Self–similar decay of spatially localized perturbations of the Nusselt solution for the inclined film problem

Hannes Uecker

Mathematisches Institut I, Universität Karlsruhe, D–76128 Karlsruhe

hannes.uecker@math.uni-karlsruhe.de

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Abstract

The Nusselt solution for the flow of a viscous incompressible fluid with a free surface down an inclined plane is at best marginally stable, i.e., the linearization has essential spectrum at least up to the imaginary axis. Nevertheless, using a renormalization group approach here we establish the stability of the Nusselt solution in the full nonlinear system in case of linear stability by proving the self similar decay of spatially localized perturbations. The asymptotic decay for $t \to \infty$ is similar to the dynamics of localized perturbations of the trivial solution in the Burgers equation on the real line which is the amplitude equation of the problem.

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1 Introduction

We consider the flow of a viscous incompressible fluid down an inclined plane, driven by gravity and governed by the Navier–Stokes equations with a free top surface. There exists a basic stationary solution with a parabolic flow profile and a flat top surface, the so called the Nusselt solution. Depending on the parameters of the system, e.g. the height of the fluid and the inclination angle, this solution is linearly stable or unstable. There are a number of experimental, numerical and analytic investigations of the system close to the first instability of the Nusselt solution [LG94, CD96, CD02, Uec03, PSU04a, PSU04b] leading to interesting dynamics, as unstable pulse dynamics for surface waves. Here we establish the stability of the Nusselt solution in the full nonlinear system in case of linear stability. The problem is non–trivial due to a complicated local existence theory and in particular due to the fact that even in the stable case the linearization around the Nusselt solution has essential spectrum up to the imaginary axis. Hence no classical argument for the nonlinear stability applies.

Although the linearization only gives marginal stability, the associated linear semigroup shows polynomial decay rates if we restrict to spatially localized perturbations. The asymptotic decay for $t \to \infty$ in the linearized system is similar to the decay in the 1–dimensional linear diffusion equation. However, in contrast to exponential decay rates the polynomial rate $t^{-\frac{1}{2}}$ is too weak to control all nonlinear terms. In fact, we show that the asymptotic dynamics is nonlinear: the so called renormalized solution does not converge towards the Gaussian profile coming from the linearized system but to a non Gaussian profile related to the Burgers equation, which is the amplitude equation of the system in the stable case. In lowest order, small spatially localized perturbations show the same asymptotics as localized perturbations of the trivial solution in the Burgers equation.

The paper is based on the renormalization group approach for the proof of diffusive behavior in nonlinear diffusion equations [BKL94], which has been transfered to more complicated systems as the Ginzburg–Landau equation and pattern forming systems in [BK92, Sch96, Sch98, Uec99, ES00, ES02, SU03, GSU04]. In contrast to these applications our system is quasilinear and the renormalized solution converges to a non Gaussian limit.

In the remainder of this introduction we first give the equations governing the inclined film flow and explain the derivation of the Burgers equation and the asymptotic behavior of localized perturbations of the trivial solution in the Burgers equation. The precise result is then stated in sec.1.4.

1.1 The problem

Figure 1 shows the geometry for a two–dimensional liquid film flowing down an inclined rigid plate with inclination angle θ. The flow is driven by gravity, the top surface is free, at the bottom we assume the no–slip boundary conditions $u_1{=}u_2{=}0$, and above the fluid we assume a constant atmospheric pressure p_a (w.l.o.g. $p_a{=}0$). The motion of the film is thus described by the incompressible Navier–Stokes equations for (u_1, u_2) and the kinematic equation for the free surface, together with boundary condition for the stress at the free surface (including surface tension, see below). Again we refer to the monograph [CD02] for a comprehensive review of

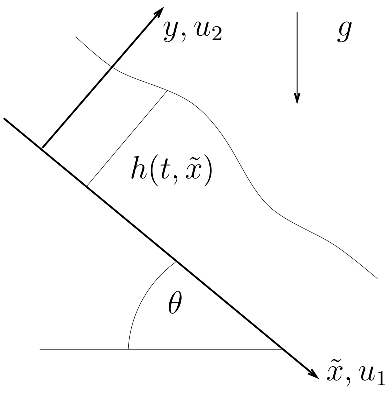

Figure 1: The inclined film problem; a fluid of height $y = h(t, \tilde{x})$ runs down a plate with inclination angle θ subject to constant gravitational force g.

this so called inclined film problem. For given parameters the problem has a basic solution, the so called Nusselt solution

$$(h, u_1, u_2, p) = (h_0, u_N, 0, p_N),\ u_N(y) = \frac{g \sin\theta}{2\nu}(2h_0 y - y^2),\ p_N(y) = \rho g \cos\theta (h_0 - y),$$

with a flat top surface and a laminar flow in the unbounded $\tilde{x}$–direction. Here ν and ρ are the kinematic viscosity and the density of the fluid, and g is the gravitational constant.

We assume that at initial time the free surface is a graph over $\tilde{x}$, and that we are close to a spectrally stable Nusselt solution, in a sense made precise below. It turns that the free surface stays a graph over $\tilde{x}$ for all times, and in fact we show that initial perturbations decay; moreover, they decay in a universal manner.

We use the height h_0 of the flat film as the characteristic length, the surface velocity $u_N^* = u_N(h_0) = gh_0^2 \sin\theta/2\nu$ of the Nusselt solution as characteristic velocity, and define the Reynolds number

$$\mathrm{R} = u_N^* h_0/\nu.$$

In dimensionless variables the Nusselt solution is

$$h \equiv 1, \quad u_N(y) = 2y - y^2, \quad p_N(y) = \frac{2\cot\theta}{\mathrm{R}}(1-y).$$

From previous work [CD02] we know that on the linear level small amplitude long surface waves travel with twice the (dimensionless) surface speed $u_N(1)$=1 of u_N. Thus we consider the dimensionless Navier–Stokes equations and boundary conditions for perturbations η, u and p of the Nusselt solution in this comoving frame: we set $x = \tilde{x} - 2t$ and obtain

$$\text{on } \Gamma_{\mathrm{f}}: \quad \partial_t \eta = u_2 + \partial_x \eta - (\partial_x \eta)(u_1 - \eta^2), \tag{1.1a}$$

$$\text{in } \Omega: \quad \partial_t u_1 - \frac{1}{\mathrm{R}}\Delta u_1 + \partial_x p + (u_N - 2)\partial_x u_1 + u_N' u_2 = -(\partial_x u_1)u_1 - (\partial_y u_1)u_2, \tag{1.1b}$$

$$\partial_t u_2 - \frac{1}{\mathrm{R}}\Delta u_2 + \partial_y p + (u_N - 2)\partial_x u_2 = -(\partial_x u_2)u_1 - (\partial_y u_2)u_2, \tag{1.1c}$$

$$\operatorname{div} u = 0, \tag{1.1d}$$

$$\text{on } \Gamma_{\mathrm{f}}: \quad 4(\partial_x\eta)(\partial_x u_1) + ((\partial_x\eta)^2 - 1)(\partial_y u_1 + \partial_x u_2 - 2\eta) = 0, \tag{1.1e}$$

$$p - g^\star\eta - \frac{2}{\mathrm{R}}\frac{(\partial_x\eta)^2(\partial_x u_1) - (\partial_x\eta)(\partial_y u_1 + \partial_x u_2 - 2\eta) + \partial_y u_2}{1 + (\partial_x\eta)^2} = -\mathrm{W}K(\eta), \tag{1.1f}$$

$$\text{on } \Gamma_{\mathrm{b}}: \quad u = 0, \tag{1.1g}$$

where u_N'=$\partial_y u_N$ and $g^\star = 2\cot\theta/\mathrm{R}$.

In (1.1), $\Gamma_{\mathrm{f}} = \Gamma_{\mathrm{f}}(t) = \{(x,y) : x \in \mathbb{R},\ y = 1 + \eta(t,x)\}$ is the free surface, $\Omega = \Omega(t) = \{(x,y) : x \in \mathbb{R},\ 0 < y < 1 + \eta(t,x)\}$ the fluid domain, $\Gamma_{\mathrm{b}} = \{(x,y) : x \in \mathbb{R},\ y = 0\}$ the bottom, $\mathrm{W} = \sigma/(\rho {u_N^*}^2 h_0)$ is the Weber number with $\sigma > 0$ being the coefficient of surface tension, and $K(\eta) = (\partial_x^2\eta)/(1 + (\partial_x\eta)^2)^{3/2}$ is the interfacial curvature; (1.1a) is the kinematic condition, (1.1b) and (1.1c) are the momentum balance, (1.1d) is the continuity equation, (1.1e) and (1.1f) are the continuity of tangential and normal stresses at the free surface, and (1.1g) is the no-slip condition at the bottom.

The evolution for (1.1) is determined by specifying the initial surface and the initial velocity field, while the pressure p is resolved a posteriori from the incompressibility condition. Using a projection onto solenoidal vector fields, the linearization of (1.1) can be written in the form

$$\partial_t U = \mathcal{A}U, \quad U = (\eta, u), \tag{1.2}$$

where $\mathcal{A}$ is a sectorial operator and $\mathrm{e}^{t\mathcal{A}}$ has certain smoothing properties (see sec.2). The nonlinear problem can then be written as a quasilinear system

$$\partial_t U = \mathcal{A}U + F(U, \nabla p), \tag{1.3}$$

where $F(U, \nabla p)$ contains the nonlinear terms and ∇p is obtained from the incompressibility condition. This formulation involves a transformation of the time dependent domain $\Omega(t)$ occupied by the fluid to the fixed domain

$$\Omega = \{(x, y) : x \in \mathbb{R}, 0 < y < 1\}.$$

Due to the transformation ∇p occurs nonlinearly in (1.3). Using maximal regularity methods [LM68] for (1.2) and the contraction mapping theorem for the nonlinear problem, the local existence in Sobolev spaces for (1.1) has been shown in [Ter92], following [Bea80, Bea84]. For this one must also impose the compatibility conditions

$$\begin{aligned} \operatorname{div} u = 0 \quad \text{in } 0 < y < 1 + \eta(x), \quad u = 0 \quad \text{on} \quad y = 0, \\ 4(\partial_x \eta)(\partial_x u_1) + ((\partial_x \eta)^2 - 1)(\partial_y u_1 + \partial_x u_2 - 2\eta) = 0 \quad \text{on} \quad y = 1 + \eta, \end{aligned} \tag{1.4}$$

on the initial data $\eta|_{t=1} = \eta_0,\ u|_{t=1} = u_0$. The initial time $t = 1$ will be convenient in the renormalization process.

1.2 Derivation of the Burgers equation

Due to the unboundedness in x of the domain and the translation invariance in x of the problem, the linearization (1.2) of (1.1) around 0 has solutions of the form

$$\mathrm{e}^{\lambda_n(k)t} \mathrm{e}^{\mathrm{i}kx} \Phi_n(k, y),$$

with $k \in \mathbb{R}$, $n \in \mathbb{N}$ and $\Phi_n(k, y) \in \mathbb{C}^3$. The Nusselt solution is spectrally stable if all eigenvalues $\lambda_n(k)$ satisfy

$$\mathrm{Re}\lambda_n(k) \leq 0$$

and if all eigenvalues with real part zero are semi simple. Let the curves of eigenvalues be ordered such that $\mathrm{Re}\lambda_n(k) \geq \mathrm{Re}\lambda_{n+1}(k)$. Then we always have the simple eigenvalue $\lambda_1(0) = 0$ for all values of the parameters, with $\Phi_1(0, y) = (1, 2y, 0)$. By a standard perturbation argument we have the smoothness of curves of simple eigenvalues, i.e. we have essential spectrum at least up to the imaginary axis.

From previous work [Ben57] it is well known that for Reynolds numbers greater than the critical Reynolds number, i.e.,

$$\mathrm{R} > \mathrm{R}_c = \frac{5}{4}\cot\theta, \tag{1.5}$$

the Nusselt solution is unstable with respect to long waves. In the spectrally stable (unstable) case we have a spectrum as sketched in fig. 2(a) (fig. 2(b)).

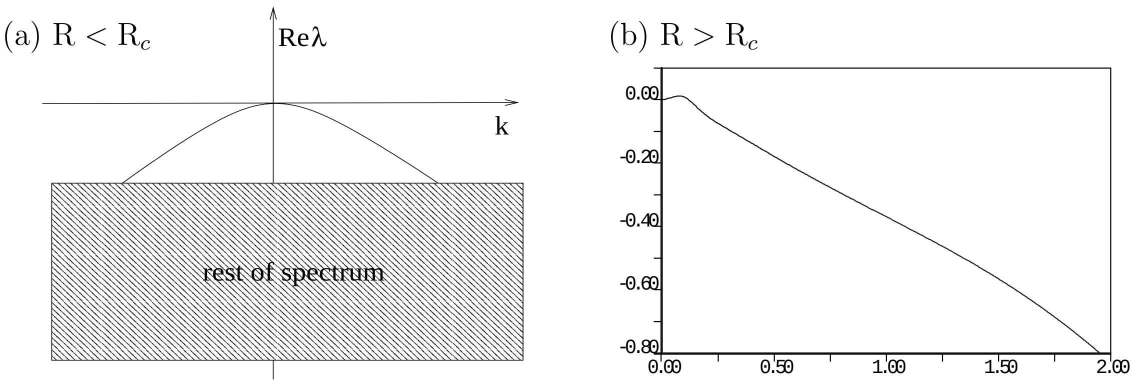

Figure 2: The spectrum drawn over the Fourier wave numbers; (a) schematic sketch in the stable case; (b) $\mathrm{Re}\lambda_1$ in the unstable case, obtained from a numerical solution of the Orr–Sommerfeld equations, see sec.2.2.

Remark 1.1 Note that the critical Reynolds number R_c in (1.5) is defined in terms of instability with respect to long waves. For very low inclination angles and high Reynolds numbers the Nusselt solution can first become unstable due to a short wave shear mode instability at a wavenumber $k_c > 0$, see [FDK87] and [CD02, sec.2.6]. Here we always exclude the extreme case of this instability. Also note that the (in)stability in (1.5) only depends on R and θ, while, e.g., the size of the unstable sideband and the growth rates also depend on the Weber number W [CD02, Uec03]. The inclusion of surface tension is also important for the existence theory of (1.1), cf. [Puk72, Bea80]. Here we assume throughout that $\mathrm{W} > 0$. ⌋

In the spectrally stable case the Burgers equation may be formally derived as an amplitude equation for (1.1). For this we substitute the long wave/small amplitude ansatz

$$\begin{pmatrix} \eta \\ u_1 \\ u_2 \\ p \end{pmatrix}(t,x,y) = \delta\Psi(T,X,y) = \begin{pmatrix} \delta\eta_1(T,X) \\ \delta u_{11}(T,X,y) + \delta^2 u_{12}(T,X,y) \\ \delta^2 u_{22}(T,X,y) + \delta^3 u_{23}(T,X,y) \\ \delta p_1(T,X,y) \end{pmatrix}, \tag{1.6}$$

$$T = \delta^2 t, \quad X = \delta x, \tag{1.7}$$

with $0 < \delta \ll 1$ a small perturbation parameter into (1.1). This yields a hierarchy of equations which can be successively solved. We first obtain

$$
\begin{aligned}
u_{11} &= 2\eta_1 y, \quad p_1 = g^\star \eta_1, \quad u_{22} = -(\partial_X \eta_1) y^2, \\
u_{12} &= \mathrm{R}(\partial_X \eta_1)\Big[\frac{1}{6}y^4 - \frac{2}{3}y^3 + \frac{1}{2}g^\star y^2 + (\frac{4}{3} - g^\star)y\Big], \\
u_{23} &= \mathrm{R}(\partial_X^2 \eta_1)\Big[-\frac{1}{30}y^5 + \frac{1}{6}y^4 - \frac{1}{6}g^\star y^3 - \frac{1}{2}(\frac{4}{3} - g^\star)y^2\Big],
\end{aligned}
\tag{1.8}
$$

and from the kinematic equation (1.1a) at order $\mathcal{O}(\delta^3)$ we find that η_1 has to satisfy the Burgers equation

$$
\partial_T \eta_1 = \alpha \partial_X^2 \eta_1 + \beta \partial_X(\eta_1^2) \tag{1.9}
$$

with

$$
\alpha = \frac{2}{3}\cot\theta - \frac{8\mathrm{R}}{15} \quad \text{and} \quad \beta = -2.
$$

Note that $\alpha > 0$ iff $\mathrm{R} < \mathrm{R}_c$, while β is independent of the parameters. A few more remarks are in order:

Remark 1.2 If, for example, we start with (1.1) in the labaratory frame, then the ansatz (1.6) with (1.7) replaced by $X = \delta(\tilde{x} - ct)$ naturally leads to $c = 2$. ⌋

Remark 1.3 Our expansion (1.6),(1.8) and (1.9) is unbalanced in the sense that we solve ((1.1b),(1.1c),(1.1e),(1.1f)) to orders $(\delta^2, \delta, \delta^2, \delta)$, respectively, and (1.1a) to order δ^3, while (1.1d) and (1.1g) hold exactly. By adding higher order terms $(0, \delta^3 u_{13}, \delta^4 u_{24}, \delta^2 p_2 + \delta^3 p_3)$ to the ansatz (1.6) and continuing similar to (1.8) we could also solve (1.1b),(1.1c),(1.1e) and (1.1f) up to order δ^3 (where in fact $\delta^4 u_{24}$ would only be needed to still satisfy the incompressibility exactly). However, at this stage we are only interested in the formal derivation of (1.9), which is unchanged by higher order terms. Essentially, to prove Theorem 1.6 we need to make precise the sense in which the dynamics of (1.1) are described by the Burgers equation (1.9). These rigorous estimates require a functional analytic frame and are therefore postponed to sec.6. See also Remark 1.8. ⌋

Remark 1.4 Above the threshold of instability, and in the limit of large Weber number W, the Kuramoto–Sivashinsky equation [Nep74, KT76, Siv77, TK78] $\partial_T \eta = -\alpha_1 \partial_X^2 \eta - \alpha_2 \partial_X^4 \eta - 2\partial_X(\eta^2)$ with $\alpha_1, \alpha_2 > 0$ can be derived from (1.1). For details and for various alternative amplitude equations for (1.1) in different scaling limits see [CD02] and the references therein, and [FI99] for an approach where no scaling for the parameters is assumed a priori and the perturbation analysis is based on a "minimal derivability" condition. Here we concentrate on the spectrally stable case which is the starting point for all rigorous analytic investigations of the problem. ⌋

1.3 Asymptotic behavior in the Burgers equation

The Burgers equation (1.9), i.e., $\partial_t\eta = \alpha\partial_x^2\eta + \beta\partial_x(\eta^2)$ after renaming $T = t$, $X = x$ and $\eta_1 = \eta$, is the amplitude equation for (1.1) in the spectrally stable case. In order to motivate our result for (1.1) we first consider the nonlinear stability of the trivial solution in (1.9). To keep track of α and β we do not rescale (1.9) to the more standard form $\partial_\tau\eta = \partial_\xi^2\eta + \partial_\xi(\eta^2)$.

The Burgers equation is transformed to the linear diffusion equation $\partial_t\psi = \partial_x^2\psi$ by the Cole–Hopf transformation

$$\psi(t,x) = \exp\left(\frac{\beta}{\alpha}\int_{-\infty}^{\sqrt{\alpha}x}\eta(t,\xi)\,\mathrm{d}\xi\right), \quad \eta(t,x) = \frac{\sqrt{\alpha}}{\beta}\frac{\psi_y(t,y)}{\psi(t,y)}, \quad y = x/\sqrt{\alpha}.$$

For $\lim_{x\to-\infty}\psi(x) = 1$ and setting $\lim_{x\to\infty}\psi(x) = z+1$, i.e., $\ln(z+1) = \frac{\beta}{\alpha}\int_{\mathbb{R}}\eta(t,\xi)\,\mathrm{d}\xi$, it is well known that

$$1 + z\,\mathrm{erf}(x/\sqrt{t}) \quad \text{with} \quad \mathrm{erf}(x) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{x}\mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi$$

is an exact solution of $\partial_t\psi = \partial_x^2\psi$. It follows that

$$\eta^{(z)}(t,x) = t^{-1/2}f_z(x/\sqrt{t}) \quad \text{with} \quad f_z(y) = \frac{\sqrt{\alpha}}{\beta}\frac{z\,\mathrm{erf}'(y/\sqrt{\alpha})}{1 + z\,\mathrm{erf}(y/\sqrt{\alpha})} \tag{1.10}$$

is an exact solution of the Burgers equation. Moreover,

$$\psi(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}\mathrm{e}^{-(x-y)^2/(4t)}\psi_0(y)\,\mathrm{d}y \to 1 + z\,\mathrm{erf}(x/\sqrt{t}) \text{ as } t\to\infty,$$

with rate $\mathcal{O}(t^{-1})$, for initial conditions $\psi_0 \in L^\infty(\mathbb{R})$ with $\lim_{\xi\to-\infty}\psi(\xi) = 1$ and $\lim_{\xi\to\infty}\psi(\xi) = 1 + z$. Therefore the so called renormalized solution satisfies

$$\lim_{t\to\infty} t^{1/2}\eta(t, t^{1/2}x) = f_z(x) \tag{1.11}$$

with rate $\mathcal{O}(t^{-1/2})$, i.e., it converges towards a non-Gaussian limit. This is illustrated in fig.3, taking into account that $\beta = -2 < 0$ and $-1 < z < 0$ if $\int\eta(1,x)\,\mathrm{d}x > 0$. The behaviour (1.11) is not true for spatially non-localized initial conditions since the Burgers equation has front solutions $\eta(t,x) = h(x-ct)$ with $|h(\xi)| \not\to 0$ as $|\xi|\to\infty$. It has been shown in [BKL94] that the self–similar dynamics (1.11) in the Burgers equation is stable under perturbation by higher order terms. Since, in a nutshell, we want to consider (1.1) for $\mathrm{R} < \mathrm{R}_c$ as a (very complicated) perturbation of the Burgers equation we note the following Theorem as a prototype of the result we show for (1.1).

Notation. Throughout this paper we denote many different constants that are independent of δ and the rescaling parameter $L > 0$ (see below) by the same symbol C.

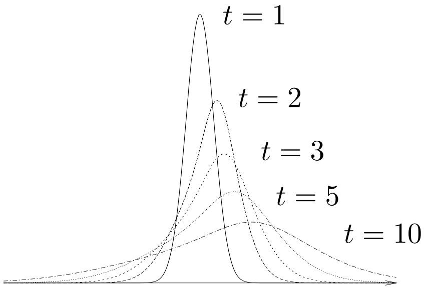

Figure 3: Sketch of self–similar decay in the Burgers equation

For $m, n \in \mathbb{N}$ we define the weighted spaces $H^m(n)=\{u \in L^2(\mathbb{R}) : \|u\|_{H^m(n)}<\infty\}$ with $\|u\|_{H^m(n)} = \|u\rho^n\|_{H^m(\mathbb{R})}$, where $\rho(x) = (1 + |x|^2)^{1/2}$ and $H^m(\mathbb{R})$ is the Sobolev space of functions with derivatives up to order m in $L^2(\mathbb{R})$. By abuse of notation we sometimes write, e.g., $\|u(t, x)\|_{H^m(n)}$ for the $H^m(n)$ norm of the function $x \mapsto u(t, x)$. Fourier transform is denoted by $\mathcal{F}$ and is always with respect to the unbounded direction x; e.g., if $u \in L^2(\mathbb{R})$, then $\hat{u}(k) := \mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} u(x)\, dx$.

