Long-time persistence of KdV solitons as transient dynamics in a model of inclined film flow

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Abstract

The KS-perturbed KdV equation (KS-KdV)

$$\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 + \partial_x^4)u,$$

with $0 < \varepsilon \ll 1$ a small parameter, arises as an amplitude equation for small amplitude long waves on the surface of a viscous liquid running down an inclined plane in certain regimes when the trivial solution, the so-called Nusselt solution, is sideband unstable. Although individual pulses are unstable due to the long-wave instability of the flat surface, the dynamics of KS-KdV is dominated by traveling pulse trains of $O(1)$ amplitude. As a step toward explaining the persistence of pulses and understanding their interactions, we prove that for $n = 1$ and 2 the KdV manifolds of n-solitons are stable in KS-KdV on an $O(1/\varepsilon)$ time scale with respect to $O(1)$ perturbations in $H^n(\mathbb{R})$.

1 The results

The Kuramoto-Sivashinsky (KS)-perturbed KdV equation

$$\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 + \partial_x^4)u, \quad u = u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t \geq 0 \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small parameter, arises for instance as an amplitude equation for small amplitude long waves on the surface of a viscous liquid running down an inclined plane [TK78, CD96]; see fig. 1 for a sketch, and the monograph [CD02] for a comprehensive review of the so-called inclined-film problem. Equation (1) describes this system in certain ranges of parameters when the trivial solution, the so-called Nusselt solution, which shows a parabolic flow profile and a flat top surface, becomes sideband unstable. For a partial result on the validity of amplitude equations in the inclined-film problem we refer to [Uec03].

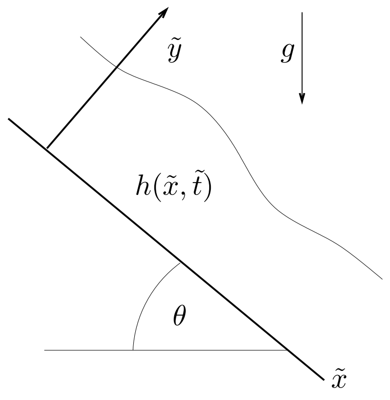

Figure 1: The inclined-film problem: A fluid of height $\tilde{y} = h(\tilde{x},\tilde{t}) = h_0 + \tilde{u}(\tilde{x},\tilde{t})$ runs down a plate with inclination angle θ subject to constant gravitational force g. In appropriate ranges of parameters (1) is the amplitude equation for this problem, where t, x, u are rescalings of $\tilde{t}, \tilde{x}$ and $\tilde{u}$.

For $\varepsilon = 0$ equation (1) is the well known KdV equation for which there exist $2n$-dimensional families M_n of n-soliton solutions; see, e.g, [AS81]. For $n = 1$ the two-dimensional family M_1 is explicitly given by

$$M_1 = \{u(x,t) = u_c(x - ct + \phi) : \phi \in \mathbb{R},\ c > 0\}, \quad u_c(y) = 3c\,\mathrm{sech}^2(\sqrt{c}y/2).$$

The amplitude parameter c also determines the speed, and ϕ is called the phase. For small $\varepsilon > 0$ there is an amplitude/speed selection principle [Oga94]: there exists a unique velocity $c_\varepsilon = 7/5 + O(\varepsilon)$ and a one-dimensional family of solitary waves for (1) of the form

$$M_\varepsilon = \{u(x,t) = u^\varepsilon(x - c_\varepsilon t + \phi) : \phi \in \mathbb{R}\}$$

with $\|u^\varepsilon - u_{c_\varepsilon}\|_{H^1} \le C_0\varepsilon$. In particular $\|u^\varepsilon\|_{L^\infty} = O(1)$ for $\varepsilon \to 0$, and $|u^\varepsilon(y)| \le C\mathrm{e}^{-\beta_0|y|}$ with constants C and $\beta_0 > 0$ both $O(1)$ for $\varepsilon \to 0$.

For all $\varepsilon > 0$ the pulse u^ε is unstable since the linearization around u^ε gives the same essential spectrum as the medium, the unstable trivial solution $u = 0$. However, a remarkable phenomenon occurs: in numerical simulations, the pulse u^ε is stable on long (but finite) time intervals. More generally speaking, the dynamics is dominated by KdV pulses over long times. On the other hand, for $t \to \infty$ the solution generally converges to a traveling pulse train consisting of (boosts of) the individually unstable pulses u^ε. See fig. 2 for an example. Such dynamics of surface waves are typical of observations in the inclined film problem [CD02], both experimentally and in numerical simulations of the free boundary Navier-Stokes problem describing this system.

