H^s, i.e., $\|e^{tM}\|_{L(H^s,H^s)} \leq 1$. We want to apply the variation of constant formula and thus it remains to estimate the inhomogeneous terms.

Since $Y_A(\cdot, T)$ depends on η_A but G_A also contains, e.g., $\partial_{\eta_B}^2 \Omega_A$ which depends on η_B we introduce the notation

$$\|A_1\|_{H^s(m,d\eta_A)} := \left(\sum_{j=0}^{s} \int |\partial_{\eta_A}^j (\rho(\eta_A)^m A_1(\eta_A, T))|^2 d\eta_A \right)^{1/2}, \quad \rho(\eta_A) = \sqrt{1 + \eta_A^2},$$

$\|A_1\|_{H^s(d\eta_A)} = \|A_1\|_{H^s(0,d\eta_A)}$, and similar for $\|B_1\|_{H^s(m,d\eta_B)}$, $\|B_1\|_{H^s(d\eta_B)}$. Since A_1, B_1 decay with some exponential rate in space we have $A_1, B_1 \in H^s(4)$. This implies $\partial_{\eta_B}^2 \Omega_A$, $\partial_{\eta_A}^2 \Omega_B \in H^s(2)$ and so terms like $i(c_B^2 - 1)(\partial_{\eta_B}^2 \Omega_A) A_1$ in G_A can be estimated as follows. We have

$$\begin{aligned} &\int_0^T \|(\partial_{\eta_B}^2 \Omega_A) A_1\|_{H^s(d\eta_A)} d\tau \\ &\leq \int_0^T \|\partial_{\eta_B}^2 \Omega_A\|_{H^s(2,d\eta_B)} \|A_1\|_{H^s(2,d\eta_A)} \|\varrho_A^{-2} \varrho_B^{-2}\|_{H^s(d\eta_A)} d\tau \\ &\leq \int_0^T \|\varrho_A^{-2} \varrho_B^{-2}\|_{H^s(d\eta_A)} d\tau \|\partial_{\eta_B}^2 \Omega_A\|_{H^s(2)} \sup_{\tau\in[0,T_0]} \|A_1\|_{H^s(2)} . \end{aligned}$$

Since $\varrho_A(\eta_A)\varrho_B(\eta_B) = (1+\eta_A^2)(1+(\eta_A-(c_A-c_B))\frac{\tau}{\varepsilon})^2)$ the time integral is of order $\mathcal{O}(\varepsilon)$. Applying the variation of constant formula to the equations for Y_A and Y_B with zero initial conditions and using Gronwall's inequality yields $\sup_{T\in[0,T_0]} \|(Y_A, Y_B)(T)\|_{H^s} \leq C = \mathcal{O}(1)$. ■

Remark 4.3 A way to increase the rate of information through the fibers is to choose the wave numbers k_A and k_B close together. For $k_A - k_B = \mathcal{O}(\varepsilon^\mu)$ with $0 \leq \mu \leq 1$ we formally find a shift of the envelope of order $\mathcal{O}(\varepsilon^{1-2\mu})$ by looking at the way (24), (25), and (26) scale. Thus we must expect a certain trade off between the wish to decrease $|k_A - k_B|$ to use more channels and the need for larger spacing of bits in a given channel, i.e., the need to increase $|x_{A,1} - x_{A,2}|$ to account for possibly larger envelope shifts.

4.3 Validity of the approximation

The proof of Lemma 3.5 is relatively easy due the fact that our choice of an original system (1) does not contain quadratic terms, cf. [KSM92]. There are a number of mathematical papers proving error estimates for the approximation of the original system by the NLS equation also in case of quadratic nonlinearities. See [Kal88, Sch98b, Sch98a, Sch05] for the spatially homogenous case and [BSTU06] for some first results in the spatially periodic case. The following proof is an easy adaption of the one from [KSM92]. A similar adaption holds in case of quadratic terms in the nonlinearity.

We define the deviation $\varepsilon^{7/2}R$ from the solution u by

$$u = \varepsilon\Psi + \varepsilon^{7/2}R$$

and find R to solve

$$\partial_t^2 R = \partial_x^2 R - R + f, \qquad R|_{t=0} = R_0, \quad \partial_t R|_{t=0} = R_1,$$

with $\|R_0\|_{H^s} = \varepsilon^{-7/2}\|u(x,0) - \varepsilon\Psi(x,0)\|_{H^s} \leq C_1$ and $\|R_1\|_{H^{s-1}} = \varepsilon^{-7/2}\|\partial_t u(x,0) - \varepsilon\partial_t\Psi(x,0)\|_{H^{s-1}} \leq C_1$, and with

$$f = -\varepsilon^{-7/2}(3\varepsilon^{11/2}\Psi^2 R + 3\varepsilon^8\Psi R^2 + \varepsilon^{21/2}R^3 + \mathrm{Res}(\varepsilon\Psi))$$

satisfying

$$\|f\|_{H^s} \leq C_3\varepsilon^2\|R\|_{H^s} + C_4(C_E)\varepsilon^{9/2}\|R\|_{H^s}^2 + C_{\mathrm{Res}}\varepsilon^2 \; , \tag{27}$$

as long as $\|R\|_{H^s} < C_E$ with C_E a constant defined below independent of $0 < \varepsilon \ll 1$.

For the time derivative of

$$E(R) = \sum_{j=0}^{s} \int_{\mathbb{R}} (\partial_t\partial_x^j R)^2 + (\partial_x^{j+1}R)^2 + (\partial_x^j R)^2 dx$$

we find, using (27) and $\|R\|_{H^s} \leq (E(R))^{1/2}$,

$$\begin{aligned}
\frac{1}{2}\,\partial_t E(R) &= \sum_{j=0}^{s}\Big\{\int (\partial_t\partial_x^j R)(\partial_x^{j+2}R - \partial_x^j R + \partial_x^j f) \\
&\qquad\qquad + (\partial_x^{j+1}R)(\partial_t\partial_x^{j+1}R) + (\partial_x^j R)(\partial_t\partial_x^j R)\Big\}dx \\
&= \sum_{j=0}^{s}\int (\partial_t\partial_x^j R)\partial_x^j f\, dx \\
&\leq E(R)^{1/2}(C_3\varepsilon^2 E(R)^{1/2} + C_4(C_E)\varepsilon^{9/2}E(R) + C_{\mathrm{Res}}\varepsilon^2) \\
&\leq (C_3 + C_{\mathrm{Res}})\varepsilon^2 E(R) + C_4(C_E)\varepsilon^{9/2}E(R)^{3/2} + C_{\mathrm{Res}}\varepsilon^2,
\end{aligned}$$

as long as $E(R) \leq C_E^2$. If we choose $\varepsilon > 0$ so small that

$$\varepsilon^{5/2}C_4(C_E)C_E < 1 \; , \tag{28}$$

then

$$\frac{1}{2}\,\partial_t E(R) \;\; \leq \;\; (C_3 + C_{\mathrm{Res}} + 1)\varepsilon^2 E(R) + C_{\mathrm{Res}}\varepsilon^2.$$

By Gronwall's inequality we find

$$E(R) \leq E(R(0))e^{2(C_3+C_{\mathrm{Res}}+1)T_0} + \frac{C_{\mathrm{Res}}}{C_3 + C_{\mathrm{Res}} + 1}(e^{2(C_1+C_{\mathrm{Res}}+1)T_0} - 1) =: C_E^2 \,.$$

Since $\|R\|_{H^s} \leq E(R)^{1/2}$ we are done if we define $\varepsilon_0 > 0$ through (28). ■

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