From $\mathcal{F}(\partial_x u)(k) = ik\hat{u}(k)$ and Parseval's identity we have that $\mathcal{F}$ is an isomorphism between $H^m(n)$ and $H^n(m)$, i.e., the weight in x–space yields smoothness in Fourier space and vice versa. This smoothness in k is essential for the proof of the following theorem, where for convenience we take initial conditions at $t = 1$. Moreover, due to the relation with (1.1), here we only consider the quasilinear case $p_3 \leq 1$. See sec.3 for the proof and more details.

Theorem 1.5 *Let $b \in (0, 1/2)$, $h(\eta, \partial_x\eta, \partial_x^2\eta) = \eta^{p_1}(\partial_x\eta)^{p_2}(\partial_x^2\eta)^{p_3}$ with $d_h = 3 - (p_1 + 2p_2 + 3p_3) \leq -1$ and $p_3 \leq 1$. There exist $C_1, C_2 > 0$ such that the following holds. If $\|\eta_0\|_{H^2(2)} \leq C_1$, then the pertubed the Burgers equation*

$$\partial_t\eta = \alpha\partial_x^2\eta + \beta\partial_x(\eta^2) + h(\eta, \partial_x\eta, \partial_x^2\eta) \tag{1.12}$$

has a unique solution η with $\eta|_{t=1} = \eta_0$, which satisfies, for a $z > -1$,

$$\|\sqrt{t}\eta(t, \sqrt{t}x) - f_z(x)\|_{H^2(2)} \leq C_2 t^{-1/2+b}, \quad t \in [1, \infty). \tag{1.13}$$

1.4 The result

Motivated by the fact that the Burgers equation is the amplitude equation for (1.1) in the spectrally stable case and shows self similar decay of small spatially localized perturbations of the trivial solution we expect a similar result for (1.1). To state this result we first need fractional Sobolev spaces with weights. In the definition of these space we in general do not distinguish between vector valued and scalar functions as this will be clear from the context. For $0 \leq r \in \mathbb{R}$, $H^r(\mathbb{R})$ is the Sobolev space of functions $u \in L^2(\mathbb{R})$ finite in the norm $\|u\|_{H^r(\mathbb{R})} = \|(1+k^2)^{r/2}\hat{u}\|_{L^2(\mathbb{R})}$. For $r \in \mathbb{N}$ this definition coincides with the usual one [LM68, chapter 1]. We let

$H^r(n) = \{u \in L^2(\mathbb{R}) : \|u\|_{H^r(n)} < \infty\}$ with $\|u\|_{H^r(n)} = \|u\rho^n\|_{H^r(\mathbb{R})}$. It follows from Parseval's identity that $\|u\|_{\hat{H}^r(n)}$ with $\|u\|^2_{\hat{H}^r(n)} = \sum_{j=0}^n \|(1+k^2)^{r/2}\partial_k^j \hat{u}\|^2_{L^2}$ defines an equivalent norm on $H^r(n)$. For $\Omega = \{(x,y) : x \in \mathbb{R}, 0 < y < h(x)\}$, real $r \geq 0$, and $n \in \mathbb{N}$, we let

$$H^r(n,\Omega) = \{u \in H^r(\Omega) : \|u\|_{H^r(n,\Omega)} < \infty\}, \quad \|u\|_{H^r(n,\Omega)} = \|u\rho^n\|_{H^r(\Omega)}.$$

Since $\|u\|_{\hat{H}^r(\Omega)}$ with $\|u\|^2_{\hat{H}^r(\Omega)} = \|\hat{u}\|^2_{L^2(\mathbb{R},H^r(\,\mathrm{d}y))} + \||k|^r\hat{u}\|^2_{L^2(\mathbb{R},L^2(\,\mathrm{d}y))}$ is an equivalent norm on $H^r(\Omega)$, where $H^r(\,\mathrm{d}y)$ denotes the Sobolev space with respect to the bounded cross section [LM68, sec.1.9], it follows that $\|u\|_{\hat{H}^r(n,\Omega)}$ with

$$\|u\|^2_{\hat{H}^r(n,\Omega)} = \sum_{j=0}^n \left(\|\partial_k^j\hat{u}\|^2_{L^2(\mathbb{R},H^r(\,\mathrm{d}y))} + \||k|^r\partial_k^j\hat{u}\|^2_{L^2(\mathbb{R},L^2(\,\mathrm{d}y))}\right)$$

is an equivalent norm on $H^r(n,\Omega)$. Finally let

$$\mathcal{H}^r(\Omega) = H^r(\mathbb{R}) \times H^{r-1/2}(\Omega) \quad \text{and} \quad \mathcal{H}^r(n,\Omega) = H^r(n) \times H^{r-1/2}(n,\Omega).$$

Our result now reads as follows, where as in Theorem 1.5 for convenience we take the initial conditions for (1.1) at $t = 1$.

Theorem 1.6 *Let* $\mathrm{R} < \mathrm{R}_c$, $b \in (0, \frac{1}{2})$, *and* $3 < r < 7/2$. *Then there exist* $C_1, C_2 > 0$ *such that the following holds. For* $(\eta_0, u_0) \in \mathcal{H}^r(2, \Omega(1))$ *satisfying*

$$\|(\eta_0, u_0)\|_{\mathcal{H}^r(2,\Omega(1))} \leq C_1$$

and the compatibility conditions (1.4), there exists a unique solution $U = (\eta, u)$ *of (1.1) with* $U|_{t=1} = (\eta_0, u_0)$. *This solution satisfies*

$$\left\|(x,y) \mapsto \left[t^{1/2}U(t, t^{1/2}x, y) - f_z(x)\Phi_1(0,y)\right]\right\|_{\mathcal{H}^r(2,\Omega(t))} \leq C_2 t^{b-\frac{1}{2}}, \tag{1.14}$$

$t \in [1, \infty)$, *where* $\Phi_1(0,y) = (1, 2y, 0)$ *and* $z > -1$ *is given by*

$$\ln(z+1) = -\frac{2}{\alpha}\int_{\mathbb{R}} \eta_0(x)\,\mathrm{d}x,$$

with α *from (1.9) and*

$$f_z(y) = -\frac{\sqrt{\alpha}}{2}\frac{z\,\mathrm{erf}'(y/\sqrt{\alpha})}{1 + z\,\mathrm{erf}(y/\sqrt{\alpha})}.$$

Remark 1.7 From (1.14) we have

$$\sup_{(x,y)\in\Omega(t)} \left|U(t,x,y) - t^{-1/2}f_z(t^{-1/2}x)\Phi_1(0,y)\right| \leq C_2 t^{b-1}.$$

The localized perturbations decay in an universal manner determined by the decay of perturbations of the trivial solution in the Burgers equation. We have the so called asymptotic $(\mathcal{H}^r(2,\Omega), C_b^0)$–stability of $(\eta, u) = 0$, i.e., for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|(\eta_0, u_0)\|_{\mathcal{H}^r(2,\Omega(1))} \leq \delta$ implies $\|\eta(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} \leq \varepsilon$ for all $t \geq 1$, and $\|\eta(t)\|_{L^\infty} + \|u(t)\|_{L^\infty} \to 0$ with rate $t^{-1/2}$. For the pressure p we obtain

$$\sup_{(x,y)\in\Omega(t)} \left|p(t,x,y) - t^{-1/2} g^\star f_z(t^{-1/2}x)\right| \leq C_2 t^{b-1}, \tag{1.15}$$

and similar estimates for the derivatives of p.

In (1.14), z can be given explicitly (in contrast to z in Theorem 1.5) due to

$$\partial_t \int_{\mathbb{R}} \eta \,\mathrm{d}x = \int_{\mathbb{R}} \big(u_2|_{\Gamma_\mathrm{f}} - (\partial_x \eta) u_1|_{\Gamma_\mathrm{f}}\big) \,\mathrm{d}x + \int_{\mathbb{R}} \partial_x \big(\eta + \frac{1}{3}\eta^3\big) \,\mathrm{d}x = 0. \tag{1.16}$$

The second integral obviously vanishes, while the first vanishes since

$$\begin{aligned}\int_{\mathbb{R}} \big(u_2|_{\Gamma_\mathrm{f}} - (\partial_x \eta) u_1|_{\Gamma_\mathrm{f}}\big) \,\mathrm{d}x &= \int_{\mathbb{R}} \left(\int_0^\eta \partial_y u_2 \,\mathrm{d}y - (\partial_x \eta) u_1|_{\Gamma_\mathrm{f}} \right) \mathrm{d}x \\ &= \int_{\mathbb{R}} -\partial_x \left(\int_0^\eta u_1 \,\mathrm{d}y \right) \mathrm{d}x = 0,\end{aligned}$$

where we used $u_2|_{\Gamma_\mathrm{b}} = 0$ and $\partial_y u_2 = -\partial_x u_1$ in $\Omega(t)$. In other words, the right hand side of (1.1a) is an x–derivative, which simply corresponds to conservation of (perturbation) mass. ⌋

Remark 1.8 To prove Theorem 1.6 we adapt the techniques used in [BKL94] for scalar nonlinear diffusion equations (see Theorem 1.5). The key idea in [BKL94] is that the Burgers equation is invariant under the parabolic rescaling

$$v(\tau, \xi) = L u(L^2 \tau, L\xi), \quad \text{i.e.} \quad u(t,x) = L^{-1} v(L^{-2} t, L^{-1} x), \tag{1.17}$$

that this rescaling produces prefactors involving L^{-1} in front of higher order terms in the equation for v, and that the exact solution η^z (cf. (1.10)) of the Burgers equation is attractive in suitable (weighted) spaces. The renormalization group approach to prove Theorem 1.5 proceeds by solving the equation for v for finite times and iterating the rescaling (1.17). Details and the functional analytic frame are given in sec.3, while the following remarks give an overview how we adapt these ideas to (1.1):

In (1.12) we can regard $h(\eta, \partial_x \eta, \partial_x^2 \eta)$ as a term in the Taylor expansion of a higher order nonlinearity $\tilde{h}$. Due to Theorem 1.5 such nonlinearities are called asymptotically irrelevant or simply irrelevant. In scalar nonlinear diffusion problems such as (1.12) irrelevant nonlinearities can be identified by a simple power counting, as expressed by the condition $d_h \leq -1$ in Theorem 1.5.

The ansatz (1.6) is related to the rescaling (1.17) with $\delta = L^{-1}$, and the formal derivation of (1.9) from (1.1) suggest that the only relevant nonlinearity in (1.1) is $(\partial_x \eta) u_1$ in the kinematic equation (1.1a). However, to prove Theorem 1.6 we first need to transform the problem to the fixed domain $\Omega = \mathbb{R} \times (0,1)$, and this transformation removes $(\partial_x \eta)u_1$ from the (transformed) kinematic equation and produces many new nonlinear terms, cf. sec. 4.1. Many of these turn out not to be irrelevant by a simple power counting argument as in Theorem 1.5. This is related to the fact that in the formal derivation of (1.9) we do not care for large errors (small order in δ) in (1.1b),(1.1c), (1.1e) and (1.1f), cf. Remark 1.3. Instead, using so called mode filters based on the spectral analysis of the linearization of (1.1), see sections 2.2 and 2.3, we shall split U in (1.2) into a critical part U_c belonging to the critical (diffusive) modes $\Phi_1(k, y)$ with k close to 0, and a stable (exponentially damped) part $U_s = U - U_c$. Using these mode–filters we can identify the relevant terms and, moreover, give rigorous estimates for the remainder.

The key observation is that the projection of the nonlinearity onto the critical mode $\Phi_1(k, y)$ vanishes at k=0. This projection is given by a scalar product (see (2.23)) with the adjoint eigenfunction $\Psi_1(k, y)$. Since $\Psi_1(0, y) = (1, 0, 0)$ the only contribution to the projection at k=0 comes from the right hand side of the kinematic equation (1.1a). This *has* to vanish at k=0 due to $\partial_t \int_{\mathbb{R}} \eta(t, x)\,\mathrm{d}x = 0$, cf. (1.16). In summary, near k=0 the critical terms correspond to a total derivative with respect to x. Heuristically, terms with derivatives with respect to x are "more irrelevant" than terms without x–derivatives, as can also be seen in the definition of $d_h = 3 - (p_1 + 2p_2 + 3p_3)$ in Theorem 1.5, and this is made rigorous in sec.6, see in particular Lemma 6.5. ⌋

Remark 1.9 The restriction $r > 3$ is used to control the nonlinearity by Sobolev embeddings. On the other hand, the restriction $r < 7/2$ is used to avoid the formulation of higher order compatibility conditions on the initial data, see sec.5. ⌋

Remark 1.10 On a bounded domain, e.g. $x \in [0, 2\pi]$, with periodic boundary conditions, the stability of the Nusselt solution, i.e., the stability of $(0, 0)$ for (1.1), has been considered in [NTW93, Sun97]. In detail, in [NTW93] the asymptotic stability of $(0, 0)$ for sufficiently small Reynolds numbers has been shown using energy estimates, while in [Sun97] the principle of linearized stability/instability is established: for $\mathrm{R} < \mathrm{R}_c$ ($\mathrm{R} > \mathrm{R}_c$), $(0, 0)$ is asymptotically stable (unstable) with exponential rate. Both, [NTW93] and [Sun97], heavily rely on the bounded domain, and $\nu, \varepsilon \to 0$ as the domain becomes large, where the decay is $\mathrm{e}^{-\nu t}$ and where ε is the size of the allowed initial perturbations. In the physical problem the ratio h/L where h is the physical film height and L the spatial length scale is typically very small. Therefore it is more natural to consider (1.1) on an unbounded domain. Then we only have algebraic decay, but our approach also gives the detailed self similar asymptotics. ⌋

Remark 1.11 The related result of Burgers–like self–similar decay in the so called Integral Boundary Layer equation (IBLe) has been shown in [Uec04]. The IBLe can be formally derived from the Navier–Stokes equations (1.1). It is a two dimensional parabolic system on the real line which is quasilinear and hence still shows many of the difficulties of (1.1). However, the analysis for the IBLe is technically simpler. Thus it may be advantageous to look at [Uec04] for an outline of the arguments used here. ⌋

The plan of the paper is as follows. In Section 2 we first review the functional analytic setup and the spectral theory for the linearization $\partial_t U = \mathcal{A}U$ of (1.1). Then we introduce the mode filters. Section 3 contains a review of the ideas from renormalization theory and a proof of Theorem 1.5. This also explains the need for the weighted spaces $H^r(n)$. In Section 4 we transform the domain $\Omega(t)$ occupied by the fluid to the fixed strip $\Omega = \mathbb{R}\times(0,1)$, and hence formulate (1.1) in the form (1.2). This transformation produces a large number of new nonlinear terms. In combination with the mode–filters, all but one of these can be seen to be irrelevant in the sense of renormalization. In Section 5 we review the local existence theory for (1.2), and in Section 6 we set up the renormalization process to prove Theorem 1.6. In the Appendix we prove some technical results.

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2 The linearized equations

With $L_N u = \big((2y{-}y^2{-}2)\partial_x u_1 + (2{-}2y)u_2,\, (2y{-}y^2{-}2)\partial_x u_2\big)$ and $g^\star = (2\cot\theta)/\mathrm{R}$, the linearization of (1.1) around $(\eta, u) = (0,0)$ reads

$$\text{in } \Omega: \quad u_t + L_N u = -\nabla p + \frac{1}{\mathrm{R}}\Delta u, \quad \operatorname{div} u = 0, \tag{2.1a}$$

$$\text{on } \Gamma_\mathrm{f}: \quad \partial_t\eta = u_2{+}\partial_x\eta, \quad \partial_y u_1{+}\partial_x u_2{-}2\eta = 0, \quad p{-}g^\star\eta{-}\frac{2}{\mathrm{R}}\partial_y u_2 = -\mathrm{W}\partial_x^2\eta, \tag{2.1b}$$

$$\text{on } \Gamma_\mathrm{b}: \quad u = 0. \tag{2.1c}$$

In subsection 2.1 we formulate (2.1) in the form $\partial_t U = \mathcal{A}U$, $U = (\eta, u)$ and, following [Bea84, Ter92], review resolvent estimates for the linear operator $\mathcal{A}$ in unweighted spaces $\mathcal{H}^r(\Omega)$, which show that $\mathcal{A}$ generates an analytic semigroup with certain smoothing properties. To prove Theorem 1.6 we need these estimates in the weighted spaces $\mathcal{H}^r(2,\Omega)$. They are proved in App.A.2. In subsections 2.2 and 2.3 we recall the spectral analysis for (2.1) and define mode filters to separate the critical from the stable modes.

2.1 The evolutionary system

To write (2.1) as an evolution equation we first need to project onto solenoidal vector fields. From

$$\int_\Omega (\operatorname{div} u)\varphi \,\mathrm{d}\Omega = \int_{\partial\Omega} (u\cdot n)\varphi \,\mathrm{d}\Gamma - \int_\Omega u\cdot\nabla\varphi \,\mathrm{d}\Omega$$

we have $\operatorname{div} u = 0$ and $u_2 = 0$ on Γ_b iff u is L^2 orthogonal to $\nabla\varphi$ for all φ with $\varphi = 0$ on Γ_f. Hence, let

$$\mathcal{G} = \{\nabla\varphi : \varphi \in H^1(\Omega),\ \varphi = 0 \text{ on } \Gamma_\mathrm{f}\},$$

and let P be the orthogonal projection of $L^2(\Omega)$ onto $\mathcal{G}^\perp$, i.e., $Pu = u$ iff $\operatorname{div} u = 0$ and $u_2|_{\Gamma_\mathrm{b}} = 0$. We start with the following Lemma, see [Bea80, Lemma 3.1].

Lemma 2.1 *For $r \ge 0$, P is a bounded linear operator on $H^r(\Omega)$ and on $H^r(2,\Omega)$. If $\varphi \in H^1(\Omega)$, then $P(\nabla\varphi) = \nabla\psi$ with $\Delta\psi = 0$, $\psi = \varphi$ on Γ_f, $\partial_y\psi = 0$ on Γ_b.*

Applying P to (2.1a) gives

$$\partial_t u = \frac{1}{\mathrm{R}} P\Delta u - PL_N u - \nabla p_1 - \nabla p_2, \tag{2.2}$$

where $\Delta p_j = 0$, $\partial_y p_j = 0$ on Γ_b, and $p_1 = 2\partial_y u_2/\mathrm{R}$, $p_2 = g^\star\eta - \mathrm{W}\partial_x^2\eta$ on Γ_f. This splitting is adapted to the inner product (2.5). Let $E : H^{r-1/2}(\Gamma_\mathrm{f}) \to H^{r-1}(\Omega)$ be defined by

$$Eh = \nabla q \quad \text{with} \quad \Delta q = 0 \text{ in } \Omega, \quad q = h \text{ on } \Gamma_\mathrm{f}, \quad \partial_y q = 0 \text{ on } \Gamma_\mathrm{b}. \tag{2.3}$$

Moreover, let A, L_0 by formally defined by

$$Au = \frac{1}{\mathrm{R}} P\Delta u - \frac{2}{\mathrm{R}} E\partial_y u_2, \quad L_0 u = PL_N u,$$

and define $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix}\eta\\ u\end{pmatrix} = \begin{pmatrix} u_2|_{\Gamma_\mathrm{f}} + \partial_x\eta \\ Au - L_0 u - E(g^\star\eta - \mathrm{W}\partial_x^2\eta)\end{pmatrix}, \tag{2.4}$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \big\{(\eta,u) :\ & \eta \in H^{5/2}(\Gamma_\mathrm{f}), \quad u \in PL^2(\Omega)\cap H^2(\Omega), \\ & \partial_y u_1 + \partial_x u_2 = 2\eta \text{ on } \Gamma_\mathrm{f}, \quad u = 0 \text{ on } \Gamma_\mathrm{b}\big\}. \end{aligned}$$

Now (2.1) can be written as

$$\partial_t U = \mathcal{A}U, \quad U = \begin{pmatrix}\eta\\ u\end{pmatrix}.$$

We define the Hilbert space

$$\mathcal{X} = \{U = (\eta, u) : \eta \in H^1(\Gamma_\mathrm{f}), u \in PL^2(\Omega)\},$$
$$\langle U, V\rangle_\mathcal{X} = \left\langle \begin{pmatrix}\eta\\u\end{pmatrix}, \begin{pmatrix}\xi\\v\end{pmatrix}\right\rangle_\mathcal{X} := \int_{\Gamma_\mathrm{f}} g^\star \eta\xi + \mathrm{W}(\partial_x\eta)(\partial_x\xi)\,\mathrm{d}\Gamma + \int_\Omega u\cdot v\,\mathrm{d}\Omega. \quad (2.5)$$

and prove the following Lemma in App. A.1.

Lemma 2.2 *$\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$ is sectorial, i.e., there exist $C, a > 0$ and $\varphi \in (0, \pi/2)$ such that the resolvent set contains the sector $S_{a,\varphi} = \{\lambda : \varphi \le |\arg(a - \lambda)| \le \pi\}$, and for $\lambda \in S_{a,\varphi}$ and $F = (\xi, f) \in \mathcal{X}$ the unique solution $U = (\eta, u)$ of the resolvent equation $(\lambda - \mathcal{A})U = F$ satisfies $\|U\|_\mathcal{X} \le \frac{C}{|a-\lambda|}\|F\|_\mathcal{X}$.*

It follows that $\mathcal{A}$ generates an analytic semigroup $\mathrm{e}^{t\mathcal{A}}$. From (A.5) in the proof we also have $\|u\|_{H^1} + \|\eta\|_{H^1} \le C\|F\|_\mathcal{X}$. Next, using a smoothing process as in the analysis of elliptic equations one can show that $\mathcal{A}$ has smoothing properties in the u component:

Theorem 2.3 *Let $r \ge 2$. There exist $C, a > 0$ and $\varphi \in (0, \pi/2)$ such that for all $\lambda \in S_{a,\varphi} = \{\lambda : \varphi \le |\arg(a - \lambda)| \le \pi\}$ and all $F = (\xi, f) \in H^{r+1/2}(\mathbb{R}) \times H^{r-2}(\Omega)$, the unique solution $U = (\eta, u)$ of the resolvent equation $(\lambda - \mathcal{A})U = F$ satisfies*

$$\begin{aligned}&\|u\|_{H^r(\Omega)} + |\lambda|^{r/2}\|u\|_{L^2(\Omega)} + \|\eta\|_{H^{r+1/2}(\mathbb{R})} + |\lambda|^{(r+1/2)/2}\|\eta\|_{L^2(\mathbb{R})}\\ &\quad\le C\left(\|f\|_{H^{r-2}(\Omega)} + |\lambda|^{(r-2)/2}\|f\|_{L^2(\Omega)} + \|\xi\|_{H^{r+1/2}(\mathbb{R})} + |\lambda|^{(r+1/2)/2}\|\xi\|_{L^2(\mathbb{R})}\right). \quad (2.6)\end{aligned}$$

Note the lack of smoothing with respect to η. Given Lemma 2.2, the proof of Theorem 2.3 works as the proof of [Ter92, Prop. 5.1] by testing with approximations of x–derivatives of the solution u and using that x–derivatives commute with $\mathcal{A}$; y–derivatives are then recovered from the incompressibility condition.