The local-in-time stability of u^ε based on spectral information has been analyzed in [CDK96, OS97, CDK98, PSU04]; additionally, see [CD02] and the references therein for the structure of families of traveling wave solutions to (1) which is a first step in the analysis of the large time behaviour of (1). To add to the understanding of the long- but

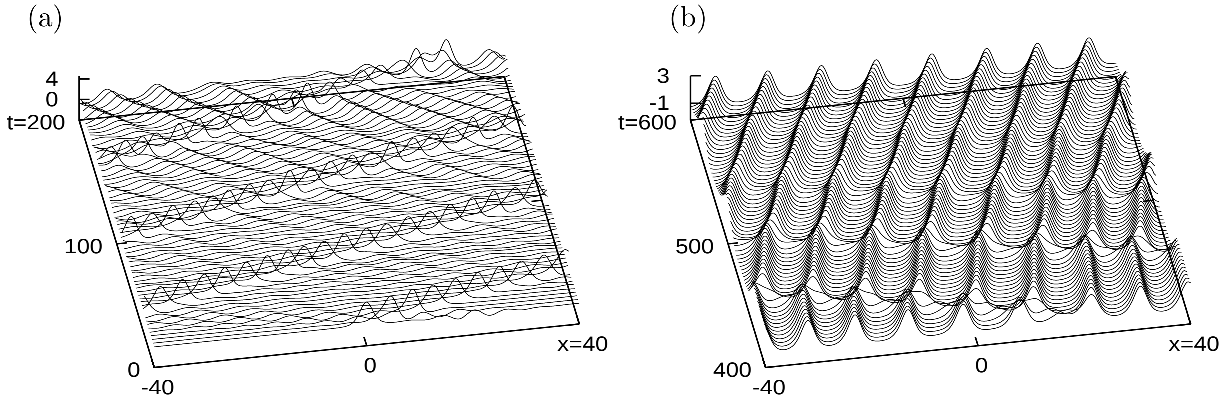

Figure 2: Numerical simulation of (1) for $\varepsilon = 0.2$ on a large domain with periodic boundary conditions: (a) illustration of long time stability of u^ε; (b) convergence towards the traveling pulse train. The initial condition in (a) is $u_{3/2}(x) + 0.8\sin(x)\mathrm{sech}(x/4-5)$. The pulse keeps its shape until $t \approx 200$, while the wave packet spreads and grows on the unstable background. In (b) the solution has converged to a pulse train with speed $c_0 \approx 0.3$ consisting of 8 copies of roughly $u^\varepsilon - c_1$ with $c_1 \approx 1.2$. Applying the boost $v(x + c_1 t, t) = u(x,t) + c_1$ we recover the expected speed $c = c_1 + c_0 \approx c_\varepsilon$.

finite-time stability of u^ε from another point of view, here we make explicit use of the first conserved quantities of the KdV. A similar approach was used in [EMR93]; there the dynamics on the attractor for the problem over a bounded domain with periodic boundary conditions is studied in terms of the perturbed dynamics of the action angle variables for the KdV over a bounded domain.

Here, over the unbounded domain, we prove results that may be paraphrased as orbital stability of KdV n-pulses (with arbitrary speed parameters c_j) on an $O(1/\varepsilon)$ time scale with respect to $O(1)$ perturbations in $H^n(\mathbb{R})$. For $n = 1$ the result is as follows.

Theorem 1.1 *Let $c_\star > 0$. For all $C_0, \delta_2 > 0$ there exist $\delta_1, T_0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let $\inf_{\phi\in\mathbb{R}} \|u_0(\cdot) - u_{c_\star}(\cdot + \phi)\|_{H^1} \leq \delta_1$, $\|u_0\|_{H^2} \leq C_0$, and let u be the solution of (1) with $u(x,0) = u_0(x)$. Then*

$$\sup_{t\in[0,T_0/\varepsilon]} \inf_{\phi\in\mathbb{R}} \|u(\cdot,t) - u_{c_\star}(\cdot + \phi)\|_{H^1} \leq \delta_2. \tag{2}$$

From Theorem 1.1 we may directly infer a result in the spirit of a stability statement.

Corollary 1.2 *For all $C_0, \delta_2 > 0$ there exist $\delta_1, T_0, C^\star, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. If $\inf_{\phi\in\mathbb{R}} \|u_0(\cdot) - u^\varepsilon(\cdot + \phi)\|_{H^1} \leq \delta_1$, $\|u_0\|_{H^2} \leq C_0$, then*

$$\sup_{t\in[0,T_0/\varepsilon]} \inf_{\phi\in\mathbb{R}} \|u(\cdot,t) - u^\varepsilon(\cdot + \phi)\|_{H^1} \leq \delta_2 + C^\star\varepsilon.$$