In order to control the pressure we use the following two standard estimates for the solution of the Stokes problem [Tem01, Chapter 1].

Lemma 2.4 *There exists a $C > 0$ such that if u, p satisfy*

$$-\frac{1}{\mathrm{R}}\Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \quad \text{in } \Omega, \quad (2.7)$$
$$u = 0 \text{ on } \Gamma_b, \quad u = \varphi \text{ on } \Gamma_f, \quad (2.8)$$

then

$$\|u\|_{H^{r+2}} + \|\nabla p\|_{H^r} \le C(\|f\|_{H^r} + \|\varphi\|_{H^{r+3/2}}) \quad \text{for all } r \ge 0. \quad (2.9)$$

Similarly, if u, p satisfy (2.7) and

$$u = 0 \text{ on } \Gamma_b, \quad u_2 = \varphi_1, \quad \partial_x u_2 + \partial_y u_1 = \varphi_2 \text{ on } \Gamma_f, \quad (2.10)$$

then, for all $r \geq 0$,

$$\|u\|_{H^{r+2}} + \|\nabla p\|_{H^r} \leq C(\|f\|_{H^r} + \|\varphi_1\|_{H^{r+3/2}} + \|\varphi_2\|_{H^{r+1/2}}). \tag{2.11}$$

The resolvent estimates (2.6) are used to show the existence of solutions to the linear inhomogeneous equation $\partial_t U - \mathcal{A}U = F(t,x,y)$ via Laplace–transform, where $U(t_0) = 0$ and F has to vanish to sufficient order at $t = t_0$, see sec.5. To prove Theorem 1.6 we shall also use direct estimates for $\mathrm{e}^{t\mathcal{A}}$ (in weighted spaces). For $(\lambda - \mathcal{A})U = (0, f)$ with $f \in PH^\alpha(\Omega)$ it can be shown, using again the same method as in [Ter92], that

$$\|u\|_{H^{2+\alpha}} + |\lambda|\|u\|_{H^\alpha} + \|\eta\|_{H^{5/2+\alpha}} + |\lambda|^{3/2}\|\eta\|_{H^{1/2+\alpha}} \leq C\|f\|_{H^\alpha}. \tag{2.12}$$

To work in the weighted spaces $H^r(2)$ and $H^r(2,\Omega)$ we prove the following Lemma in App.A.2.

Lemma 2.5 *Theorem 2.3, Lemma 2.4 and the estimate (2.12) also hold with the spaces* $H^r(\mathbb{R})$ *and* $H^r(\Omega)$ *replaced by the weighted spaces* $H^r(2)$ *and* $H^r(2,\Omega)$, *i.e., (2.6) becomes*

$$\begin{aligned} &\|u\|_{H^r(2,\Omega)} + |\lambda|^{r/2}\|u\|_{H^0(2,\Omega)} + \|\eta\|_{H^{r+1/2}(2)} + |\lambda|^{(r+1/2)/2}\|\eta\|_{H^0(2)} \\ &\quad \leq C\left(\|f\|_{H^{r-2}(2,\Omega)} + |\lambda|^{(r-2)/2}\|f\|_{H^0(2,\Omega)} + \|\xi\|_{H^{r+1/2}(2)} + |\lambda|^{(r+1/2)/2}\|\xi\|_{H^0(2)}\right), \end{aligned} \tag{2.13}$$

and similar for (2.9),(2.11) and (2.12).

Hence let $U_0 = (\eta_0, u_0) \in H^{r+1/2}(2) \times PH^{r-2}(2,\Omega)$. By shifting the path of integration in the representation

$$(\eta, u)(t) = \mathrm{e}^{t\mathcal{A}}U_0 = \lim_{\tau\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\tau}^{a+\mathrm{i}\tau} \mathrm{e}^{\lambda t}(\lambda - \mathcal{A})^{-1}U_0 \, \mathrm{d}\lambda,$$

we obtain, for instance,

$$\begin{aligned} &\|\eta(t)\|_{H^{r+1/2}(2)} + \|u(t)\|_{H^r(2,\Omega)} \leq \\ &\quad C\mathrm{e}^{at}t^{-1}\left(\|u_0\|_{H^{r-2}(2,\Omega)} + t^{-(r-2)/2}\|u_0\|_{H^0(2,\Omega)} + \|\eta_0\|_{H^{r+1/2}(2)} + t^{-(r/2+1/4)}\|\eta_0\|_{H^0(2)}\right), \end{aligned} \tag{2.14}$$

and, in case $\eta_0 = 0$,

$$\|u(t)\|_{H^\alpha(2,\Omega)} \leq C\mathrm{e}^{at}\|u_0\|_{H^\alpha(2,\Omega)}, \quad \|u(t)\|_{H^{2+\alpha}(2,\Omega)} \leq C\mathrm{e}^{at}t^{-1}\|u_0\|_{H^\alpha(2,\Omega)}, \tag{2.15}$$

$$\|\eta(t)\|_{H^{1/2+\alpha}(2)} \leq C\mathrm{e}^{at}t^{1/2}\|u_0\|_{H^\alpha(2,\Omega)}, \quad \|\eta(t)\|_{H^{5/2+\alpha}(2)} \leq C\mathrm{e}^{at}t^{-1}\|u_0\|_{H^\alpha(2,\Omega)}. \tag{2.16}$$

Due to $a>0$ from the proof of Lemma 2.2, and since there is no smoothing in η, these semigroup estimates are bad a priori. However, $\mathrm{e}^{t\mathcal{A}}$ has local in time parabolic smoothing properties if $\eta_0 = 0$. To take advantage of this we shall remove the nonlinear terms from the η component of (1.3). Moreover, we split $\mathcal{A} = \mathcal{A}_c + \mathcal{A}_s$ where $\mathcal{A}_c$ is one–dimensional in Fourier space, and $\mathrm{e}^{t\mathcal{A}_s}$ fulfills (2.14)–(2.16) with $a<0$. The part belonging to $\mathcal{A}_c$ will be treated explicitly in Fourier space, see sec.6.

2.2 Spectral analysis

To calculate the eigenvalues and eigenfunctions of $\mathcal{A}$ we use the Fourier ansatz

$$\begin{pmatrix} \eta \\ u_1 \\ u_2 \end{pmatrix}(t,x,y) = \begin{pmatrix} 1 \\ \varphi'(k,y) \\ -\mathrm{i}k\varphi(k,y) \end{pmatrix} \mathrm{e}^{\mathrm{i}k(x-\omega t)} = \Phi(k,y)\mathrm{e}^{\mathrm{i}k(x-\omega t)}. \tag{2.17}$$

By cross–differentiating (2.1a) we obtain

$$\partial_t(\partial_y u_1 - \partial_x u_2) + (u_N - 2)\partial_x(\partial_y u_1 - \partial_x u_2) - 2u_2 = \frac{1}{\mathrm{R}}(\partial_y \Delta u_1 - \partial_x \Delta u_2),$$

hence

$$\partial_y^4 \varphi - 2k^2\varphi'' + k^4\varphi = \mathrm{i}k\mathrm{R}\big[(u_N - 2 - \omega)(\varphi'' - k^2\varphi) + 2\varphi\big], \tag{2.18a}$$

where $\varphi = \varphi(k,y)$ and $\varphi' = \partial_y\varphi$. In order to eliminate p from the normal stress in (2.1b) we use

$$\partial_x p|_{y=1} = (\frac{1}{\mathrm{R}}\Delta u_1 - \partial_t u_1 - (u_n - 2)\partial_x u_1)|_{y=1}.$$

This yields the boundary conditions (from (2.1b) and (2.1c), in the respective order)

$$\varphi(1) - \omega - 1 = 0, \quad \varphi''(1) + k^2\varphi(1) - 2 = 0, \tag{2.18b}$$

$$\varphi'''(1) - 3k^2\varphi'(1) + \mathrm{i}k\mathrm{R}\big[(\omega+1)\varphi'(1) - g^\star - \mathrm{W}k^2\big] = 0, \tag{2.18c}$$

$$\varphi(0) = \varphi'(0) = 0. \tag{2.18d}$$

This non–constant coefficient eigenvalue problem with the wavenumber k as parameter is called Orr–Sommerfeld equation, and $\lambda(k) = -\mathrm{i}k\omega(k)$ is called the associated eigenvalue.

At $k = 0$ there is the critical mode $\varphi(0,y) = y^2$, $\omega(0) = 0$. Moreover, it is well known (e.g. [BLDB99] and the references therein) that for small to intermediate Reynolds numbers there exists one isolated curve $(-\delta,\delta) \ni k \mapsto \lambda_1(k)$, $0 < \delta$ small, of eigenvalues with small and possibly positive real part $\lambda(k)$. This curve belongs to the so called surface mode, and gives an instability iff $\mathrm{R} > \mathrm{R}_c$. In fig.2(b) on page 6 we show the real part of $\lambda_1(k) = -\mathrm{i}k\omega(k)$ as calculated from a numerical solution of (2.18) using `AUTO97` [DCF+97]. As explained in Remark 1.1, at very low θ and for high Reynolds numbers, a so called shear mode with wave number $k \neq 0$ can first become unstable, but we exclude this exceptional case here. For later reference we expand $\omega(k) = \mathrm{i}k\omega_1 + \mathcal{O}(k^2)$, $\varphi(k,y) = \varphi_0(y) + \mathrm{i}k\varphi_1(y) + \mathcal{O}(k^2)$ to obtain

$$\Phi_1(k,y) = \begin{pmatrix} 1 \\ 2y + \mathrm{i}k\varphi_1'(y) \\ -\mathrm{i}ky^2 \end{pmatrix} + \mathcal{O}(k^2), \tag{2.19}$$

$$\varphi_1(y) = \mathrm{R}\big[\frac{1}{30}y^5 - \frac{1}{6}y^4 + \frac{1}{6}g^\star y^3 + \frac{1}{2}(\frac{4}{3} - g^\star)y^2\big], \quad \omega_1 = \varphi_1(1) = \mathrm{R}\big[\frac{8}{15} - \frac{1}{3}g^\star\big].$$

Note that, e.g., $\omega_1 = \frac{8\mathrm{R}}{15} - \frac{2}{3}\cot\theta = -\alpha$ and $(\partial_X \eta_1)\varphi_1' = u_{12}$ in accordance with (1.8) and (1.9).

2.3 The mode filters

The adjoint of $\mathcal{A}$ with respect to the inner product (2.5) is given by

$$\mathcal{A}^* \begin{pmatrix} \xi \\ v \end{pmatrix} = \begin{pmatrix} Hv_1 - \partial_x \xi \\ Av - L_0^* v + E(g^\star \eta - \mathrm{W}\partial_x^2 \eta) \end{pmatrix},$$

with domain

$$D(\mathcal{A}^*) = \big\{(\xi, v):\ \xi \in H^{5/2}(\Gamma_\mathrm{f}), \quad v \in PL^2(\Omega) \cap H^2(\Omega), \\ \partial_y v_1 + \partial_x u_2 = 0 \text{ on } \Gamma_\mathrm{f}, \quad u = 0 \text{ on } \Gamma_\mathrm{b}\big\},$$

where

$$L_0^* v = P \begin{pmatrix} (u_N - 2)\partial_x v_1 \\ (u_N - 2)\partial_x v_2 - u_N' v_1 \end{pmatrix} \quad \text{and} \quad Hv_1 = \frac{2}{\mathrm{R}}(g^\star - \mathrm{W}\partial_x^2)^{-1} v_1.$$

The ansatz

$$\begin{pmatrix} \xi \\ v_1 \\ v_2 \end{pmatrix}(t,x,y) = \begin{pmatrix} 1 \\ \psi'(k,y) \\ -\mathrm{i}k\psi(k,y) \end{pmatrix} \mathrm{e}^{\mathrm{i}k(x-\omega t)} = \Psi(k,y)\mathrm{e}^{\mathrm{i}k(x-\omega t)},$$

yields the adjoint Orr–Sommerfeld equations

$$\begin{aligned} &\partial_y^4 \psi - 2k^2 \psi'' + k^4 \psi = \mathrm{i}k\mathrm{R}\big[-(u_N - 2 + \omega)(\psi'' - k^2\psi) - 2u_N'\psi'\big], \\ &k(\omega - 1) = \frac{2\mathrm{i}}{\mathrm{R}}(g^\star + \mathrm{W}k^2)^{-1}\psi'(1), \quad \psi''(1) + k^2\psi(1) = 0, \\ &\psi'''(1) - 3k^2\psi'(1) + \mathrm{i}k\mathrm{R}\big[g^\star + \mathrm{W}k^2 + (\omega - 1)\psi'(1)\big] = 0, \\ &\psi(0) = \psi'(0) = 0. \end{aligned}$$

The critical solution is $\omega = -\mathrm{i}k\omega_1 + \mathcal{O}(k^2)$ and $\psi(k,y) = \mathrm{i}k\psi_1(y) + \mathcal{O}(k^2)$, i.e,

$$\Psi_1(k,y) = \begin{pmatrix} 1 \\ \mathrm{i}k\psi_1'(y) \\ 0 \end{pmatrix} + \mathcal{O}(k^2) \quad \text{with} \quad \psi_1(k) = \mathrm{R}g^\star(\frac{1}{2}y^2 - \frac{1}{6}y^3). \tag{2.21}$$

Let $\rho > 0$ be sufficiently small, and let χ be a smooth cutoff function with

$$\chi(k) = \begin{cases} 1 & |k| \le \rho, \\ \in (0,1) & \rho < |k| < 2\rho \\ 0 & 2\rho \le |k|, \end{cases} \tag{2.22}$$

Corresponding to the inner product (2.5), for $\hat{U} = (\hat{\eta}, \hat{u}), \hat{V} = (\hat{\xi}, \hat{v}) \in \mathbb{C} \times L^2((0,1))$ let

$$\langle \hat{U}, \hat{V} \rangle_{(k)} = (g^\star + \mathrm{W}k^2)\hat{\eta}\overline{\hat{\xi}} + \int_0^1 \hat{u}(y) \cdot \overline{\hat{v}}(y)\,\mathrm{d}y. \tag{2.23}$$

Then

$$\hat{E}_c(k)\hat{U}(k) = c(k)\chi_c(k)\langle\hat{U}(k), \Psi_1(k)\rangle_{(k)}\Phi_1(k) \tag{2.24}$$

with

$$c(k) = 1/\langle\Phi_1(k), \Psi_1(k)\rangle_{(k)} = 1/g^\star + \mathcal{O}(|k|) \tag{2.25}$$

defines the so called central modefilter with $\|\hat{E}_c\|_{\mathbb{C}\times H^r((0,1))\to\mathbb{C}\times H^r((0,1))} \leq C$. By construction

$$(\hat{\mathcal{A}}\hat{E}_c\hat{U})(k) = (\hat{E}_c\hat{\mathcal{A}}\hat{U})(k) = \lambda_1(k)\hat{U}(k)$$

where $\hat{\mathcal{A}}\hat{U} = \mathcal{F}(\mathcal{A}U)$. The corresponding operators in x–space are

$$E_cU = \mathcal{F}^{-1}(\hat{E}_c\hat{U}) \quad \text{and} \quad \mathrm{E}_s = \mathrm{Id} - E_c, \tag{2.26}$$

and it follows that $\|E_c\|_{\mathcal{H}^r(2,\Omega)\to\mathcal{H}^2(2,\Omega)} \leq C$. Moreover, $(\hat{E}_c\hat{U})(k,y) = a(k)\Phi_1(k,y)$, hence by construction

$$\begin{aligned}
&(E_cU|_{y=0})_i{=}\mathcal{F}^{-1}(a(k)\Phi_{1i}(k,0)){=}0, \quad i = 2,3,\\
&\operatorname{div}(E_cU) = \mathcal{F}^{-1}(a(k)(\mathrm{i}k\varphi'{-}\mathrm{i}k\varphi')){=}0,\\
&\partial_y(E_cU)_2{+}\partial_x(E_cU)_3{-}2(E_cU)_1 = \mathcal{F}^{-1}(a(k)(\partial_y^2\varphi(k,1){+}k^2\varphi(k,1){-}2)) = 0.
\end{aligned} \tag{2.27}$$

Finally, define the auxiliary modefilters

$$\begin{aligned}
\hat{E}_c^h\hat{U}(k) &= c(k)\chi_c(k/2)\langle\hat{U}(k), \Psi_1(k)\rangle\Phi_1(k),\\
\hat{E}_s^h\hat{U}(k) &= \hat{U}(k) - c(k)\chi_c(2k)\langle\hat{U}(k), \Psi_1(k)\rangle\Phi_1(k).
\end{aligned}$$

Then $\hat{E}_c^h\hat{E}_c = \hat{E}_c$ and $\hat{E}_s^h\hat{E}_s = \hat{E}_s$ which will be used to replace the missing projection properties of $\hat{E}_c, \hat{E}_s$ and E_c, E_s.

3 Ideas from renormalization theory

Before transforming (1.1) to the form (1.2) we explain the idea of renormalization [BK92, BKL94] and prove Theorem 1.5. This, together with the mode–filters from sec. 2.3, will (heuristically) explain which nonlinear terms in (1.2) are irrelevant and therefore do not need to be written explicitly in the transformations in sec. 4. This section also explains the need for the weighted spaces $H^m(n)$.

3.1 Basics

Consider

$$\partial_t u = \partial_x^2 u + f(u, \partial_x u, \partial_x^2 u), \quad u = u(t,x) \in \mathbb{R}, \quad u(1,x) = u_0(x), \tag{3.1}$$

where $f(a,b,c) = a^{d_1}b^{d_2}c^{d_3}$ is a monomial, and where the initial conditions are given at $t=1$ for convenience. For $L>0$ define the rescaling operators

$$\mathcal{R}_L u(x) = u(Lx),$$

and for $n \in \mathbb{N}$ and $L>1$ sufficiently large (see below) let $u_n(\tau) = L^n \mathcal{R}_{L^n} u(L^{2n}\tau)$, i.e., $u_n(\tau,\xi) = L^n u(L^{2n}\tau, L^n\xi)$. Then

$$\partial_\tau u_n = \partial_\xi^2 u_n + f_n(u_n, \partial_\xi u_n, \partial_\xi^2 u_n), \tag{3.2}$$

with

$$f_n(a,b,c) = L^{nd_f} a^{d_1}b^{d_2}c^{d_3}, \quad d_f = 3 - d_1 - 2d_2 - 3d_3. \tag{3.3}$$

Moreover, solving $\partial_t u = \partial_x^2 u + f$ for $t \in [1,\infty)$ is equivalent to iterating the renormalization process

$$\text{solve (3.2) on } \tau \in [L^{-2},1] \text{ with initial data } u_n(L^{-2},\xi) = Lu_{n-1}(1,L\xi) \in X, \tag{3.4}$$

where X is a Banach space such that we can solve the in general quasilinear or fully nonlinear problem (3.2).

First assume $d_f<0$. In this case the factor L^{nd_f} in (3.3) goes to 0 as $n\to\infty$, and in the limit we obtain $\partial_\tau u_n = \partial_\xi^2 u_n$ with $u_n(L^{-2},\xi) = Lu_{n-1}(1,L\xi)$. This problem has the line of (Gaussian) fixed points $z\mathrm{e}^{-\xi^2/4}$, $z\in\mathbb{R}$, which is attractive in suitable spaces. To see this, let $u(\xi) = z\mathrm{e}^{-\xi^2/4} + g(\xi)$ with $\hat g(0)=0$ by choice of z. Due to

$$\mathcal{F}(L\mathcal{R}_L u) = \mathcal{R}_{1/L}\hat u, \tag{3.5}$$

we have $\mathrm{e}^{(1-L^{-2})\partial_\xi^2} L\mathcal{R}_L(\mathrm{e}^{-\xi^2/4}) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{4\pi}}\mathrm{e}^{-(1-L^{-2})k^2}\mathrm{e}^{-k^2/L^2}\right) = \mathrm{e}^{-\xi^2/4}$ and

$$\begin{aligned}
\|\mathrm{e}^{(1-L^{-2})\partial_\xi^2} L\mathcal{R}_L g\|_{H^m(2)} &\le C\|\mathrm{e}^{-(1-L^{-2})k^2}\hat g(k/L)\|_{H^2(m)} \\
&\le C\left(\int (1+k^2)^m \sum_{j=0}^{2}\left(\partial_k^j\left(\mathrm{e}^{-(1-L^{-2})k^2}\hat g(k/L)\right)\right)^2 \mathrm{d}k\right)^{1/2} \\
&\le CL^{-1}(\|\hat g\|_{C^1(\mathbb{R})} + \|\hat g\|_{H^2(m)}).
\end{aligned}$$

This is obtained from writing $\hat g(k/L) = \hat g(0) + \hat g(\tilde k)\frac{k}{L}$ and using $\hat g(0)=0$ and $\|(1+k^2)^{m/2}\partial_k^j(\mathrm{e}^{-(1-L^{-2})k^2})\|_{L^2} \le C$. Hence we obtain

$$\|\mathrm{e}^{(1-L^{-2})\partial_\xi^2} L\mathcal{R}_L g\|_{H^m(2)} \le CL^{-1}\|g\|_{H^m(2)} \tag{3.6}$$

from Sobolev embedding since $\|\hat g\|_{C^1(\mathbb{R})} \le C\|\hat g\|_{H^n}$ for $n-1>1/2$, i.e. $n>3/2$. This is where we need the weight in x; see also [BKL94, Uec04] and Remark 3.1 for alternative Banach spaces that directly assume smoothness in Fourier space.

By (3.3) we may assign each derivative ∂_x the order L^{-n}. Equivalently, due to (3.5) we may assign each factor k in the Fourier transform of (3.1) the order L^{-n}. Hence the basic idea is that by a power–counting argument one can easily identify nonlinearities f that are "asymptotically irrelevant" ($d_f < 0$), while a nonlinearity with $d_f > 0$ would be called "relevant". Indeed, relevant nonlinearities, and also the "marginal" case $f = u^3$ (with $d_f = 0$) may lead to finite–time blow up of the solution, see, e.g., [Wei81]. The advantage of the discrete renormalization approach is that the large time behavior of (3.1) is split into the sequence (3.4) of finite time problems and that it uses only few special features of the equation. Therefore it can be applied to a variety of problems; see the references in the Introduction. A related approach is the continuous rescaling of time and space used in [CEE92, Way97, EWW97, GM98].

3.2 Proof of Theorem 1.5

Now let

$$f(u, \partial_x u, \partial_x^2 u) = \partial_x(u^2) + h(u, \partial_x u, \partial_x^2 u)$$

with $h(a, b, c) = a^{d_1} b^{d_2} c^{d_3}$ and $d_h = 3 - d_1 - 2d_2 - 3d_3 < 0$. Then (3.1) is a rescaling of the perturbed Burgers equation (1.12), and (3.4) becomes

$$\partial_\tau u_n = \partial_\xi^2 u_n + \partial_\xi(u_n^2) + L^{nd_h} h(u_n, \partial_\xi u_n, \partial_\xi^2 u_n), \quad u_n(1/L^2) = L\mathcal{R}_L u_{n-1}(1). \quad (3.7)$$

Remark 3.1 In [BKL94], (3.7) is treated for $u_n \in C([1/L^2, 1], X)$ where

$$X = \{f : \hat{f} \in C^1 \text{ and } \|f\|_X = \sup_{k\in\mathbb{R}} (1 + k^4)(|\hat{f}(k)| + |\hat{f}'(k)|) < \infty\},$$

This space allows the solution of the quasilinear or fully nonlinear parabolic problems (3.7) directly by the variation of constant formula using the explicit formula for $\mathrm{e}^{\tau\partial_x^2} u$. This yields the analog of Theorem 1.5 based in the space X [BKL94, Theorem 4]. Here, for suitable r we want to use the spaces $X = H^r(2)$. These are more natural for the Navier–Stokes problem (1.1) where we want to use resolvent estimates instead of explicit formulas for the linear semigroup. As a consequence, we first have to use maximal regularity methods to obtain existence of solutions to (3.7). A posteriori we can then use the variation of constant formula to obtain improved estimates. The following proof of Theorem 1.5, which merely adapts [BKL94] to the different spaces, explains this idea and gives a guideline for the proof of Theorem 1.6. ⌋

For $\tau_0 < \tau_1$, $n \in \mathbb{N}$ and $r \geq 0$ we define

$$H^{r,s}((\tau_0, \tau_1), n) = L^2((\tau_0, \tau_1), H^r(2)) \cap H^s((\tau_0, \tau_1), H^0(n)). \quad (3.8)$$

Since (3.7) is parabolic these spaces will only occur with $s = r/2$. Hence we set

$$K^r((\tau_0, \tau_1), n) = H^{r,r/2}((\tau_0, \tau_1), n),$$

For $u \in K^r((\tau_0,\tau_1),n)$, r not a half integer, which we will always assume in the following, there exists traces $\partial_x^\alpha u \in K^{r-\alpha-1/2}((\tau_0,\tau_1),n)$ for $\alpha < r-1/2$ and $\partial_t^j u(\tau_0,\cdot) \in H^{r-2k-1}(2)$, for $2j < r-1$. Conversely, for $u \in H^r(2)$ there exist extensions $u \in K^{r+1}(\mathbb{R},2)$; see [LM68, Thm. 4.2.1 and 4.2.3] for the unweighted case. We define the subspaces $K_0^r((\tau_0,\tau_1),n)$ of functions $u \in K^r((\tau_0,\tau_1),n)$ with $\partial_t^j u(\tau_0,\cdot) = 0$ for $2j < s-1$.

For $u \in K^r(\mathbb{R},2)$, let $\tilde{u}(\lambda) = \int \mathrm{e}^{\lambda\tau} u(\tau)\,\mathrm{d}\tau$ be its Laplace transform. For $u \in K_0^r((\tau_0,\infty),2)$ we have (via extension of u by 0 for $\tau < \tau_0$) the equivalence

$$\|u\|^2_{K^r((\tau_0,\infty),2)} \sim \int_{\mu\in\mathbb{R}} \|\tilde{u}(\mathrm{i}\mu)\|^2_{H^r(2)} + |\mu|^r \|\tilde{u}(\mathrm{i}\mu)\|^2_{H^0(2)}\,\mathrm{d}\mu. \tag{3.9}$$

Finally, note that

$$\|L\mathcal{R}_L u\|_{H^r(2)} \le C L^{r+1/2} \|u\|_{H^r(2)}, \tag{3.10}$$

due to the scaling properties of Sobolev spaces, and let

$$\rho_n = \|u_n(1)\|_{H^2(2)}. \tag{3.11}$$

Then we have the following essentially classical [LM68] existence result for (3.7); for convenience we review the main steps of the proof in App.A.3.

Lemma 3.2 *There exist $L_0 > 1, C_1, C_2 > 0$ such that for all $L > L_0$ the following holds. If $\rho_{n-1} \le C_1 L^{-5/2}$, then there exists a unique solution $u_n \in K^3([1/L^2,1],2)$ of (3.7), and $\|u_n\|_{K^3([1/L^2,1],2)} \le C_2\rho_{n-1}$. Moreover, for any $m \in \mathbb{N}$, $u_n \in K^{3+m}([\frac{1}{2},1],2)$, and there exists a $C_3 = C_3(m)$ such that $\|u_n\|_{K^{3+m}([1/2,1],2)} \le C_3\rho_{n-1}$.*

To iterate Lemma 3.2 we shall use the integral equation satisfied by u_n to obtain improved estimates. Therefore let

$$u_n(\tau,\xi) = w_{z_n}(\tau,\xi) + v_n(\tau,\xi), \tag{3.12}$$

where $w_z(\tau,\xi) = \tau^{-1/2} f_z(\xi/\sqrt{\tau})$, $f_z(x) = z\,\mathrm{erf}'(x)/(1+z\,\mathrm{erf}(x))$ and where $z_n > -1$ is defined by

$$\ln(1+z_n) = \int u_{n-1}(1,\xi)\,\mathrm{d}\xi = \int u_n(1/L^2,\xi)\,\mathrm{d}\xi. \tag{3.13}$$

Consequently

$$\hat{v}_n(1/L^2,0) = \int v_n(1/L^2,\xi)\,\mathrm{d}\xi = 0. \tag{3.14}$$

Since w_z is an exact solution of the Burgers equation (cf. sec. 1.3) we obtain

$$\begin{aligned} v_n(1) =& \mathrm{e}^{(1-L^{-2})\partial_\xi^2} L\mathcal{R}_L v_{n-1}(1) \\ &+ \int_{1/L^2}^1 \mathrm{e}^{(1-\tau)\partial_\xi^2} \big[B(v_n(\tau)) + Q_{z_n}(v_n(\tau)) + L^{nd_h} h(u_n(\tau))\big]\,\mathrm{d}\tau \end{aligned} \tag{3.15}$$

where $B(v) = \partial_\xi(v^2), Q_{z_n}(v) = \partial_\xi(w_z v)$.

Finally, let $\tilde\rho_n = \|v_n(1)\|_{H^2(2)}$ and assume that $\rho_{n-1} \le C|z_n| + \tilde\rho_n \le L^{-m_0}$ with $m_0 > 0$ chosen below. By Lemma 3.2 there exists a unique solution u_n of (3.7) with

$$\|u_n\|_{K^3([1/L^2,1],2)} + \|u_n\|_{K^4([1/2,1],2)} \le CL^{5/2-m_0} \le 1 \tag{3.16}$$

and clearly we may assume the same estimate for v_n. By (3.14), the first term on the right hand side in (3.15) yields

$$\left\| \mathrm{e}^{(1-L^{-2})\partial_\xi^2} L\mathcal{R}_L v_{n-1}(1) \right\|_{H^2(2)} \le CL^{-1}\|v_{n-1}(1)\|_{H^2(2)}, \tag{3.17}$$

cf. (3.6). The linear semigroup $\mathrm{e}^{\tau\partial_\xi^2}$ fulfills

$$\|\mathrm{e}^{\tau\partial_\xi^2} u\|_{H^{r+j}(2)} \le \max\{1, \tau^{-j/2}\}\|u\|_{H^r(2)}. \tag{3.18}$$

Therefore, using $\sup_{\tau\in[1/L^2,1]} \|v_n(\tau)\|_{H^2(2)} \le CL^{5/2-m_0}$, the first two terms in the integral in (3.15) can be directly estimated as

$$\left\| \int_{1/L^2}^1 \mathrm{e}^{(1-\tau)\partial_\xi^2} \left[B(v_n(\tau)) + Q_{z_n}(v_n(\tau)) \right] \mathrm{d}\tau \right\|_{H^2(2))} \le CL^{5/2-m_0}(|z_n| + L^{5/2-m_0}). \tag{3.19}$$

For terms in $h(u_n(\tau))$ which contain $\partial_\xi^2 v_n$ we split the integral as

$$\int_{1/L^2}^1 \ldots \, \mathrm{d}\tau = \int_{1/L^2}^{1/2} \ldots \, \mathrm{d}\tau + \int_{1/2}^1 \ldots \, \mathrm{d}\tau \tag{3.20}$$

and use the higher regularity of u_n on $[1/2, 1]$ in (3.16) to obtain

$$\left\| \int_{1/L^2}^1 \mathrm{e}^{(1-\tau)\partial_\xi^2} L^{nd_h} f(u_n(\tau)) \, \mathrm{d}\tau \right\|_{H^2(2)} \le CL^{nd_h} L^{5/2-m_0}(|z_n| + L^{5/2-m_0}). \tag{3.21}$$

Hence, for $m_0 > 7/2$, small $b > 0$, and $L > L_0$ sufficiently large we obtain, since $d_h < 0$,

$$\tilde\rho_n = \|v_n(1)\|_{H^2(2)} \le L^{-(1-b)}(\tilde\rho_{n-1} + |z_n|). \tag{3.22}$$

By definition of z_n we also have

$$\begin{aligned} |\ln(1 + z_{n+1}) - \ln(1 + z_n)| &= \left| \int \int_{1/L^2}^1 \mathrm{e}^{(1-\tau)\partial\xi^2} L^{nd_h} f_n(u_n(\tau)) \, \mathrm{d}\tau \, \mathrm{d}\xi \right| \\ &\le CL^{nd_h}(L^{(5/2-m_0)})^2 \le L^{-n} \end{aligned} \tag{3.23}$$

where the term involving $B(v_n) + Q_{z_n}(v_n)$ drops out of the integral in ξ since it is a total derivative, and where again we used the splitting (3.20) and $\|u\|_\infty \le C\|u\|_{H^r}$ for $r > 1/2$. By (3.23) there exists a z_* with $|z_* - z_n| \le CL^{-n}$. Thus, for $t = L^{2n}$ we have

$$\begin{aligned} \|t^{1/2}u(t, t^{1/2}x) - f_{z_*}(x)\|_{H^2(2)} &= \|u_n(1) - f_{z_n} + f_{z_n} - f_{z_*}\|_{H^2(2)} \\ &\le \|v_n(1)\|_{H^2(2)} + \|f_{z_n} - f_{z_*}\|_{H^2(2)} \le CL^{-(1-b)n}, \end{aligned}$$

and for $t \in (L^{2n-1}, L^{2n})$ the estimate (1.13) in Theorem 1.5 follows from Lemma 3.2. The proof of Theorem 1.5 is complete. □

4 Transformations

4.1 Transformation to fixed domain

Following [Bea80, Bea84] we first transform (1.1) to a new problem over the fixed domain $\Omega = \mathbb{R} \times (0,1)$. Here we use the sum convention and write $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ and $x = (x_1, x_2)$ for the independent variables in Ω and $\Omega(t)$, respectively. We start with an extension of $\eta(t,\cdot)$, defined on $\mathbb{R}$, to $\tilde{\eta}(t,\cdot) = S\eta(t,\cdot)$ defined on Ω. Therefore, let

$$\tilde{\eta}(t,\tilde{x}_1,\tilde{x}_2) = (S\eta(t,\cdot))(\tilde{x}_1,\tilde{x}_2) = \mathcal{F}^{-1}\left(\frac{1}{1+k^2(\tilde{x}_2-1)^2}\hat{\eta}(t,k)\right)(\tilde{x}_1). \tag{4.1}$$

Then $S : H^r(\mathbb{R}) \to H^{(r+1/2)}(\Omega)$ is a bounded linear operator, and

$$\|\nabla\tilde{\eta}\|_{L^2(\Omega)} \le C\|\partial_x\eta\|_{L^2(\mathbb{R})} \tag{4.2}$$

since

$$\partial_{\tilde{x}_2}\tilde{\eta} = \mathcal{F}^{-1}\left(\frac{-2k^2(\tilde{x}_2-1)}{(1+k^2(\tilde{x}_2-1))^2}\hat{\eta}(t,k)\right) = \mathcal{F}^{-1}\left(\frac{\mathrm{i}2k(\tilde{x}_2-1)}{(1+k^2(\tilde{x}_2-1))^2}(\mathcal{F}\partial_x\eta)(t,k)\right).$$

For each $t > 0$ as long as $\eta(t,\cdot)$ exists and is sufficiently smooth (this will be justified a posteriori), the fluid domain $\Omega(t)$ is given by a diffeomorphism

$$\vartheta : \Omega \to \Omega(t), \quad \vartheta(\tilde{x}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2(1+\tilde{\eta}(t,\tilde{x}_1,\tilde{x}_2)) \end{pmatrix}. \tag{4.3}$$

To conserve the incompressibility condition we transform u, p as follows. Let $(\vartheta_{ij}) = (\partial_{\tilde{x}_j}\vartheta_i)$ be the Jacobian of ϑ, $d = \det(\vartheta_{ij})$, $(\alpha_{ij}) = \frac{1}{d}(\vartheta_{ij})$ and $(\zeta_{ij}) = (\vartheta_{ij})^{-1}$, i.e.,

$$d = 1 + \partial_{\tilde{x}_2}(\tilde{x}_2\tilde{\eta}), \quad (\alpha_{ij}) = \begin{pmatrix} 1/d & 0 \\ \tilde{x}_2\tilde{\eta}_{\tilde{x}_1}/d & 1 \end{pmatrix}, \quad (\zeta_{ij}) = \begin{pmatrix} 1 & 0 \\ -\tilde{x}_2\tilde{\eta}_{\tilde{x}_1}/d & 1/d \end{pmatrix}, \tag{4.4}$$

and define v and $\tilde{p}$ by

$$u_i(t,\vartheta(t,\tilde{x})) = \alpha_{ij}(t,\tilde{x})v_j(t,\tilde{x}), \quad p(t,\vartheta(t,\tilde{x})) = \tilde{p}(t,\tilde{x}). \tag{4.5}$$

Then v is divergence–free iff u is, [Boc77]. With $\partial_j = \partial_{x_j}$ and $\tilde{\partial}_k = \partial_{\tilde{x}_k}$ the spatial derivatives transform as

$$\partial_j = \zeta_{kj}\tilde{\partial}_k,$$

and since ϑ depends on time we obtain

$$\partial_t u_i = \alpha_{ij}\partial_t v_j + \left(\frac{\mathrm{d}}{\mathrm{d}t}\vartheta^{-1}\right)_l \left((\tilde{\partial}_l\alpha_{ij})v_j + \alpha_{ij}\tilde{\partial}_l v_j\right),$$

with

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\vartheta^{-1}\Big)_1 = 0, \quad \Big(\frac{\mathrm{d}}{\mathrm{d}t}\vartheta^{-1}\Big)_2 = \frac{1}{d}\tilde{x}_2(1+\partial_t\tilde{\eta}) = \frac{1}{d}\tilde{x}_2(1+S(Rv_2+\eta^2\partial_x\eta)).$$

Here we used that the kinematic equation simplifies to

$$\partial_t\tilde{\eta} = v_2|_{\tilde{x}_2=1} + \tilde{\partial}_1\tilde{\eta} + \tilde{\eta}^2\tilde{\partial}_1\tilde{\eta}.$$

Plugging (4.5) into (1.1) and renaming $\tilde{\partial}_i = \partial_i$, $(\tilde{x}_1,\tilde{x}_2) = (x,y)$, $v = u$ and $\tilde{p} = p$ yields

$$\begin{aligned} \text{at } y=1: &\quad \partial_t\eta - \partial_x\eta - u_2 = \eta^2\partial_x\eta, && (4.6a)\\ \text{in } \Omega: &\quad \partial_t u_i - \mathcal{L}_i u_i + \partial_i p = f_i(\eta,u,\nabla p) \quad i=1,2, && (4.6b)\\ \text{at } y=1: &\quad \partial_y u_1 + \partial_x u_2 - 2\eta = f_3(\eta,u), && (4.6c)\\ &\quad p - \frac{2}{\mathrm{R}}\partial_y u_2 - g^\star\eta + \mathrm{W}\partial_x^2\eta = f_4(\eta,u), && (4.6d) \end{aligned}$$

and $\operatorname{div} u = 0$ in Ω, $u = 0$ on Γ_b. Here $\mathcal{L}_i u = \frac{1}{\mathrm{R}}\Delta u_i - (u_N-2)\partial_x u_i - \delta_{1i}u_N' u_2$, $i=1,2$, and, for instance

$$\begin{aligned} f_i(\eta,u,\nabla p) = &-d\zeta_{ij}(\partial_t\alpha_{ik})u_k + yS(Ru_2\partial_x\eta + \eta^2\partial_1\eta)\zeta_{ij}\partial_y(\alpha_{jk}u_k)\\ &+ \frac{1}{\mathrm{R}}(\zeta_{al}\zeta_{bl} - \delta_{al}\delta_{bl})\partial_a\partial_b u_i + \frac{d}{\mathrm{R}}\zeta_{ij}\zeta_{al}\big((\partial_b u_k)\partial_a(\zeta_{bl}\alpha_{jk}) + \partial_a(\zeta_{bl}(\partial_b\alpha_{jk})u_k)\big)\\ &- (u_N-2)(\delta_{a1} - \zeta_{a1})\partial_a u_i + (2(y-1)yS\eta + y^2(S\eta)^2)\zeta_{a1}\partial_a u_i\\ &+ (y-1+yS\eta)^2 d\zeta_{ij}\zeta_{a1}(\partial_a\alpha_{jk})u_k + 2(y-1)(d\zeta_{i1}\alpha_{2k} - \delta_{i1}\delta_{2k})u_k\\ &+ 2yS\eta d\zeta_{i1}\alpha_{2k}u_k + (\delta_{ia} - d\zeta_{ij}\zeta_{aj})\partial_a p + \zeta_{ij}u_l\partial_l(\alpha_{jk}u_k), \end{aligned}$$

$i=1,2$, where $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$. The compatibility conditions for $(\eta,u)|_{t=1} = (\eta_0,u_0)$ are $\operatorname{div} u = 0$ in Ω, $u=0$ on $y=0$ and $\partial_y u_1 + \partial_x u_2 - 2\eta = f_3(\eta,u)$ on $y=1$.

Due to the lack of smoothing in η in (2.14) it is useful to remove the nonlinear terms from (4.6a). To do so we set $u = \tilde{u} + v$ with

$$\tilde{u}(t,x,y) = \begin{pmatrix} y\tilde{\eta}^3(t,x,y) + \frac{y^2}{2}\partial_y(\tilde{\eta}^3(t,x,y))\\ -\frac{y^2}{2}\partial_x(\tilde{\eta}^3(t,x,y)) \end{pmatrix}. \qquad (4.7)$$

After renaming $v{=}u$ this yields a system like (4.6) with (4.6a) replaced by

$$\partial_t\eta - \partial_x\eta - u_2|_{\Gamma_\mathrm{f}}{=}0$$

and f_i, $i=1,\dots,4$ changed by at least cubic terms, including terms coming from $\partial_t\tilde{\eta}$ in (4.6b). In order not to proliferate symbols we denote these new nonlinear terms again by f_i. Then

$$\begin{aligned} \text{at } y=1: &\quad \partial_t\eta - \partial_x\eta - u_2 = 0, && (4.8a)\\ \text{in } \Omega: &\quad \partial_t u_i - \mathcal{L}_i u_i + \partial_i p = f_i(\eta,u,\nabla p) \quad i=1,2, && (4.8b)\\ \text{at } y=1: &\quad \partial_y u_1 + \partial_x u_2 - 2\eta = f_3(\eta,u), && (4.8c)\\ &\quad p - \frac{2}{\mathrm{R}}\partial_y u_2 - g^\star\eta + \mathrm{W}\partial_x^2\eta = f_4(\eta,u), && (4.8d) \end{aligned}$$

and $\operatorname{div} v = 0$ in Ω, $v = 0$ on Γ_{b}. Moreover we write

$$f_3(\eta, u) = 2\eta\partial_y u_1 + \tilde{f}_3(\eta, u). \tag{4.9}$$

The following remark explains the structure of the nonlinear terms in (4.8), and the reason for displaying $2\eta\partial_y u_1$ explicitly in (4.9).

Remark 4.1 The functions f_i, $i = 1, \ldots, 4$ in (4.8) contain quadratic and higher order terms. In order to write (4.8) in the form (1.2) and hence treat (4.8) in fixed function spaces we still need to remove the nonlinear terms from the tangential stress (4.8c). Direct calculation shows that the nonlinear term $2\eta\partial_y u_1$ written explicitly in (4.9) is the only nonlinear term that is relevant in the derivation of the Burgers equation from (4.8), see also Remark 4.2. It follows that $2\eta\partial_y u_1$ is also the only relevant (marginal) nonlinear term in the proof of Theorem 1.6. Here the heuristic argument is as follows. In the proof of Theorem 1.6 we apply a renormalization argument similar to (3.4) to a system of the form (1.2), i.e., formally $\partial_t U = \mathcal{A}U + F(U, \nabla p)$ with $U = (\eta, u)$. Due to the rescaling we may assign orders $\partial_x = \mathcal{O}(\delta)$, $\delta = L^{-n}$, and similarly $k = \mathcal{O}(\delta)$ where again k is the dual variable to x under Fourier transform, cf. sec.3, while $\partial_y = \mathcal{O}(1)$. Moreover, roughly speaking, due to the splitting of $U = U_c + U_s$ the main problem is in equation (6.1a) for $\partial_t U_c$ below. Due to $k = \mathcal{O}(\delta)$ and $\hat{U}_c(t, k, y) = a(t, k)\Phi_1(k, y) = a(t, k)(1, 2y + \mathcal{O}(|k|), \mathcal{O}(|k|)$ we may assign orders $(\delta, \delta, \delta^2, \delta)$ to (η, u_1, u_2, p). Similarly, due to $\Psi_1(k, y) = (1, \mathcal{O}(|k|), \mathcal{O}(k^2))$ we only need to keep track of terms of order $(\delta^3, \delta^2, \delta, \delta^2)$ in the equations ((4.15a),(4.15b),(4.15c), (4.15e)) obtained below from (4.8) by removing the nonlinear terms from (4.8c). This will be explained in more detail and be made rigorous in sec.6. ⌋

4.2 Linearization of the tangential stress

From Remark 4.1 it follows that in removing the nonlinear terms from (4.8c) we need to take special care of the term $2\eta\partial_y u_1$ in f_3. From (1.8) we know that in lowest order in δ we have $u_1(T, X, y) = 2y\eta(T, X)$. Hence, in lowest order, $\partial_y u_1 = 2\eta$. Therefore we split

$$u = u^{(1)} + \tilde{u} \tag{4.10}$$

and choose $\tilde{u}$ such that

$$\begin{aligned} \operatorname{div} \tilde{u} = 0 \text{ in } \Omega, \qquad \tilde{u} = 0 \text{ on } y = 0, \\ \tilde{u}_2 = 0 \quad \text{and} \quad \partial_y \tilde{u}_1 + \partial_x \tilde{u}_2 = 4\eta^2 \text{ on } y = 1. \end{aligned} \tag{4.11}$$

A solution $\tilde{u}$, with moreover $\partial_y \tilde{u}_2|_{y=1} = 0$, is given by $\tilde{u} = (\partial_y w, -\partial_x w)$ with

$$w(t, x, y) = h(y)\tilde{\eta}^2(t, x, y), \quad \text{where} \quad h(y) = 2y^4 - 4y^3 + 2y^2. \tag{4.12}$$

For $\eta(t,\cdot) \in H^{r+1/2}(2,\Gamma_\mathrm{f})$ this yields $\tilde{u}(t,\cdot,\cdot) \in H^r(2,\Omega)$. This regularity is not optimal but sufficient in the following, and for explicit calculations it is useful to know $\tilde{u}$ explicitly, see Remark 4.2 below. We obtain

$$\begin{aligned}
\text{at } y=1: &\quad \partial_t\eta - \partial_x\eta - u_2 = 0, && (4.13a)\\
\text{in } \Omega: &\quad \partial_t u_i^{(1)} - \mathcal{L}_i u^{(1)} + \partial_i p = f_i(\eta, u^{(1)}+\tilde{u}, p) - \partial_t\tilde{u}_i + \mathcal{L}_i\tilde{u} \quad i=1,2, && (4.13b)\\
\text{at } y=1: &\quad \partial_y u_1^{(1)} + \partial_x u_2^{(1)} - 2\eta = \tilde{f}_3(\eta, u^{(1)}+\tilde{u}) + 2\eta\partial_y u_1^{(1)} - 4\eta^2, && (4.13c)\\
&\quad p - \frac{2}{\mathrm{R}}\partial_y u_2^{(1)} - g^\star\eta + \mathrm{W}\partial_x^2\eta = f_4(\eta, u^{(1)}+\tilde{u}), && (4.13d)
\end{aligned}$$

and $\operatorname{div} u = 0$ in Ω, $u=0$ on Γ_b. Since $\partial_y\tilde{u}_2|_{y=1}=0$, this term drops out of (4.13d).

Finally we remove the nonlinear terms from (4.13c). Therefore we split $u^{(1)} = u^{(2)} + u^{(3)}$ and choose $u^{(3)}$ such that

$$\begin{aligned}
&\operatorname{div} u^{(3)} = 0 \ \text{ in } \Omega, \qquad u^{(3)} = 0 \ \text{ on } y=0, \quad \text{and}\\
&\left.\begin{aligned} &u_2^{(3)} = 0,\\ &\partial_y u_1^{(3)} + \partial_x u_2^{(3)} = g_3(\eta, u^{(1)}) := \tilde{f}_3(\eta, \tilde{u}+u^{(1)}) + 2\eta\partial_y u_2^{(1)} - 4\eta^2 \end{aligned}\right\} \text{ on } y=1.
\end{aligned} \tag{4.14}$$

For $\eta \in H^{r+1/2}(\Gamma_\mathrm{f})$ and $u^{(1)} \in H^r(\Omega)$ we have $g_3(\eta, u^{(1)}) \in H^{r-3/2}(\Gamma_\mathrm{f})$. If $g_3 = g_3(x) \in H^{r-3/2}(\Gamma_\mathrm{f})$ is a given function, then a solution $u^{(3)} \in H^{r+1}(\Omega)$ of (4.14) can be obtained from the ansatz $\tilde{u} = (\partial_y w, -\partial_x w)$ with $w|_{y=1} = \partial_y w|_{y=1} = 0$, $\partial_y^2 w|_{y=1} = g(x)$ [LM68, Theorem 1.4.2]. The existence of a solution $u^{(3)} \in H^{r+1}(\Omega)$ of (4.14) then follows for $\tilde{u}, u^{(1)}$ and η sufficiently small by the contraction mapping theorem since $g_3(\eta, u)$ is quadratic and higher order. It is clear that all this also holds in the weighted spaces.

After renaming $u^{(2)} = u$ we obtain

$$\begin{aligned}
\text{at } y=1: &\quad \partial_t\eta - \partial_x\eta - u_2 = 0, && (4.15a)\\
\text{in } \Omega: &\quad \partial_t u_1 - \mathcal{L}_1 u + \partial_x p = \tilde{b}(\eta) + g_1(\eta, u, \nabla p), && (4.15b)\\
&\quad \partial_t u_2 - \mathcal{L}_2 u + \partial_y p = g_2(\eta, u, \nabla p), && (4.15c)\\
\text{at } y=1: &\quad \partial_y u_1 + \partial_x u_2 - 2\eta = 0, && (4.15d)\\
&\quad p - \frac{2}{\mathrm{R}}\partial_y u_2 - g^\star\eta + \mathrm{W}\partial_x^2\eta = g_4(\eta, u), && (4.15e)
\end{aligned}$$

together with $\operatorname{div} u = 0$ and $u=0$ on Γ_b, and where

$$\tilde{b}(\eta) = \frac{1}{\mathrm{R}}(\partial_y^3 h(y))\tilde{\eta}^2, \quad g_4(\eta, u) = f_4(\eta, u+u^{(3)}+\tilde{u}), \quad \text{and}$$
$$g_i(\eta, u, \nabla p) = f_i(\eta, u+u^{(3)}+\tilde{u}, p) - \partial_t(\tilde{u}_i + u_i^{(3)}) + \mathcal{L}_i(\tilde{u}_i + u_i^{(3)}) - \delta_{i1}\tilde{b}(\eta), \quad i=1,2.$$

The splitting of the right hand side of (4.15b) is explained in Remark 4.2. The compatibility conditions for (4.15) at $t=1$ are

$$\operatorname{div} u = 0 \ \text{ in } \Omega, \quad u=0 \ \text{ on } \ y=0, \quad \partial_y u_1 + \partial_x u_2 - 2\eta = 0 \ \text{ on } \ y=1. \tag{4.16}$$

Remark 4.2 It is instructive and a good check of the above calculations to re-derive the Burgers equation (1.9) from (4.15). This also explains the splitting of the right hand side of (4.15b) into the term $\frac{1}{\mathrm{R}}(\partial_y^3 h)\tilde{\eta}^2$ which is relevant in the sense of renormalization and the remainder g_1 which contains only higher order terms. As in the formal derivation of (1.9) from (1.1) in sec. 1.2, here again we only consider the lowest order terms needed to derive the Burgers equation. Rigorous estimates for the remaining terms are given in Lemma 6.5. We substitute (1.6), i.e.,

$$\begin{pmatrix} \eta \\ u_1 \\ u_2 \\ p \end{pmatrix}(t,x,y) = \delta\Psi(T,X,y) = \begin{pmatrix} \delta\eta_1(T,X) \\ \delta u_{11}(T,X,y) + \delta^2 u_{12}(T,X,y) \\ \delta^2 u_{22}(T,X,y) + \delta^3 u_{23}(T,X,y) \\ \delta p_1(T,X,y) \end{pmatrix},$$

with $T = \delta^2 t,\ X = \delta x$ into (4.15). At $\mathcal{O}(\delta)$ this yields $u_1 = 2\eta_1 y$, $p_1 = g^\star \eta_1$ and $u_{22} = -y^2 \partial_X \eta_1$ as in (1.8). Then (4.15b) at $\mathcal{O}(\delta^2)$ yields

$$-\frac{1}{\mathrm{R}}\partial_y^2 u_{12} + \partial_X p_1 + (u_N - 2)\partial_X u_{11} + u_N' u_{22} \stackrel{!}{=} \frac{1}{\mathrm{R}}\partial_y^3 h(y)\eta_1^2 = \frac{24}{\mathrm{R}}(2y-1)\eta_1^2$$

which together with $u = 0$ on Γ_{b} and (4.15d) gives

$$u_{12} = \mathrm{R}(\partial_X \eta_1)\left[\frac{1}{6}y^4 - \frac{2}{3}y^3 + \frac{1}{2}g^\star y^2 + (\frac{4}{3} - g^\star)y\right] - 24\left(\frac{1}{3}y^3 - \frac{1}{2}y^2\right)\eta_1^2. \quad (4.17)$$

From $\operatorname{div} u = 0$ we obtain

$$u_{23} = \mathrm{R}(\partial_X^2 \eta_1)\left[-\frac{1}{30}y^5 + \frac{1}{6}y^4 - \frac{1}{6}g^\star y^3 - \frac{1}{2}(\frac{4}{3} - g^\star)y^2\right] + 4(y^4 - 2y^3)\eta_1 \partial_X \eta_1. \quad (4.18)$$

Then (4.15a) at $\mathcal{O}(\delta^3)$, i.e. $\partial_T \eta_1 - u_{23}|_{y=1} = 0$, yields the Burgers equation (1.9). Note how by the transformations (4.5) and (4.10) the relevant nonlinear term $(\partial_x \eta)u_1$ from the original kinematic condition (1.1a) is first moved into the tangential stress in (4.8c) and then into the first component (4.15b) of the momentum balance. Consequently, in the new coordinates we have the quadratic dependence of first u_{12} and then u_{23} on η_1 in (4.17) and (4.18). ⌋

5 Local existence

We review the main steps for proving local existence for (4.15), which also explains how to obtain local existence for the rescaled system (6.13a),(6.13b) below. The local existence for the 3–dimensional version of (1.1) has been shown in [Ter92]. There, following [Bea84], the problem is solved for small initial data (η, u) in Sobolev spaces $H^r(\Gamma_{\mathrm{f}}) \times H^{r-1/2}(\Omega)$ with $3 < r < 7/2$. Here, in order to prove Theorem 1.6 we need contraction properties of the operator $U \mapsto \mathrm{e}^{(1-L^{-2})\mathcal{A}} L\mathcal{R}_L E_c U$ when acting on functions U with $\hat{U}|_{k=0} = 0$, as in (3.6). This is why we search for solutions $U = (\eta, u)$ of (4.15) in the weighted space $\mathcal{H}^r(2,\Omega)$.

Applying P to (4.15b),(4.15c) we can rewrite (4.15) as

$$\partial_t U - \mathcal{A}U = F(U,\nabla p), \quad F(U,\nabla p) = B(U) + H(U,\nabla p), \tag{5.1}$$

together with the compatibility conditions (4.16) on $U|_{t=1} = U_0$, with $\mathcal{A}$ from (2.4),

$$B(U) = \begin{pmatrix} 0 \\ b(\eta) \end{pmatrix}, \quad b(\eta) = \begin{pmatrix} \tilde{b}(\eta) \\ 0 \end{pmatrix}, \quad \text{and} \quad H(U,\nabla p) = \begin{pmatrix} 0 \\ h(\eta,u,\nabla p) \end{pmatrix}, \tag{5.2}$$

where

$$h(\eta,u,\nabla p) = (P-\mathrm{Id})b(\eta) + P\begin{pmatrix} g_1(\eta,u,\nabla p) \\ g_2(\eta,u,\nabla p) \end{pmatrix} + Eg_4(\eta,u). \tag{5.3}$$

The splitting of $Pb(\eta) = b(\eta)+(P-\mathrm{Id})b(\eta)$ has no importance for the local existence; however, it will be useful in the proof of Theorem 1.6, and is therefore introduced here for later convenience, see Remark 6.1.

Formally (5.1) is solved by the variation of constant formula

$$U(t) = \mathrm{e}^{t\mathcal{A}}U_0 + \int_0^t \mathrm{e}^{(t-s)\mathcal{A}}F(U(s),\nabla p(s))\,\mathrm{d}s. \tag{5.4}$$

However, since we have a quasilinear problem and since ∇p appears on the right hand side, (5.4) cannot be used to construct a solution. Therefore we proceed as in sec.3.2. To obtain existence for (5.1) in the weighted space $\mathcal{H}^r(2,\Omega)$ we again use maximal regularity methods from [LM68] as in [Ter92], based on Lemma 2.3. Then (5.4) can be used to estimate the solution a posteriori, which we shall do for the rescaled systems (6.13) below. See also [BN85] for a similar approach.

Therefore, to show local existence we first consider the linear inhomogeneous problem

$$\text{in } \Omega: \quad \partial_t u - \mathcal{L}u + \nabla p = g(t,x,y), \tag{5.5a}$$

$$\text{on } \Gamma_\mathrm{f}: \quad \partial_t\eta - \partial_x\eta - u_2 = 0, \quad p - \frac{2}{\mathrm{R}}\partial_y u_2 - g^\star\eta + \mathrm{W}\partial_x^2\eta = g_4(t,x), \tag{5.5b}$$

together with $\operatorname{div} u = 0$, $\partial_y u_1 + \partial_x u_2 - 2\eta = 0$ on Γ_f, and $u = 0$ on Γ_b, where we shall assume that η, u and p vanish sufficiently fast at initial time $t = t_0$. For $t_0 < t_1$ we define (cf. (3.8))

$$H^{r,s}((t_0,t_1),n,\Omega) = L^2((t_0,t_1),H^r(n,\Omega)) \cap H^s((t_0,t_1),H^0(n,\Omega)). \tag{5.6}$$

We set $K^r((t_0,t_1),n,\Omega) = H^{r,r/2}((t_0,t_1),n,\Omega)$, define the subspaces $K_0^r((t_0,t_1),n,\Omega)$ of functions $u \in K^r((t_0,t_1),n,\Omega)$ with $\partial_t^j u(t_0,\cdot,\cdot) = 0$ for $2j < s-1$, and introduce the abbreviation

$$\mathcal{K}^r((t_0,t_1),2,\Omega) = K^r((t_0,t_1),2,\mathbb{R}) \times K^{r-1/2}((t_0,t_1),2,\Omega).$$

Let $r \geq 2$ and as usual not a half integer, $t_1 > t_0$, $n \in \mathbb{N}$ and $g \in K_0^{r-2}((t_0,t_1),n,\Omega)$ and $g_4 \in K_0^{r-3/2}((t_0,t_1),n,\mathbb{R})$. Applying P to (5.5) yields

$$\partial_t U - \mathcal{A}U = \begin{pmatrix} 0 \\ g_0 \end{pmatrix}, \quad g_0 = Pg - Eg_4. \tag{5.7}$$

Since $g_0 \in K_0^{r-2}((t_0,t_1),n,\Omega)$ there exists a continuation $g_c \in K_0^{r-2}(\mathbb{R},n,\Omega)$ with $g_c(t)$=0 for $t \leq t_0$. Therefore $\mathrm{e}^{-\sigma(t-t_0)}g_c \in L^1(\mathbb{R},n,\mathcal{H}^{r-2}(\Omega)) \cap L^2(\mathbb{R},n,\mathcal{H}^{r-2}(\Omega))$ and we can solve (5.7) by Laplace transform. For $\lambda = \sigma + \mathrm{i}\tau$ this yields

$$(\lambda - \mathcal{A})\tilde{U} = \begin{pmatrix} 0 \\ \tilde{g}_c \end{pmatrix}, \tag{5.8}$$

where $\tilde{U}, \tilde{g}_c$ denote the Laplace transformed functions. Due to Lemma 2.5 we can solve (5.8) for $\sigma > a$; the resolvent estimates (2.13), together with the Paley–Wiener Theorem and the fact that $t_1 - t_0$ is finite, yield the following Lemma, where p is obtained a posteriori from the weighted version Lemma 2.5 of Lemma 2.4.

Lemma 5.1 *Let $r \geq 2$ and not a half integer, and fix some $t_1 > t_0$. Then there exists a $C = C(t_1) > 0$ such that the following holds. If $g \in K_0^{r-2}((t_0,t_1),2,\Omega)$ and $g_4 \in K_0^{r-3/2}((t_0,t_1),2,\mathbb{R})$ then there exists a unique solution $(\eta,u) \in \mathcal{K}_0^r((t_0,t_1),2,\Omega)$ and $p \in K_0^{r-1}((t_0,t_1),2,\Omega)$ of (5.7), with*

$$\|(\eta,u)\|_{\mathcal{K}^r((t_0,t_1),2,\Omega)} + \|p\|_{K^{r-1}((t_0,t_1),2,\Omega)} \leq C\rho, \tag{5.9}$$

where $\rho = \|(g,g_4)\|_{\mathcal{K}^{r-2}((t_0,t_1),2,\Omega)\times\mathcal{K}^{r-3/2}((t_0,t_1),2,\mathbb{R})}$.

To solve the nonlinear problem (1.3), let $3 < r < 7/2$ and

$$\begin{aligned} X = \Big\{ &(\eta,u,p) : (\eta,u,p) \in \mathcal{K}^{r+1/2}((t_0,t_1),2,\mathbb{R}) \times K^{r-1}((t_0,t_1),2,\mathbb{R}), \\ &\operatorname{div} u = 0, \quad \partial_y u_1 + \partial_x u_2 - 2\eta = 0 \text{ on } \Gamma_{\mathrm{f}}, \quad u = 0 \text{ on } \Gamma_{\mathrm{b}} \Big\}, \\ Y = \{&(g,g_4) : g \in K^{r-2}((t_0,t_1),2,\Omega), \quad g_4 \in K^{r-3/2}((t_0,t_1),2,\mathbb{R})\}. \end{aligned}$$

Let X_0 and Y_0 be the subspaces with K^r replaced by K_0^r, let $M : X \to Y$ be the linear operator defined by the left hand side of (5.5), and let M_0 be its restriction to X_0. Then $M_0^{-1} : Y_0 \to X_0$ exists due to Lemma 5.1, and the idea to solve (4.15) is as follows. First let

$$(\eta,u,p) = (\eta^{(1)},u^{(1)},p^{(1)}) + (\eta^{(2)},u^{(2)},p^{(2)}) = Z^{(1)} + Z^{(2)} \in X$$

where $Z^{(1)}$ depends on (η_0,u_0) in such a way that $Z^{(2)} \in X_0$. Then solve (4.15) in the form

$$M_0 Z^{(2)} = F(Z^{(1)} + Z^{(2)}) - MZ^{(1)}. \tag{5.10}$$

For $(\eta_0, u_0) \in \mathcal{H}^r(\Omega)$ sufficiently small and satisfying the compatibility conditions (4.16) the right hand side will be a contraction in X_0. First we need the following standard Sobolev lemma, cf., e.g., [Bea84, Lemma 5.1].

Lemma 5.2 *(a) If $f \in H^r(\Omega)$ with $r > 1$ then f is continuous on $\overline{\Omega}$. If also $g \in H^s(\Omega)$, $r \geq s \geq 0$ then $fg \in H^s(\Omega)$ and $\|fg\|_{H^s} \leq C\|f\|_{K^r}\|g\|_{K^s}$. The analogous result holds for $\eta \in H^r(\mathbb{R})$ and $\xi \in H^s(\mathbb{R})$ for $r > 1/2$.*
(b) If $f \in K^r((t_0,t_1),\Omega)$ with $r > 2$ then f is continuous on $[t_0,t_1] \times \overline{\Omega}$. If also $g \in K^s((t_0,t_1),\Omega)$, $r \geq s \geq 0$, then $fg \in K^s((t_0,t_1),\Omega)$, and $\|fg\|_{K^s} \leq C\|f\|_{K^r}\|g\|_{K^s}$. The analogous result holds for $\eta \in K^r((t_0,t_1),\mathbb{R})$, $r > 3/2$, $\xi \in K^s((t_0,t_1),\Omega)$, $r \geq s \geq 0$.
(c) The same holds for $H^r(\Omega)$ and $H^r(\Gamma_f)$ replaced with the respective weighted spaces.

To construct $Z^{(1)}$ set $\rho = \|\eta_0\|_{H^r(2,\mathbb{R})} + \|u_0\|_{H^{r-1/2}(2,\Omega)}$. Choose a continuation $\eta^{(1)} \in K^{r+1/2}((t_0,t_1),2,\mathbb{R})$ with $\eta^{(1)}(t_0) = \eta_0$, $\partial_t\eta^{(1)}(t_0)(0) = \partial_x\eta_0 + (u_0)_2|_{\Gamma_\mathrm{f}}$, and $\|\eta^{(1)}\|_{K^{r+1/2}((t_0,t_1),2,\mathbb{R})} \leq C\rho$. Next, $P(\nabla p^{(1)}(t_0)) \in H^{r-3/2}(2,\Omega)$ is determined from

$$p^{(1)}(t_0,1) = g_4(\eta_0,u_0) + \frac{2}{\mathrm{R}}\partial_y(u_0)_2|_{\Gamma_\mathrm{f}} + (g^\star - \mathrm{W}\partial_x^2)\eta_0 \in H^{r-2}(2,\Gamma_\mathrm{f}) \tag{5.11}$$

using Lemma 2.1. Moreover, $(\mathrm{Id}-P)(\nabla p^{(1)}(t_0)) \in H^{r-5/2}(\Omega)$ fulfills

$$(\mathrm{Id}-P)(\nabla p^{(1)}(t_0)) = (\mathrm{Id}-P)\left[\mathcal{L}u_0 + \begin{pmatrix} \tilde{b}(\eta_0) + g_1(\eta^{(1)}(t_0), u_0, \nabla p^{(1)}(t_0)) \\ g_2(\eta^{(1)}(t_0), u_0, \nabla p^{(1)}(t_0)) \end{pmatrix}\right], \tag{5.12}$$

assuming $(\mathrm{Id}-P)\partial_t u = 0$. For $p^{(1)}(t_0) \in H^{r-3/2}(2,\Omega)$ the right hand side of (5.12) is in $H^{r-5/2}(2,\Omega)$ by Lemma 5.2, and $\tilde{b}, g_1, g_2$ are nonlinear. Hence, for η_0, u_0 sufficiently small the system (5.11),(5.12) can be solved for $p^{(1)}(t_0)$ as a function of η_0, u_0 by the implicit function theorem. Hence there exists a $p^{(1)}$ with $\|p^{(1)}\|_{K^{r-1}((t_0,t_1),2,\Omega)} \leq C\rho$ fulfilling (5.11) and (5.12). Finally, we may choose $u^{(1)}$ with $u^{(1)}(t_0) = u_0$, $\operatorname{div} u^{(1)} = 0$, $\partial_y u_1^{(1)} + \partial_x u_2^{(1)} - 2\eta^{(1)} = 0$ on Γ_f, $u^{(1)} = 0$ on Γ_b, $\|u^{(1)}\|_{K^r(0,t_0),2,\Omega)} \leq C\rho$, and

$$\partial_t u^{(1)}(t_0) = \mathcal{L}u_0 - \nabla p^{(1)}(t_0) + \begin{pmatrix} \tilde{b}(\eta_0) + g_1(\eta^{(1)}(t_0), u_0, \nabla p^{(1)}(t_0)) \\ g_2(\eta^{(1)}(t_0), u_0, \nabla p^{(1)}(t_0)) \end{pmatrix}.$$

Here the restriction $r < 7/2$ arises since we do not want to impose compatibility conditions on, e.g., $\partial_t u^{(1)}(t_0)$ on Γ_b.

Using Lemma 5.2 and going through (4.7), (4.11) and (4.14) it is a straightforward though lengthy task to check that if $r > 3$, and $\eta \in K^{r+1/2}((t_0,t_1),2,\mathbb{R})$, $u \in K^r((t_0,t_1),2,\Omega)$ and $p \in K^{r-1}((t_0,t_1),2,\Omega)$, then the right hand side in (5.1) fulfills $F(U,\nabla p) \in \{0\} \times K^{r-2}((t_0,t_1),2,\Omega)$. The condition $r > 3$ is forced by, for instance, the term

$$h = \frac{1}{\mathrm{R}}(\partial_y u_1)\left(\frac{y\partial_x^2\tilde{\eta}}{d} + \frac{2y(\partial_x\tilde{\eta})\partial_y(y\partial_x\tilde{\eta})}{d^2}\right), \quad d = 1 + \partial_y(y\tilde{\eta}),$$

obtained from setting $i = j = l = a = 1$ and $b = 2$ in (4.6b). For $r > 3$ this can be bounded by

$$\begin{aligned}\|h\|_{K^{r-2}((t_0,t_1),2,\Omega)} &\le C\|\nabla u\|_{K^{r-1}((t_0,t_1),2,\Omega)}\|\tilde{\eta}\|_{C^2((t_0,t_1)\times\Omega)} \\ &\le C\|u\|_{K^r((t_0,t_1),2,\Omega)}\|\eta\|_{K^{r+1/2}((t_0,t_1),2,\Omega)}\end{aligned}$$

with, for $|\alpha| = 2$, $\partial^\alpha_{(x,y)}\tilde{\eta} \in K^{r-1}((t_0,t_1),2,\Omega) \subset C([t_0,t_1] \times \overline{\Omega})$ if $r > 3$.

Hence, for $Z^{(2)} \in X_0$ the right hand side $F(Z^{(1)}+Z^{(2)}) - MZ^{(1)}$ of (5.10) as a function of $Z^{(2)}$ maps X_0 into Y_0 by construction of $Z^{(1)}$. Combining this with Lemma 5.1 and the fact that the right hand side is at least quadratic, an application of the contraction mapping theorem together with a bootstrapping argument to obtain higher regularity yields the following result.

Theorem 5.3 *Let $3<r<7/2$ and fix some $t_1 > t_0$. Then there exist $C_1, C_2 > 0$ such that the following holds. If $(\eta_0, u_0) \in \mathcal{H}^r(2,\Omega) = H^r(2,\mathbb{R}) \times H^{r-1/2}(2,\Omega)$ satisfy (4.16) and $\rho = \|(\eta_0,u_0)\|_{\mathcal{H}^r(2,\Omega)} \le C_1$, then there exists a unique solution (η,u,p) of (5.7), $(\eta,u) \in \mathcal{K}^{r+1/2}((t_0,t_1),2,\Omega)$, $(\eta,u)|_{t=t_0} = (\eta_0,u_0)$, $p \in K^{r-1}((t_0,t_1),2,\Omega)$, and*

$$\|(\eta,u)\|_{\mathcal{K}^{r+1/2}((t_0,t_1),2,\Omega)} + \|p\|_{K^{r-1}((t_0,t_1),2,\Omega)} \le C_2\rho. \tag{5.13}$$

Moreover, for $t_0 < \tilde{t}_0 < t_1$ and any $m \in \mathbb{N}$ we have

$$(\eta,u) \in \mathcal{K}^{r+1/2+m}((\tilde{t}_0,t_1),2,\Omega), \quad p \in K^{r+m-1}((\tilde{t}_0,t_1),2,\Omega),$$

i.e., the solution becomes smooth for $t > t_0$, and there exists a $C_3 = C_3(m,\tilde{t}_0)$ such that

$$\|(\eta,u)\|_{\mathcal{K}^{r+1/2+m}((\tilde{t}_0,t_1),2,\Omega)} + \|p\|_{K^{r+m-1}((\tilde{t}_0,t_1),2,\Omega)} \le C_3\rho. \tag{5.14}$$

6 Renormalization

Let $U = U_c + U_s$ where U_c, U_s solve

$$\partial_t U_c = \mathcal{A}U_c + B_c(U) + \tilde{H}_c(U,\nabla p), \tag{6.1a}$$

$$\partial_t U_s = \mathcal{A}U_s + B_s(U) + \tilde{H}_s(U,\nabla p), \tag{6.1b}$$

where $B_\star = E_\star B$ and $\tilde{H}_\star = E_\star H$, $\star = c, s$, with E_c, E_s from sec.2.3 and B, H from (5.1), and where $(U_c,U_s)|_{t=1} = (E_cU, E_sU)|_{t=1}$. Then, by definition (2.26) of E_c, E_s,

$U = U_c + U_s$ solves (5.1). The idea of this splitting into central modes U_c and stable (exponentially damped) modes U_s is as follows. By construction, the function

$$\hat{W}_z(t,k,y) = \hat{f}_z(t^{1/2}k)\chi(k)\Phi_1(k,y)$$

with $\hat{f}_z$ from (1.10) fulfills

$$\partial_t \hat{W}_z = \hat{\mathcal{A}}\hat{W}_z + \hat{E}_c\hat{B}(\hat{W}_z) + \mathcal{O}(|k|^2).$$

This holds since $\hat{u}_z(t,k) = \hat{f}_z(t^{1/2}k)$ fulfills $\partial_t \hat{u} = -\alpha k^2 \hat{u} + \beta \mathrm{i}k(u_z^{*2})$, $\beta = -2$, since $\hat{\mathcal{A}}\hat{W}_z = \lambda_1(k)\hat{W}_z = (-\alpha k^2 + \mathcal{O}(k^3))\hat{W}_z$, and since

$$\begin{aligned}
\hat{E}_c\hat{B}(\hat{W}_z)(k) &= c(k)\chi(k)\big\langle \hat{B}(\hat{W}), \overline{\Psi_1}(k,y)\big\rangle_{(k)}\Phi_1(k,y) \\
&= (4/g^\star)(\hat{f}_z * \hat{f}_z)(1+\mathcal{O}(|k|))\int_0^1 \left(\frac{1}{\mathrm{R}}\partial_y^3 h(y)\right)(-\mathrm{i}k\partial_y\psi_1')\,\mathrm{d}y\,\chi(k)\Phi_1(k,y) \\
&= 4\mathrm{i}k(\hat{f}_z * \hat{f}_z)(1+\mathcal{O}(|k|))\int_0^1 6(2y-1)(y^2/2-y)\,\mathrm{d}y\,\chi(k)\Phi_1(k,y) \\
&= -2\mathrm{i}k(\hat{f}_z * \hat{f}_z)(1+\mathcal{O}(|k|))\chi(k)\Phi_1(k,y), \qquad (6.2)
\end{aligned}$$

where we used (2.19),(2.21) and (2.24). This also shows the "derivative like" structure of B_c. Then splitting $\hat{U}_c(t,k,y) = \hat{W}_z(t,k,y) + \hat{V}(t,k,y)$ with $\hat{V}|_{(t,k)=(1,0)} = 0$ we will obtain $\hat{V}(t) \to 0$. On the other hand, there exists a $\gamma_0 > 0$ such that

$$\mathrm{Re}\lambda_j^s(k) < -\gamma_0 \qquad (6.3)$$

for all $k \in \mathbb{R}$ for the eigenvalues of λ_j^s of $\mathcal{A}E_s$, such that U_s is linearly exponentially damped. Also note that reasoning as in (6.2) we have

$$(B_c + \tilde{H}_c)|_{k=0} = 0. \qquad (6.4)$$

Thus the whole nonlinearity $B_c + \tilde{H}_c$ locally at $k = 0$ corresponds to an x–derivative.

Remark 6.1 Formula (6.2) shows the reason for splitting $Pb(\eta){=}b(\eta){+}(P{-}\mathrm{Id})b(\eta)$ in (5.1). The idea is that $\operatorname{div} b(\eta){=}\partial_x\tilde{b}(\eta){=}2\partial_y^3 h(y)\eta\partial_x\eta{=}\mathcal{O}(\delta^3)$ if as in Remark 4.2 we assume $\eta = \mathcal{O}(\delta)$ and $\partial_x = \mathcal{O}(\delta)$. Then we also have $(P - \mathrm{Id})b(\eta) = \mathcal{O}(\delta^3)$, hence $Pb(\eta)$ splits into the relevant term $b(\eta)$ in $B(U)$ and the remainder contained in $H(U, \nabla p)$. ⌋

Before making these arguments rigorous we apply one more transformation. In the equation (6.1b) for U_s we remove the quadratic terms in U_c by setting

$$U_s = V_s - \frac{1}{2}\mathcal{A}^{-1}E_s\big(D_U^2 B_s(0)[U_c,U_c] + D_U^2\tilde{H}_s(0,\nabla p)[U_c,U_c]\big). \qquad (6.5)$$

Here $\mathcal{A}^{-1}$ exists on $E_s^h\mathcal{H}^{r-2}(2,\Omega)$ due to (6.3). Moreover,

$$\left\| \mathcal{A}^{-1}E_s^h \begin{pmatrix} 0 \\ u \end{pmatrix} \right\|_{\mathcal{H}^r(2,\Omega)} \leq C\|u\|_{H^{r-5/2}(2,\Omega)} \tag{6.6}$$

due to (2.13) and since the η component of $E_s^h(0,u)$ has finite support in Fourier space and hence can be estimated in $H^r(2)$ by $\|u\|_{H^0(2,\Omega)}$. Therefore,

$$\|V_s\|_{\mathcal{H}^r(2,\Omega)} \leq C(\|U_s\|_{\mathcal{H}^r(2,\Omega)} + \|U_c\|_{\mathcal{H}^r(2,\Omega)} + \|\nabla p\|_{H^{r-1}(2,\Omega)}) \tag{6.7}$$

in (6.5) since there are no nonlinear terms in the η component. We obtain

$$\partial_t U_c = \mathcal{A}U_c + B_c(U_c + V_s) + H_c(U_c, V_s, \nabla p), \tag{6.8}$$

$$\partial_t V_s = \mathcal{A}V_s + H_s(U_c, V_s, \nabla p), \tag{6.9}$$

with

$$\begin{aligned} H_c(U_c, V_s, \nabla p) = \;& \tilde{H}_c(U_c{+}U_s, \nabla p) + (B_c(U_c{+}U_s) - B_c(U_c{+}V_s)), \\ H_s(U_c, V_s, \nabla p) = \;& B_s(U_c{+}U_s) + \tilde{H}_s(U_c{+}U_s, \nabla p) \\ & -\tfrac{1}{2}E_s\big(D_U^2 B_s(0)[U_c, U_c] + D_U^2\tilde{H}_s(0,\nabla p)[U_c, U_c]\big) \\ & +\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}E_s\big(D_U^2 B_s(0)[U_c, U_c] + D_U^2\tilde{H}_s(0,\nabla p)[U_c, U_c]\big). \end{aligned} \tag{6.10}$$

In (6.10) U_s has to be replaced everywhere by the right hand side of (6.5).

Remark 6.2 The idea for (6.5) is as follows. We expect that $U_c \to 0$ with rate $t^{-1/2}$, hence $U_s \to 0$ with rate t^{-1} due to the terms in $\partial_t U_s$ which are quadratic in U_c. By eliminating these terms we may expect $V_s \to 0$ with rate $t^{-3/2}$, which we take into account in the scaling (6.12) below to simplify the analysis. ⌋

6.1 The rescaled systems

Following the ideas outlined in sec.3 and Remark 6.2, for some $L > 1$ chosen below we set

$$U_{n,c}(\tau,\xi,y) = L^n U_c(L^{2n}\tau, L^n\xi, y) = L^n\mathcal{R}_{L^N}U_c(L^{2n}\tau,\xi,y), \tag{6.11}$$

$$U_{n,s}(\tau,\xi,y) = L^{2n} V_s(L^{2n}\tau, L^n\xi, y) = L^{2n}\mathcal{R}_{L^N}U_c(L^{2n}\tau,\xi,y), \tag{6.12}$$

and $p_n(\tau,\xi,y) = L^n p(L^{2n}\tau, L^n\xi, y)$. Then

$$\partial_\tau U_{n,c} = \mathcal{A}_n U_{n,c} + L^{3n}B_{n,c}(U_n) + L^{3n}H_{c,n}(U_n, \nabla_n p_n), \tag{6.13a}$$

$$\partial_\tau U_{n,s} = \mathcal{A}_n U_{n,s} + L^{4n}H_{n,s}(U_n, \nabla_n p_n), \tag{6.13b}$$

where $U_n = U_{n,c} + L^{-n}U_{n,s}$, $\mathcal{A}_n = L^{2n}\mathcal{R}_{L^n}\mathcal{A}\mathcal{R}_{L^{-n}}$, and

$$\begin{aligned} B_{n,c}(U_n) &= \mathcal{R}_{L^n}B_c(L^{-n}U_n), \qquad \nabla_n p_n = (L^{-n}\partial_\xi p_n, \partial_y p), \\ H_{c,n}(U_n, \nabla_n p_n) &= \mathcal{R}_{L^n}H_c(L^{-n}\mathcal{R}_{L^{-n}}U_n, L^{-n}\mathcal{R}_{L^{-n}}\nabla_n p_n), \\ H_{s,n}(U_n, \nabla_n p_n) &= \mathcal{R}_{L^n}H_s(L^{-n}\mathcal{R}_{L^{-n}}U_n, L^{-n}\mathcal{R}_{L^{-n}}\nabla_n p_n). \end{aligned}$$

As in sec. 3 the idea is that solving (6.8),(6.9) on $t \in (1, \infty)$ is equivalent to iterating

$$\text{solve (6.13) on } \tau{\in}[L^{-2}, 1] \text{ with initial data } \begin{pmatrix} U_{n,c}(L^{-2}){=}L\mathcal{R}_L U_{n-1,c}(1) \\ U_{n,s}(L^{-2}){=}L^2\mathcal{R}_L U_{n-1,s}(1) \end{pmatrix}. \quad (6.14)$$

A local solution to (6.13) can be obtained as in sec.5 by first solving the linear inhomogeneous problems

$$\partial_\tau U_{n,c} - \mathcal{A}_n U_{n,c} = E_{c,n} \begin{pmatrix} 0 \\ g(\tau) \end{pmatrix}, \quad \partial_\tau U_{n,s} - \mathcal{A}_n U_{n,s} = E_{s,n} \begin{pmatrix} 0 \\ g(\tau) \end{pmatrix},$$

with $g \in K_0^{r-2}([1/L^2, 1], 2, \Omega)$ and $(U_{n,c}, U_{n,s})(1/L^2) = 0$ for $(U_{n,c}, U_{n,s}) \in \mathcal{K}_0^{r+1/2}$, recovering p_n as in Lemma 2.4, and then applying a fixed point argument and the contraction mapping theorem as in (5.10). Note that, due to (2.27), $U_c|_{t=1}$ and $U_s|_{t=1}$ both satisfy the same compatibilty conditions as $U|_{t=1} = (U_c + U_s)|_{t=1}$. The crucial step is to obtain estimates independent of n. Since we are going to refine these estimates in sec.6.3 (see Lemma 6.4, Lemma 6.5) here we only state the result.

Let

$$\rho_{n,c} = \|U_{n,c}(1)\|_{\mathcal{H}^r(2,\Omega)}, \quad \rho_{n,s} = \|U_{n,s}(1)\|_{\mathcal{H}^r(2,\Omega)}, \quad \rho_n = \rho_{n,c} + \rho_{n,s}. \quad (6.15)$$

and note that

$$\|L\mathcal{R}_L U\|_{\mathcal{H}^r(2,\Omega)} \le CL^{r+1/2}\|U\|_{\mathcal{H}^r(2,\Omega)}$$

due to the rescaling properties of Sobolev spaces.

Theorem 6.3 *Let $3 < r < 7/2$. There exist $L_0 > 1$, $C_1, C_2 > 0$ such that for all $L{>}L_0$ the following holds. If $\rho_{n-1} \le C_1 L^{-5}$ then there exists a unique solution $(U_{n,c}, U_{n,s}) \in [\mathcal{K}^{r+1/2}((L^{-2}, 1), 2, \Omega)]^2$ and $p_n \in K^{r-1}([L^{-2}, 1], 2, \Omega)$ of (6.13) with $(U_{n,c}, U_{n,s})|_{\tau=L^{-2}} = \big(L\mathcal{R}_L U_{n-1,c}, L^2\mathcal{R}_L U_{n-1,s}\big)|_{\tau=1}$, and*

$$\|U_{n,c}\|_{\mathcal{K}^{r+1/2}((L^{-2},1),2,\Omega)} + \|U_{n,s}\|_{\mathcal{K}^{r+1/2}((L^{-2},1),2,\Omega)} + \|p_n\|_{K^{r-1}((L^{-2},1),2,\Omega)} \le C_2 L^5 \rho_{n-1}. \quad (6.16)$$

For any $m > 0$, $(U_{n,c}, U_{n,s}) \in [\mathcal{K}^{r+1/2+m}((1/2, 1), 2, \Omega)]^2$, $p_n \in K^{r+m-1}((1/2, 1), 2, \Omega)$ and there exists a $C_3(m)$, independent of L, n, such that

$$\|U_{n,c}\|_{\mathcal{K}^{r+1/2+m}((1/2,1),2,\Omega)} + \|U_{n,s}\|_{\mathcal{K}^{r+1/2+m}((1/2,1),2,\Omega)} + \|p_n\|_{K^{r+m-1}((1/2,1),2,\Omega)} \le C_3 L^5 \rho_{n-1}. \quad (6.17)$$

6.2 Estimates for the linear semigroup and the nonlinear terms

Due to the loss of L^5 in (6.16) we need better control of ρ_n from (6.15) to iterate (6.14). Given a local solution from Theorem 6.3 we use the variation of constant formula to obtain improved estimates. We have to take care of the different roles of x (rescaled) and y (not rescaled). Therefore we introduce the notation

$$\|u\|_{H^{r_1,r_2}(\Omega)} := \|\hat{u}\|_{L^2(\mathbb{R},H^{r_2}(\,\mathrm{d}y))} + \||k|^{r_1}\hat{u}\|_{L^2(\mathbb{R},L^2(\,\mathrm{d}y))}. \tag{6.18}$$

We also let

$$E^h_{c,n} = \mathcal{R}_{L^n} E^h_c \mathcal{R}_{L^{-n}}, \quad E^h_{s,n} = \mathcal{R}_{L^n} E^h_s \mathcal{R}_{L^{-n}}.$$

Lemma 6.4 *There exists a $C > 0$ such that for all $L > 1$ we have*

$$\|\mathrm{e}^{\tau\mathcal{A}_n} E^h_{c,n} U_0\|_{\mathcal{H}^r(2,\Omega)} \leq C \max\{1, \tau^{-j/2}\} \|U_0\|_{\mathcal{H}^{r-j}(2,\Omega)}, \quad j = 0, 1, 2. \tag{6.19}$$

For $U_0 = (\eta_0, u_0) \in H^{r+1/2}(2) \times H^{r-i,r-j}(2,\Omega)$ we have

$$\begin{aligned} &\|\mathrm{e}^{\tau\mathcal{A}_n} E^h_{s,n} U_0\|_{\mathcal{H}^r(\Omega)} \\ &\quad \leq C\mathrm{e}^{-L^{2n}\gamma_0\tau} \left(\max\{1, \tau^{-i/2}(L^{2n}\tau)^{-j/2}\} \|u_0\|_{H^{r-i,r-j}(2,\Omega)} + \|\eta_0\|_{H^{r+1/2}(2)} \right). \end{aligned} \tag{6.20}$$

Proof. To prove (6.19) we write

$$E^h_{c,n} U_0(\xi, y) = \mathcal{F}^{-1}\big(a(\ell)\Phi_1(\ell/L^n, y)\big)$$

with $\mathrm{supp}(a) \subset \{|\ell| \leq 4\rho\}$ with $\rho = \mathcal{O}(1)$ from (2.22). Then

$$\mathrm{e}^{\tau\mathcal{A}_n} E^h_{c,n} U_0 = \mathcal{F}^{-1}\left(\mathrm{e}^{L^{2n}\lambda_1(\ell/L^n)\tau} a(\ell)\Phi_1(\ell/L^n, y)\right)$$

and the estimate follows from $\mathrm{Re}(L^{2n}\lambda_1(\ell/L^n)) = -\alpha\ell^2 + \mathcal{O}(\ell^4/L^{2n}) \leq -\tilde{\alpha}\ell^2$ for $|\ell| \leq 4\rho$.

The estimate (6.20) follows from (2.14) and the fact that $(\lambda - \mathcal{A})E^h_s$ is invertible for $\mathrm{Re}\lambda > -\gamma_0$. Since

$$E^h_{s,n} \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} (\xi, y) = \begin{pmatrix} \eta_0 \\ u_0 \end{pmatrix} - \mathcal{F}^{-1}\left(c(\ell)\chi(2\ell) \left\langle \begin{pmatrix} \hat{\eta}_0(\ell, \cdot) \\ \hat{u}_0(\ell, \cdot) \end{pmatrix}, \Psi_1(\ell/L^n, \cdot) \right\rangle \Phi_1(\ell/L^n, y) \right),$$

and since Ψ_1 and Φ_1 are smooth functions, the term involving u_0 in the η component of $E^h_{s,n}U_0$ is controlled by the L^2 norm of u_0. □.

Lemma 6.5 *Let $3 < r < 7/2$ and $\|U_{n,\star}\|_{\mathcal{H}^r(2,\Omega)} \leq R_n \leq 1$. There exists a $C > 0$ such that*

$$L^{3n} \|B_{c,n}(U_{n,c} + L^{-n}U_{n,s})\|_{\mathcal{H}^{r-1}(2,\Omega)} \leq CR_n^2. \tag{6.21}$$

Moreover, let $\|p_n\|_{H^{r-3/2}(2,\Omega)} \leq R_n$. *Then*

$$L^{3n}\|H_{c,n}(U_{n,c} + L^{-n}U_{n,s}, \nabla_n p_n)\|_{\mathcal{H}^{r-2}(2,\Omega)} \leq CL^{-n}R_n^2. \tag{6.22}$$

Finally, $H_{s,n}(U_n, \nabla_n p_n)$ *can be split according to the order of* ξ*–derivatives and* y*–derivatives in the form*

$$L^{4n}H_{s,n}(U_n, \nabla_n p_n) = \sum_{i+j\leq 2} \begin{pmatrix} g_n^{i,j} \\ h_{s,n}^{i,j} \end{pmatrix} (U_n, \nabla_n p_n)$$

with

$$\|g_n^{i,j}\|_{H^{r+1/2}(2)} \leq CL^{n(1-i)}R_n^2, \tag{6.23}$$
$$\|h_{s,n}^{i,j}\|_{H^{r-i,r-j}(2,\Omega)} \leq CL^{n(1-i)}R_n^2. \tag{6.24}$$

Proof. The argument for (6.21) has essentially been given in (6.2). Naively we have $B_{c,n}(U_{n,c}) = \mathcal{O}(L^n R_n^2)$. But using

$$\mathcal{F}(\mathcal{R}_{L^n}((\mathcal{R}_{L^{-n}}\eta_n)^2)) = L^n\mathcal{R}_{L^{-n}}(\mathcal{R}_L^n\hat{\eta}_n)^{*2} = \hat{\eta}_n^{*2} \tag{6.25}$$

and the fact that $\mathcal{F} : H^r(n, \mathbb{R}) \to H^n(r, \mathbb{R})$ is an isomorphism we obtain

$$\begin{aligned}
&L^{3n}\|B_{c,n}(U_n)\|_{\mathcal{H}^{r-1}(2,\Omega)} \\
=&L^n \left\| \mathcal{F}^{-1}\left[\frac{4}{\mathrm{R}}c\left(\frac{\ell}{L^n}\right)\chi\left(\frac{\ell}{L^n}\right)\int_0^1 \widehat{S\eta_n}^{*2} h'''(y)\psi'\left(\frac{\ell}{L^n}, y\right)\,\mathrm{d}y\ \Phi_1\left(\frac{\ell}{L^n}, y\right)\right]\right\|_{\mathcal{H}^{r-1}(2,\Omega)} \\
\leq& C\|\mathrm{i}\ell(\hat{\eta}_n^{*2})\|_{H^2(r-1,\mathbb{R})} \leq C\|\hat{\eta}_n^{*2}\|_{H^2(r,\mathbb{R})} \leq C\|\eta_n^2\|_{H^r(2,\mathbb{R})} \\
\leq& CR_n^2.
\end{aligned} \tag{6.26}$$

We used that $\psi(k) = \mathrm{i}k\psi_1(y) + \mathcal{O}(k^2)$, and that $c(k)$ and $\|\Phi_1(k,\cdot)\|_{H^r((0,1))}$ are uniformly bounded on $|k| \leq \rho$. This means that here we obtain the needed additional factor L^{-n} from the mode filter E_c. Due to the finite support of $\chi(k)$, E_cU is actually a smooth function for $U \in \mathcal{H}^0(m, \Omega)$. However, we do not use this smoothing since we explicitly need the $\mathcal{O}(k)$ terms to obtain the factor L^{-n} via rescaling.

To obtain (6.22) first consider a term in, e.g., (4.6b) with $i = 1$. For instance consider $b_1(\eta, u) = -\frac{4}{\mathrm{R}}\eta\partial_y^2 u_1$, obtained from setting $c = d = 2$ in

$$\begin{aligned}
\tfrac{1}{\mathrm{R}}(\zeta_{cl}\zeta_{dl} - \delta_{cl}\delta_{dl})\partial_c\partial_d u_i &= \tfrac{1}{\mathrm{R}}(1/d^2 - 1)\partial_y^2 u_1 \qquad (\text{for } d = c = 2) \\
&= b_1(\eta, u) + \text{h.o.t},
\end{aligned}$$

where h.o.t denotes higher order terms (either cubic or, via (4.2), containing ∂_x), and where other combination of c, d also yield higher order terms. By omitting P with $\|P\|_{H^r(2,\Omega)\to H^r(2,\Omega)} \leq C$ in $H(U)$ in (5.1) and rescaling, b_1 yields the term

$$\mathcal{R}_{L^n}E_c \begin{pmatrix} 0 \\ -\frac{4}{\mathrm{R}}(L^n h_1 + h_2 + L^{-n}h_3) \\ 0 \end{pmatrix} \tag{6.27}$$

in $L^{3n} H_{c,n}$ with

$$h_1 = (\mathcal{R}_{L^{-n}}\eta_{n,c})(\mathcal{R}_{L^{-n}}\partial_y^2(U_{n,c})_2),$$
$$h_2 = (\mathcal{R}_{L^{-n}}\eta_{n,s})(\mathcal{R}_{L^{-n}}\partial_y^2(U_{n,c})_2) + (\mathcal{R}_{L^{-n}}\eta_{n,c})(\mathcal{R}_{L^{-n}}\partial_y^2(U_{n,s})_2),$$
$$h_3 = (\mathcal{R}_{L^{-n}}\eta_{n,s})(\mathcal{R}_{L^{-n}}\partial_y^2(U_{n,s})_2).$$

Similar to (6.26), in estimating (6.27) we obtain an additional L^{-n} from $\mathcal{R}_{L^n} E_c$, but for h_1 this is not yet enough to obtain (6.22). However, writing $\hat{U}_{n,c}(\ell, y) = a(\ell)\Phi_1(\ell/L^n, y)$ and noting that

$$\partial_y^2 \Phi_{12}(k, y) = \partial_y^2(2y + \mathcal{O}(|k|)) = \mathcal{O}(|k|) \tag{6.28}$$

we obtain

$$\begin{aligned} &L^n \|\mathcal{R}_{L^n} E_c h_1\|_{\mathcal{H}^{r-2}(2,\Omega)} \\ &\quad = L^n \left\| \mathcal{F}^{-1} \left((\hat{\eta}_{c,n} * \partial_y^2(\hat{U}_{n,c})_2) c(\frac{k}{L^n}) \chi(\frac{k}{L^n}) \Phi_1(\frac{k}{L^n}, y) \int_0^1 h'''(y)\psi'(\frac{k}{L^n}, y)\,\mathrm{d}y \right) \right\|_{\mathcal{H}^{r-2}(2,\Omega)} \\ &\quad \leq C L^{-n} \|a(k)(1 + \mathcal{O}(k)) * (k a(k))\|_{H^2(r-2,\mathbb{R})} \leq C L^{-n} \|a(k)\|^2_{H^2(r,\mathbb{R})} \\ &\quad \leq C L^{-n} R_n^2. \end{aligned} \tag{6.29}$$

Note that the additional L^{-n} due to (6.28) is absent in $(\mathcal{R}_{L^{-n}}\eta_{n,c})(\mathcal{R}_{L^{-n}}\partial_y^2(U_{n,s})_2)$ which is why we used the scaling (6.12).

Estimates similar to (6.26),(6.29) can be used for all "a priori low order terms" in $H_{c,n}$. For instance, in estimating the "dangerous" terms coming from $(\partial_y u_1)u_2$ in (4.6b) with $i = 1$ we obtain an additional L^{-n} in $\partial_y(U_{n,c})_2(U_{n,c})_3$ from $\Phi_{13} = \mathcal{O}(|k|)$. All terms containing ∂_x in g_0, g_1, g_2, g_4 in (4.15) a forteriori yield sufficient powers of L^{-n} in $H_{c,n}$, while the terms in $H_{c,n}$ generated by replacing U_s in (6.10) by the right hand side of (6.5) are either cubic in U_c or contain a factor V_s which yields L^{-n} via (6.12).

The proof of (6.23) and (6.24) follows similar lines. Here we cannot gain an additional factor L^{-n} via E_c, but there are no quadratic terms in $U_{n,c}$ due to the transformation (6.5). Thus we directly have

$$\|h_{s,n}^{0,0}\|_{H^r(2,\Omega)} = \mathcal{O}(L^n R_n^2).$$

The same estimate holds for y derivatives (in $H^{r,r-j}(2,\Omega)$), while ξ derivatives yield a factor L^{-n}. The terms coming from

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[E_s\big(\big(D_U^2 B_s(0)[U_c, U_c] + D_U^2 H_s(0)[U_c, U_c]\big)\big)\big] \tag{6.30}$$

in (6.10) can be controlled by replacing $\partial_\tau U_{n,c}$ by the right hand side of (6.13a) and using (6.21),(6.22). Finally, the η component of $H_{s,n}$ is only generated by projection. Therefore it has finite support in Fourier space and can be controlled by R_n^2. □

6.3 Splitting, iteration and conclusion

We split

$$U_{n,c}(\tau,\xi,y) = W_n^{(z)}(\tau,\xi,y) + V_n(\tau,\xi,y),$$

where

$$W_n^{(z)} = \mathcal{F}^{-1}\big(\hat{a}_z(\tau,\ell)\chi(\ell/L^n)\Phi_1(\ell/L^n)\big), \quad \hat{a}_z(\tau,\ell) = \hat{f}_z(\tau^{1/2}\ell),$$

with f_z from (1.10) and z defined by

$$\ln(z+1) = \frac{\beta}{\alpha}\int \eta(1,x)\,\mathrm{d}x = \frac{\beta}{\alpha}\hat{\eta}(1,0).$$

Then

$$\partial_\tau V_n = \mathcal{A}_n V_n + L^{3n}(B_{c,n}(U_n) - B_{c,n}(W_n^{(z)})) + L^{3n}H_c^n(U_n) + \mathrm{Res}_n \tag{6.31}$$

where

$$\mathrm{Res}_n = -\partial_\tau W_n^{(z)} + \mathcal{A}_n W_n^{(z)} + L^{3n}B_{c,n}(W_n^{(z)}).$$

Lemma 6.6 *Let $|z|<1$. There exists a $C>0$ such that*

$$\sup_{\tau\in[L^{-2},1]} \|\mathrm{Res}_n\|_{\mathcal{H}^r(2,\Omega)} \le CL^{-n}|z|.$$

Proof. We have $\mathcal{A}_n W_n^{(z)} = (-\alpha\ell^2 + \mathcal{O}(l^3/L^n))W_n^{(z)}$ as $|\ell|\to 0$, and, as in (6.2) and similar to (6.26),

$$L^{3n}\hat{B}_{c,n}(W_n^{(z)}) = -2\mathrm{i}\ell(\hat{a}_z * \hat{a}_z)(1+\mathcal{O}(|l|/L^n))\Phi_1(\ell/L^n).$$

Combining this with

$$\begin{aligned}\partial_\tau W_n^{(z)} &= \mathcal{F}^{-1}(\partial_\tau \hat{a}_z(\tau,\ell)\chi(\ell/L^n)\Phi_1(\ell/L^n))\\ &= \mathcal{F}^{-1}((-\alpha\ell^2\hat{a}_z + \mathrm{i}\beta\ell(\hat{a}_z*\hat{a}_z))\chi(\ell/L^n)\Phi_1(\ell/L^n))\end{aligned}$$

yields

$$\mathrm{Res}_n(\ell) = CL^{-n}(\mathcal{O}(\ell^3)W_n^{(z)} + \mathcal{O}(\ell^2)(\hat{a}_z*\hat{a}_z)\varphi^1(\ell/L^n)).$$

This can be estimated in $\mathcal{H}^r(2,\Omega)$ by $CL^{-n}|z|$ since a_z is an analytic and exponentially decaying function. □

To proceed we write

$$U_{n,c}(1,\xi,y) = W_n^{(z)}(1,\xi,y) + G_{n,c}(\xi,y), \quad U_{n,s}(1,\xi,y) = G_{n,s}(\xi,y).$$

By the choice of z we have $\hat{g}_{0,c}(0,\cdot)=0$. Moreover, as already explained in (6.4), $\tilde{B}_{c,n}(U_n)$, $H_{c,n}(u_n)$ and Res_n locally at ℓ=0 have the form of a total derivative, therefore $\partial_\tau\hat{V}_n(\tau,0,y)=0$. This gives

$$\hat{V}_n(\tau,0,\cdot) = 0 \quad \forall\tau\in[L^{-2},1], \quad \text{hence} \quad \hat{G}_{n,c}(0,\cdot) = 0 \quad \forall n\in\mathbb{N}. \tag{6.32}$$

To control $(G_{n,c}, G_{n,s})$ we now use the integral equation satisfied by V_n and $U_{n,s}$,

$$G_{n,c} = \mathrm{e}^{(1-L^{-2})\mathcal{A}_n} E^h_{c,n} L\mathcal{R}_L G_{n-1,c} + L^{3n}\int_{L^{-2}}^1 \mathrm{e}^{(1-s)\mathcal{A}_n}\big[B_{c,n}(U_n) - B_{c,n}(W_n^{(z)}) + H_{c,n}(U_n) + \mathrm{Res}_n\big](s)\,\mathrm{d}s, \tag{6.33}$$

$$G_{n,s} = \mathrm{e}^{(1-L^{-2})\mathcal{A}_n} E^h_{s,n} L^2\mathcal{R}_L G_{n-1,s} + \int_{L^{-2}}^1 \mathrm{e}^{(1-s)\mathcal{A}_n} L^{4n} H_{s,n}(U_n(s))\,\mathrm{d}s, \tag{6.34}$$

where, as before, $U_n = W_n^{(z)} + V_n + L^{-n}U_{n,s}$.

The first term on the right hand side of (6.33) is estimated using

$$\Big\|\mathrm{e}^{(1-L^{-2})\mathcal{A}_n} E^h_{c,n} L\mathcal{R}_L G\Big\|_{\mathcal{H}^r(2,\Omega)} \le CL^{-1}\|G\|_{\mathcal{H}^r(2,\Omega)} \tag{6.35}$$

for $G \in \mathcal{H}^r(2,\Omega)$ with $\hat{G}(0) = 0$, similar to (3.6). Again the idea is to write $\hat{G}(\ell/L) = \hat{G}(0) + \partial_\ell \hat{G}(0)L^{-1}\tilde{\ell} = \partial_\ell\hat{G}(0)L^{-1}\tilde{\ell}$ with $\tilde{\ell} \in (0,\ell)$.

In the stable part we have, due to (6.20) and for L sufficiently large,

$$\begin{aligned}\Big\|\mathrm{e}^{(1-L^{-2})\mathcal{A}_n} E^h_{s,n} L^2\mathcal{R}_L G\Big\|_{\mathcal{H}^r(2,\Omega)} &\le CL^5\mathrm{e}^{-\gamma_0 L^{2n}(1-L^{-2})}\|G\|_{\mathcal{H}^r(2,\Omega)} \\ &\le L^{-1}\|G\|_{\mathcal{H}^r(2,\Omega)}.\end{aligned} \tag{6.36}$$

The integral in (6.34) is of the form

$$\sum_{i+j\le 2} I^{i,j} \quad \text{where} \quad I^{i,j} = \int_{L^{-2}}^1 \mathrm{e}^{(1-s)\mathcal{A}_n}\begin{pmatrix} g_n^{i,j} \\ h_{n,s}^{i,j}\end{pmatrix}(s)\,\mathrm{d}s,$$

with, due to Theorem 6.3 and Lemma 6.5,

$$\begin{aligned}\|g_n^{i,j}(s)\|_{H^{r+1/2}(2)} &\le CL^{n(1-i)}\|U_n\|^2_{\mathcal{H}^r(2,\Omega)} \le CL^{n(1-i)}(L^5\rho_{n-1})^2, \\ \|h_{n,s}^{i,j}(s)\|_{H^{r-i,r-j}(2,\Omega)} &\le CL^{n(1-i)}\|U_n(s)\|^2_{\mathcal{H}^r} \le CL^{n(1-i)}(L^5\rho_{n-1})^2,\end{aligned}$$

for $s \in [L^{-2}, 1]$, and we want to estimate in $\mathcal{H}^r(2,\Omega)$. Using Lemma 6.4 we see that $I^{2,0}$, $I^{1,1}$ and $I^{0,2}$ diverge at $s = 1$ due to the u component. However, due to the higher regularity in Theorem 6.3 we additionally have

$$\|h_{n,s}^{i,j}(s)\|_{H^r(2,\Omega)} \le C(C_3L^5\rho_{n-1})^2 \quad \text{for} \quad s \in [1/2, 1],$$

with C_3 from (6.17). Therefore, as in (3.20) we split the integrals $I^{2,0}, I^{1,1}$ and $I^{0,2}$ as $\int_{L^{-2}}^1 \cdots\,\mathrm{d}s = \int_{L^{-2}}^{1/2}\cdots\,\mathrm{d}s + \int_{1/2}^1 \cdots\,\mathrm{d}s$.

The integrals in (6.34) can then be estimated as

$$\begin{aligned}\|I^{0,j}\|_{\mathcal{H}^r(2,\Omega)} &\le CL^n(L^5\rho_{n-1})^2\int_{L^{-2}}^1 \mathrm{e}^{-\gamma_0 L^{2n}(1-s)}\max\{1, (L^{2n}(1-s))^{-j/2}\}\,\mathrm{d}s \\ &\le CL^{-n}(L^5\rho_{n-1})^2, \quad j = 0,1,\end{aligned} \tag{6.37}$$

$$\begin{aligned}\|I^{1,0}\|_{\mathcal{H}^r(2,\Omega)} &\le C(L^5\rho_{n-1})^2\int_{L^{-2}}^1 \mathrm{e}^{-\gamma_0 L^{2n}(1-s)}\max\{1, (1-s)^{-1/2}\}\,\mathrm{d}s \\ &\le CL^{-n}(L^5\rho_{n-1})^2,\end{aligned} \tag{6.38}$$

$$\begin{aligned}\|I^{0,2}\|_{\mathcal{H}^r(2,\Omega)} &\leq CL^n(L^5\rho_{n-1})^2\bigg[\int_{L^{-2}}^{1/2} \mathrm{e}^{-\gamma_0 L^{2n}(1-s)}\max\{1,(L^{2n}(1-s))^{-1}\}\,\mathrm{d}s \\ &\qquad\qquad + C_3^2\int_{1/2}^1 \mathrm{e}^{-\gamma_0 L^{2n}(1-s)}\,\mathrm{d}s\bigg] \\ &\leq C(1+C_3^2)L^{-n}(L^5\rho_{n-1})^2, \end{aligned} \tag{6.39}$$

and similarly

$$\|I^{1,1}\|_{\mathcal{H}^r(2,\Omega)} + \|I^{2,0}\|_{\mathcal{H}^r(2,\Omega)} \leq C(1+C_3^2)L^{-n}(L^5\rho_{n-1})^2. \tag{6.40}$$

To estimate the integrals in (6.33) we write

$$B_{c,n}(U_n) - B_{c,n}(W_n^{(z)}) = Q_n(W_n^{(z)}, V_n + L^{-n}U_{n,s}) + B_{c,n}(V_n + L^{-n}U_{n,s})$$

where $Q_{c,n}(W_n^{(z)}, V_n)$ is bilinear. Then, similar to (6.21) in Lemma 6.5, we obtain

$$\begin{aligned} L^{3n}\|Q_{c,n}(W_n^{(z)}, V_n + L^{-n}U_{n,s})\|_{\mathcal{H}^{r-1}(2,\Omega)} &\leq CL^5|z|\rho_{n-1}, \\ L^{3n}\|B_{c,n}(V_n + L^{-n}U_{n,s})\|_{\mathcal{H}^{r-1}(2,\Omega)} &\leq C(L^5\rho_{n-1})^2. \end{aligned}$$

Therefore, using Lemma 6.4,

$$\begin{aligned} &\left\| L^{3n}\int_{L^{-2}}^1 \mathrm{e}^{(1-s)\mathcal{A}_n}\left[B_{c,n}(U_n(s)) - B_{c,n}(W_n^{(z)}(s)) + \mathrm{Res}_n(s)\right]\mathrm{d}s\right\|_{\mathcal{H}^r(2,\Omega)} \\ &\qquad\qquad \leq C(L^5|z|\rho_{n-1} + (L^5\rho_{n-1})^2 + L^{-n}|z|). \end{aligned} \tag{6.41}$$

Finally we have

$$L^{3n}\left\|\int_{L^{-2}}^1 \mathrm{e}^{(1-s)\mathcal{A}}\mathcal{H}_{c,n}(U_n(s))\,\mathrm{d}s\right\|_{\mathcal{H}^r(2,\Omega)} \leq CL^{-n}(1+C_3^2)(L^5\rho_{n-1})^2$$

by splitting $\int_{L^{-2}}^1 \cdots \,\mathrm{d}s = \int_{L^{-2}}^{1/2} \cdots \,\mathrm{d}s + \int_{1/2}^1 \cdots \,\mathrm{d}s$. Combining this with (6.35)–(6.40) and setting

$$\tilde{\rho}_{n,c} = \|G_{n,c}\|_{\mathcal{H}^r(2,\Omega)}$$

we obtain

$$\begin{aligned} \rho_{n,s} &\leq L^{-1}\rho_{n-1,s} + CL^{-n}(L^5\rho_{n-1})^2, \\ \tilde{\rho}_{n,c} &\leq CL^{-1}\tilde{\rho}_{n-1,c} + C\big(|z|L^5\rho_{n-1} + (L^5\rho_{n-1})^2 + L^{-n}(L^5\rho_{n-1})^2 + |z|L^{-n}\big). \end{aligned} \tag{6.42}$$

Thus we can complete the **proof of Theorem 1.6:** Let $L \geq L_0$ with L_0 sufficiently large such that $CL^{-1} \leq L^{-(1-\delta)}$ for some small $\delta > 0$. Let $\|(\eta_0, u_0)\|_{\mathcal{H}^r(2,\Omega)}$ be so small that $|z| + \tilde{\rho}_{0,c} + \rho_{0,s} \leq L^{-m_0}$ with m_0 to be chosen below, hence also $\rho_0 \leq CL^{-m_0}$. Then (6.42) implies $\tilde{\rho}_{n,c} + \rho_{n,s} \leq L^{-(m_n - n\delta)}$ with

$$m_n = \min\{m_{n-1}+1, m_0 + m_{n-1} - 5, 2m_{n-1} - 10, m_0 + n\}.$$

Choosing, for instance, $m_0 = 11$ yields $m_1 = 12, m_2 = 13, \ldots$, hence $\rho_n \le CL^{-n(1-\delta)}$. Therefore,

$$\|L^n \mathcal{R}_{L^n} U(L^{2n}) - W_n^{(z)}(1)\|_{\mathcal{H}^r(2,\Omega)} = \|G_{n,c} + L^{-n} G_{n,s}\|_{\mathcal{H}^r(2,\Omega)} \le CL^{-n(1-\delta)}. \quad (6.43)$$

Using

$$\left\|\mathcal{F}^{-1}\big(\hat{a}_z(L^{2n}, \ell)(\Phi_1(0,\cdot) - \chi(\ell/L^n)\Phi^1(\ell/L^n,\cdot))\big)\right\|_{\mathcal{H}^r(2,\Omega)} \le CL^{-n},$$

setting $t = L^{2n}\tau$, using Theorem 6.3 for $\tau \in (1/L^2, 1)$, and doing back the transformations in sec.4 the proof of Theorem 1.6 is complete. □

A Appendix

A.1 Proof of Lemma 2.2

The proof that the operator $\mathcal{A}$ in the linearization $\partial_t U = \mathcal{A}U$ of (1.1) is sectorial in $\mathcal{X} = \{U = (\eta, u) : \eta \in H^1(\Gamma_{\rm f}), u \in PL^2(\Omega)\}$ (see (2.5)) is based on Korn's inequality. It works as in, e.g., [Sun97, section 3] and is sketched here for convenience. For $u, v \in H^2(\Omega)$, and $p \in H^1(\Omega)$ we have

$$\begin{aligned}\int_\Omega \left(\frac{1}{\mathrm{R}}\Delta u - \nabla p\right) v \,\mathrm{d}\Omega &= -\frac{1}{2\mathrm{R}}\langle u, v\rangle - \int_{\partial\Omega} S(u,p) v \,\mathrm{d}\Gamma + \int_\Omega p \operatorname{div} v \,\mathrm{d}\Omega, \qquad (A.1)\\ \langle u, v\rangle &= \int_\Omega \sum_{i,j=1,2} (\partial_{x_j} u_i + \partial_{x_i} u_j)(\partial_{x_j} v_i + \partial_{x_i} v_j)\,\mathrm{d}\Omega,\\ S(u,p)_i &= p n_i - \frac{1}{\mathrm{R}}\sum_{j=1,2}(\partial_{x_j} u_i + \partial_{x_i} u_j) n_j,\end{aligned}$$

where n is the unit outer normal on $\partial\Omega$. Moreover, for $u \in H^1(\Omega)$ with $\operatorname{div} u = 0$ and $u = 0$ on $\Gamma_{\rm b}$ we have Korn's inequality in the form [Ito93]

$$\begin{aligned}&\frac{4}{3}\|\nabla u\|_0^2 \le \langle u, u\rangle \le 4\|\nabla u\|_0^2,\\ &\langle u, u\rangle \ge \frac{\pi^2}{2}\|u\|_0^2 \quad \text{and} \quad \langle u, u\rangle \ge 2\int_\Gamma u_1^2|_{\Gamma_{\rm f}} + u_2^2|_{\Gamma_{\rm f}}\,\mathrm{d}\Gamma,\end{aligned} \qquad (A.2)$$

where $\|u\|_m = \|u\|_{H^m(\Omega)}$ and in the following $(u,v)_m = (u,v)_{H^m}$.

Let $F = (\xi, f) \in \mathcal{X}$. First we show that for $\mathrm{Re}\lambda > a$ with a sufficiently large there exists a (weak) solution $U = (\eta, u) \in H^1(\Gamma_{\rm f}) \times H^1(\Omega)$ of $(\lambda - \mathcal{A})U = F$. For $v \in PH^1(\Omega)$ with $v|_{\Gamma_{\rm b}} = 0$ we have, using (A.1),

$$\begin{aligned}(f,v)_0 &= ((\lambda - A + L_0)u - E(g^\star\eta - \mathrm{W}\partial_x^2\eta), v)_0\\ &= \lambda(u,v)_0 + (L_0 u, v)_0 + \frac{1}{2\mathrm{R}}\langle u, v\rangle + \int_{\Gamma_{\rm f}} -\frac{2}{\mathrm{R}}\eta v_1 + g^\star\eta v_2 + \mathrm{W}\partial_x\eta\partial_x v_2\,\mathrm{d}\Gamma \quad (A.3)\end{aligned}$$

From $(\lambda+\partial_x)\eta = u_2+\xi$ we obtain

$$\lambda(u,v)+(L_0u,v)_0+\frac{1}{2\mathrm{R}}\langle u,v\rangle \int_{\Gamma_\mathrm{f}} \big(-\frac{2}{\mathrm{R}}v_1+g^\star v_2\big)(\lambda-\partial_x)^{-1}u_2+\mathrm{W}\partial_x v_2\partial_x(\lambda-\partial_x)^{-1}u_2\,\mathrm{d}\Gamma$$
$$= (f,v)_0+\int_{\Gamma_\mathrm{f}}\big(\frac{2}{\mathrm{R}}v_1-g^\star v_2\big)(\lambda-\partial_x)^{-1}\xi-\mathrm{W}\partial_x v_2\partial_x(\lambda-\partial_x)^{-1}\xi\,\mathrm{d}\Gamma. \quad (\mathrm{A.4})$$

Let $B_\lambda(u,v)$ be the left hand side of (A.4) and $H(F,v)$ be the right hand side, and define the Hilbert space $\mathcal{H}\subset H^1(\Omega)$ via the inner product

$$(u,v)_\mathcal{H} = (u,v)_1+\int_{\Gamma_\mathrm{f}} u_2v_2+\partial_x u_2\partial_x v_2\,\mathrm{d}\Gamma.$$

Then $B_\lambda(\cdot,\cdot)$ is a bounded bilinear form in $\mathcal{H}$ and $H(F,\cdot)$ a bounded linear functional. Moreover, from Korn's inequality (A.2) it follows that there exist an $a>0$ such that for $\mathrm{Re}\lambda>a$ we have $\mathrm{Re}(B_\lambda(u,u))\ge C\|u\|_\mathcal{H}^2$, i.e., B is coercive. By the Lax–Milgram Theorem we have a unique solution u with $\|u\|_1+\|u|_{\Gamma_\mathrm{f}}\|_1\le\|F\|_\mathcal{X}$. From $\eta=(\lambda+\partial_x)^{-1}(u_2+\xi)$ we obtain $\|\eta\|_1\le C\|F\|_\mathcal{X}$, and hence $\lambda-\mathcal{A}$ is invertible in $\mathcal{X}$.

Thus, let $U=(\lambda-\mathcal{A})^{-1}F$. Choosing $v=u$ in (A.3) we obtain

$$\lambda\|u\|_0^2+(L_0u,u)_0+\frac{1}{2\mathrm{R}}\langle u,u\rangle+\int_{\Gamma_\mathrm{f}}-\frac{2}{\mathrm{R}}\eta u_1+\lambda(g^\star\eta^2+\mathrm{W}(\partial_x\eta)^2)\,\mathrm{d}\Gamma$$
$$=(f,u)_0+\int_{\Gamma_\mathrm{f}} g^\star\eta\xi+\mathrm{W}\partial_x\eta\partial_x\xi\,\mathrm{d}\Gamma. \quad (\mathrm{A.5})$$

Then, using again (A.2) and $|bc|\le\delta b^2+\frac{1}{4\delta}c^2, \delta>0$, and choosing a sufficiently large we obtain from the real part of (A.5)

$$\|U\|_\mathcal{X}\le\frac{1}{\mathrm{Re}(\lambda-a)}\|F\|_\mathcal{X}$$

for $\mathrm{Re}(\lambda)>a$. Since $\mathcal{A}$ is closed and densely defined in $L^2(\Gamma_\mathrm{f})\times L^2(\Omega)$, the Hille–Yosida Theorem yields that $\mathcal{A}$ generates a C_0 semigroup $\mathrm{e}^{t\mathcal{A}}:\mathcal{X}\to\mathcal{X}$. Similarly, the imaginary part of (A.5) gives

$$\|U\|_\mathcal{X}\le\frac{C}{|\mathrm{Im}\lambda|}\|F\|_\mathcal{X},$$

which implies that $\mathcal{A}$ is sectorial [Paz83, Theorem 2.5.2]. □

A.2 Resolvent estimates in the weighted spaces

To prove Lemma 2.5 we first transfer the resolvent estimates from Lemma 2.3 into the weighted spaces $H^r(2)$ and $H^r(2,\Omega)$. Therefore, let

$$\rho_b(x)=(1+(bx)^2)$$

where $b > 0$ will conveniently be chosen sufficiently small below. Clearly, for all $b > 0$ there exist $C_1, C_2 > 0$ such that $C_1\|\rho_b u\|_{H^r(\Omega)} \le \|u\|_{H^r(2,\Omega)} \le C_2\|\rho_b u\|_{H^r(\Omega)}$ for all $u \in H^r(2,\Omega)$, and similar for $H^r(2)$, i.e., the norms are equivalent.

Substituting $(\eta, u, p, \xi, f)=\rho_b(\alpha, \tilde{u}, q, \beta, g)$ into (2.1), the resolvent equation

$$(\lambda - \mathcal{A})\begin{pmatrix}\eta\\ u\end{pmatrix} = \begin{pmatrix}\xi\\ f\end{pmatrix}$$

is equivalent to

$$(\lambda - \tilde{\mathcal{A}})\begin{pmatrix}\alpha\\ \tilde{u}\\ q\end{pmatrix} = G + B_1\begin{pmatrix}\alpha\\ \tilde{u}\\ q\end{pmatrix}, \tag{A.6a}$$

$$\partial_y\tilde{u}_1 + \partial_x\tilde{u}_2 - 2\alpha = \varphi_1(\tilde{u}), \quad q - g^\star\alpha - \frac{2}{\mathrm{R}}\partial_y\tilde{u}_2 + \mathrm{W}\partial_x^2\alpha = \varphi_2(\alpha) \quad \text{on } \Gamma_\mathrm{f}, \tag{A.6b}$$

$$\operatorname{div}\tilde{u} = -\frac{\rho_b'}{\rho_b}\tilde{u}_1 \quad \text{in } \Omega, \qquad \tilde{u} = 0 \quad \text{on } \Gamma_\mathrm{b}, \tag{A.6c}$$

where $\rho_b' = \partial_x\rho_b = 2bx$, $\varphi_1(\tilde{u}) = -\frac{\rho_b'}{\rho_b}\tilde{u}_2$, $\varphi_2(\alpha) = -\frac{1}{\rho_b}\mathrm{W}(\rho_b''\alpha + \rho_b'\partial_x\alpha)$ and

$$(\lambda - \tilde{\mathcal{A}})\begin{pmatrix}\alpha\\ \tilde{u}\\ q\end{pmatrix} = \begin{pmatrix}\lambda\alpha - \partial_x\alpha - \tilde{u}_2\\ \lambda\tilde{u} - \frac{1}{\mathrm{R}}\Delta\tilde{u} + L_N\tilde{u} + \nabla q\end{pmatrix},$$

$$G = \begin{pmatrix}\beta\\ g_1\\ g_2\end{pmatrix}, \quad B_1\begin{pmatrix}\alpha\\ \tilde{u}\\ q\end{pmatrix} = \begin{pmatrix}\frac{1}{\rho_b}\rho_b'\alpha\\ \frac{1}{\rho_b}\left(\frac{1}{\mathrm{R}}(2\rho_b'\partial_x\tilde{u}_1 + \rho_b''\tilde{u}_1) - \rho_b'u_N\tilde{u}_1 - \rho_b'q\right)\\ \frac{1}{\rho_b}\left(\frac{1}{\mathrm{R}}(2\rho_b'\partial_x\tilde{u}_2 + \rho_b''\tilde{u}_2) - \rho_b'u_N\tilde{u}_2\right)\end{pmatrix}.$$

For $\lambda \in S_{a,\varphi} = \{\lambda : \varphi \le |\arg(a-\lambda)| \le \pi\}$ and $(\beta, g) \in H^{r+1/2}(\mathbb{R}) \times H^{r-2}(\Omega)$ we prove the resolvent estimate (2.6) for the solution $(\alpha, \tilde{u})$ of (A.6) which implies (2.13), i.e.,

$$\begin{aligned}&\|u\|_{H^r(2,\Omega)} + |\lambda|^{r/2}\|u\|_{L^2(2,\Omega)} + \|\eta\|_{H^{r+1/2}(2)} + |\lambda|^{(r+1/2)/2}\|\eta\|_{H^0(2)}\\ &\le C\left(\|f\|_{H^{r-2}(2,\Omega)} + |\lambda|^{(r-2)/2}\|f\|_{H^0(2,\Omega)} + \|\xi\|_{H^{r+1/2}(2)} + |\lambda|^{(r+1/2)/2}\|\xi\|_{H^0(2)}\right).\end{aligned} \tag{A.7}$$

Let $u = \mathcal{K}h$ be the solution of $\operatorname{div} u = h$, $u|_{\Gamma_\mathrm{b}} = 0$. Due to, e.g., [Tem01, Prop.1.2.3] this satisfies $\|\mathcal{K}h\|_{H^{r+1}(\Omega)} \le C\|h\|_{H^r(\Omega)}$. Set $\tilde{u} = v + u^{(1)}$ with

$$u^{(1)} = \mathcal{K}^*v := \mathcal{K}\left(-\frac{\rho_b'}{\rho_b}(u_1^{(1)} + v_1)\right),$$

where the operator $\mathcal{K}^*$ with $\|\mathcal{K}^*v\|_{H^{r+1}(\Omega)} \le Cb\|v\|_{H^r(\Omega)}$ exists for b sufficiently small due to the contraction mapping theorem. This yields

$$(\lambda - \tilde{\mathcal{A}}) \begin{pmatrix} \alpha \\ v \\ q \end{pmatrix} = G + B_2 \begin{pmatrix} \alpha \\ v \\ q \end{pmatrix}, \tag{A.8a}$$

$$\partial_y v_1 + \partial_x v_2 - 2\alpha = \varphi_3(v) := \varphi_1(v + \mathcal{K}^* v) \quad \text{on } \Gamma_{\mathrm{f}}, \tag{A.8b}$$

$$q - g^\star \alpha - \frac{2}{\mathrm{R}} \partial_y v_2 + \mathrm{W} \partial_x^2 \alpha = \varphi_4(\alpha, v) := \varphi_2(\alpha) + \frac{2}{\mathrm{R}} \partial_y (\mathcal{K}^* v)_2 \quad \text{on } \Gamma_{\mathrm{f}}, \tag{A.8c}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \Gamma_{\mathrm{b}}, \tag{A.8d}$$

where

$$B_2 \begin{pmatrix} \alpha \\ v \\ q \end{pmatrix} = B_1 \begin{pmatrix} \alpha \\ v + \mathcal{K}^* v \\ q \end{pmatrix} - \begin{pmatrix} (\mathcal{K}^* v)_2 \\ (\lambda - \frac{1}{\mathrm{R}} \Delta + L_n) \mathcal{K}^* v \end{pmatrix}.$$

Next we remove φ_3 from (A.8b), similar to (4.14). Therefore, let $v = w + u^{(2)}$ where $u^{(2)} = \mathcal{M}^* w$ solves

$$\begin{gathered} \operatorname{div} u^{(2)} = 0 \text{ in } \Omega, \qquad u^{(2)} = 0 \text{ on } y = 0, \\ u_2^{(2)} = 0, \quad \partial_y u_1^{(2)} + \partial_x u_2^{(2)} = \varphi_3(w + u^{(2)}) \text{ on } y = 1. \end{gathered} \tag{A.9}$$

Again, for b sufficiently small $\mathcal{M}^* w$ exists by the contraction mapping theorem and fulfills $\|\mathcal{M}^* w\|_{H^{r+1}(\Omega)} \leq Cb \|w\|_{H^r(\Omega)}$ due to $\|\varphi_3(v)\|_{H^{r-1/2}(\Gamma_{\mathrm{f}})} \leq Cb \|v\|_{H^r(\Omega)}$. This yields

$$(\lambda - \tilde{\mathcal{A}}) \begin{pmatrix} \alpha \\ w \\ q \end{pmatrix} = G + B_3 \begin{pmatrix} \alpha \\ w \\ q \end{pmatrix} \tag{A.10a}$$

$$\partial_y w_1 + \partial_x w_2 - 2\alpha = 0, \quad q - g^\star \alpha - \frac{2}{\mathrm{R}} \partial_y w_2 + \mathrm{W} \partial_x^2 \alpha = \varphi_5(\alpha, w) \quad \text{on } \Gamma_{\mathrm{f}}, \tag{A.10b}$$

$$\operatorname{div} w = 0 \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \Gamma_{\mathrm{b}}, \tag{A.10c}$$

where $\varphi_5(\alpha, w) = \varphi_4(\alpha, w + \mathcal{M}^* w) + \frac{2}{\mathrm{R}} \partial_y (\mathcal{M}^* v)_2$ and

$$B_3 \begin{pmatrix} \alpha \\ w \\ q \end{pmatrix} = B_2 \begin{pmatrix} \alpha \\ w + \mathcal{K}^* w \\ q \end{pmatrix} - \begin{pmatrix} (\mathcal{M}^* w)_2 \\ (\lambda - \frac{1}{\mathrm{R}} \Delta + L_N) \mathcal{M}^* w \end{pmatrix}.$$

For $b = 0$ we have $B_3 = 0$, and $\varphi_5 = 0$ and (A.10) is equivalent to $(\lambda - \mathcal{A}) \left(\begin{smallmatrix} \alpha \\ w \end{smallmatrix} \right) = G$. Thus, for $\lambda \in S_{a,\varphi} = \{\lambda : \varphi \leq |\arg(a - \lambda)| \leq \pi\}$ there exists a unique solution (α, w) which fulfills (2.6). From Lemma 2.4 we obtain $q = q(\alpha, w)$ with

$$\begin{aligned} &\|w\|_{H^r(\Omega)} + |\lambda|^{r/2} \|w\|_{L^2(\Omega)} + \|\alpha\|_{H^{r+1/2}(2)} + |\lambda|^{(r+1/2)/2} \|\alpha\|_{H^0(2)} \\ &\qquad\qquad + \|\nabla q\|_{H^{r-2}(\Omega)} + \|q\|_{H^{r-3/2}(\Gamma_{\mathrm{f}})} \\ &\leq C \left(\|g\|_{H^{r-2}(\Omega)} + |\lambda|^{(r-2)/2} \|g\|_{H^0(\Omega)} + \|\beta\|_{H^{r+1/2}(\mathbb{R})} + |\lambda|^{(r+1/2)/2} \|\beta\|_{H^0(\mathbb{R})} \right) \end{aligned} \tag{A.11}$$

For $b > 0$ we may then write (A.10) as

$$((\lambda - \mathcal{A}) - B_4)\begin{pmatrix}\alpha\\ w\end{pmatrix} = G, \quad B_4\begin{pmatrix}\alpha\\ w\end{pmatrix} = B_3\begin{pmatrix}\alpha\\ w\\ q(\alpha,w)\end{pmatrix} + \begin{pmatrix}0\\ E\varphi_5(\alpha,w)\end{pmatrix}. \quad \text{(A.12)}$$

Collecting the above estimates for $\mathcal{K}^*$, $\mathcal{M}^*$ and (A.11) we see that $\mathrm{Id} - (\lambda - \mathcal{A})^{-1}B_4$ is invertible for b sufficiently small due to Neumann's series. Hence we obtain a solution

$$(\alpha, w) = (\mathrm{Id} - (\lambda - \mathcal{A})^{-1}B_4)^{-1}(\lambda - \mathcal{A})^{-1}G$$

of (A.10) which again fulfills (2.6). This shows (A.7).

The remaining statements from Lemma 2.5 are proved the same way. □

A.3 Proof of Lemma 3.2

For simplicity, throughout the proof we omit the index n for u. For $r \geq 2$, we first consider the linear inhomogenous equation

$$Mu := (\partial_\tau - \partial_\xi^2)u = f(\tau), \quad f \in K_0^{r-2}([1/L^2, 1], 2), \quad u(1/L^2) = 0. \quad \text{(A.13)}$$

For a $\sigma_0 > 0$ we let $v(\tau) = u(\tau - L^{-2})\mathrm{e}^{-\sigma_0\tau}$. We identify f with its continuation for $\tau > 1$ and let $g(\tau) = f(\tau - L^{-2})\mathrm{e}^{-\sigma_0\tau}$. Then $(\partial_\tau + \sigma_0 - \partial_\xi^2)v = g$ which under Laplace transform becomes $(\lambda + \sigma_0 - \partial_\xi^2)\tilde{v}(\lambda) = \tilde{g}(\lambda)$. For $\mathrm{Re}\lambda \geq 0$ we have the resolvent estimate

$$\|\tilde{v}\|_{H^r(2)} + |\lambda|^r\|\tilde{v}\|_{H^0(2)} \leq C(\|\tilde{g}\|_{H^{r-2}(2)} + |\lambda|^{(r-2)/2}\|\tilde{g}\|_{H^0(2)}). \quad \text{(A.14)}$$

Moreover, $\tilde{v}$ is analytic in $\mathrm{Re}\lambda \geq 0$ because so is $\tilde{v}$. Thus, due to the Paley–Wiener Theorem $v(\tau) = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\mu\tau}\tilde{v}(\mathrm{i}\mu)\,\mathrm{d}\mu = 0$ for $\tau < 0$, and by (3.9) and (A.14) we have $\|v\|_{K^r([0,\infty],2)} \leq C\|g\|_{K^{r-2}([0,\infty],2)}$. This immediately yields $u \in K_0^r([1/L^2, 1], 2)$ with $\|u\|_{K^r([1/L^2,1],2)} \leq C\|f\|_{K^{r-2}([1/L^2,1],2)}$.

We write $u = M_0^{-1}f$ for the solution operator of (A.13). To solve the nonlinear problem (3.7) we set $u = v + w$ where $v \in K^r([1/L^2, 1], 2)$ is a continuation of $u(1/L^2) \in H^{r-1}(2)$. Then $Mw = f(w + v) - Mv$ where $f(u) = u\partial_\xi u + L^{nd_h}h(u, \partial_\xi u, \partial_\xi^2 u)$. By standard Sobolev embeddings (cf. Lemma 5.2) the nonlinearity maps $K_0^r([1/L^2, 1], 2)$ into $K_0^{r-2}([1/L^2, 1], 2)$ for $r > 5/2$. Here we use that (3.2) is quasilinear, while the fully nonlinear case would require $r > 7/2$. This is not important for the present problem, but in more complicated problems larger r may require more compatibility conditions.

Here, choosing $r = 3$, additional to $v(1/L^2) = u(1/L^2)$ we need to choose v in such a way that for $w \in K_0^r([0, 1/L^2], 2)$ we have $f(w+v) - Mv \in K_0^{r-2}([1/L^2, 1], 2)$.

Hence we also require $\partial_\tau v = f(v) - \partial_\xi^2 v$ at $\tau = 1/L^2$. In summary, we consider (3.7) in the form

$$w = M_0^{-1}(f(v+w) - Mw), \tag{A.15}$$

and the above estimates and the fact that f is at least quadractic imply that the right hand side of (A.15) defines a contraction in a sufficiently small ball in $K_0^{r+1}([1/L^2,1],2)$. This shows the existence of $u = v + w \in K^3([1/L^2,1],2)$ with $\|u\|_{K^3} \leq C\|u(1/L^2)\|_{H^2} \leq CL^{5/2}\rho_{n-1} \leq C_2\rho_n$.

The higher regularity follows from a standard bootstrapping argument: for $u \in K^3([1/L^2,1],2)$ we have $u(\tau_1) \in H^3(2)$ for almost all $\tau_1 \in [1/L^2,1]$. Starting again at such τ_1 we obtain $u \in K^4([\tau_1,1],2)$, and iteratively $u \in K^{3+m}([\tau_m,1],2)$ with the given estimates. □

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