The proof of Theorem 1.1 is based on the orbital stability proof for KdV 1-solitons given in [Ben72, Bo75], i.e., the orbital stability of u_c in the case $\varepsilon = 0$. There it is shown that the Hamiltonian

$$H(u) = \int \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{6}u^3 \right) \mathrm{d}x$$

of the KdV equation has a line of minima along the orbit $\{\tau_\phi u_c : \phi \in \mathbb{R}\}$, $\tau_\phi u_c(\cdot) = u_c(\cdot + \phi)$, under the constraint

$$E(u) = \int \frac{1}{2} u^2 \,\mathrm{d}x = \text{const.}$$

In fact, this constraint yields the first part in the inequality

$$C_3 \inf_{\phi\in\mathbb{R}} \|u - \tau_\phi u_c\|_{H^1}^2 \le H(u) - H(u_c) \le C_4 \|u - u_c\|_{H^1}^2, \tag{3}$$

with $C_3, C_4 > 0$, which implies the orbital stability of a pulse u_c in the KdV-equation. Here we adapt this proof to (1) with $\varepsilon > 0$ by proving a priori estimates

$$H(u(t)) - H(u_0) + |E(u(t)) - E(u_0)| \le C\varepsilon t. \tag{4}$$

On the other hand, the mass

$$M(u) = \int u(x) \,\mathrm{d}x$$

is conserved also for $\varepsilon > 0$. The idea for using (3) and (4) to prove Theorem 1.1 is sketched in fig. 3, where M_1 symbolizes the one dimensional family of KdV 1-solitons obtained from varying c, and where $c(0), c(t) > 0$ are the unique numbers such that $E(u_{c(0)}) = E(u_0)$ and $E(u_{c(t)}) = E(u(t))$. Given ① $:= \inf_{\phi\in\mathbb{R}} \|u(0) - \tau_\phi u_{c_\star}\|_{H^1}$ we want

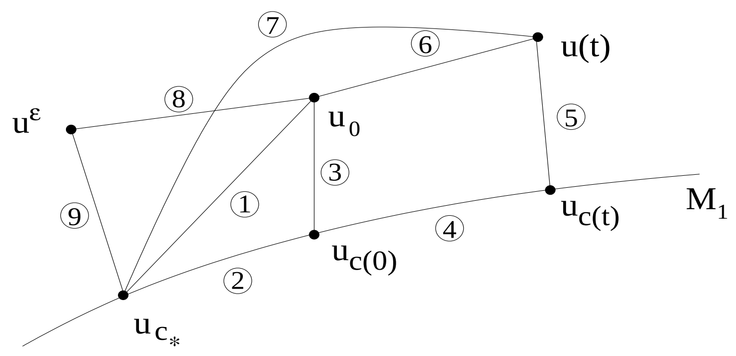

Figure 3: Scheme for estimating ⑦ $:= \inf_{\phi\in\mathbb{R}} \|u(t) - \tau_\phi u^\varepsilon\|_{H^1} \le$ ② + ④ + ⑤ in the proof of Theorem 1.1. For Corollary 1.2 we additionally assume $c_\star = c_\varepsilon$.

to estimate ⑦ $:= \inf_{\phi\in\mathbb{R}} \|u(t) - \tau_\phi u_{c_\star}\|_{H^1} \le$ ②+ ④+ ⑤ with appropriate norms on the right hand side. Estimates on ②, ③ and ④ follow from the explicit shape of u_c, while (4) yields an estimate on ⑥ in the sense given by the Hamiltonian. Combining this with (3) we then obtain an estimate on ⑤ $\leq$ ⑥ + ③ + ④ in the H^1 sense. If $c_\star = c_\varepsilon$, then the estimate in Corollary 1.2 follows from ⑨ $= O(\varepsilon)$ in H^1. In order to prove (4) we additionally need a priori estimates on the next integral H_2 of the unperturbed KdV equation.

Remark 1.3 Theorem 1.1 improves the local in time and space stability result from [PSU04, Theorem 5.1] in two directions: Theorem 1.1 is global and not only local in space; i.e., no weight in space is needed, and the allowed magnitude of the initial perturbations in Theorem 1.1 is $O(1)$ and not $O(\varepsilon)$ as in [PSU04]. ⌋

Remark 1.4 Besides the local-in-time stability stated in Theorem 1.1, also the local-in-space attractivity of the pulses proved in [CDK96, PSU04] helps to explain why the dynamics of (1) is dominated by essentially unstable pulses. Due to the fact that the local-in-space attractive two-dimensional structure found in [PSU04] is not invariant under the flow of (1) and does not lie in the phase space $H^1(\mathbb{R})$, the method of [PW94] cannot be applied directly. Therefore, the attractivity result [PSU04, Theorem 5.1] is not improved substantially using the a priori estimates from the proof of Theorem 1.1. However, the coefficients $\delta_v(0)$ and $\delta_w(0)$ from [PSU04, Theorem 5.1], which describe the magnitude of the initial perturbations in an unweighted and a weighted norm, can now be chosen up to order $O(1)$ in $H^1(\mathbb{R})$ instead of $O(\varepsilon)$, and $O(\varepsilon^2)$ in $H^n(\mathbb{R})$ for general n, respectively. ⌋

The orbital H^1-stability result for KdV 1-solitons has been generalized to H^n-stability for n-solitons in [MS93]. (Recently, higher-order H^m-stability of 1-solitons was studied in [BLN04].) For fixed $n \geq 2$, KdV n-solitons are given by a $2n$-parameter family of profiles $u^{(n)}(y; c_1, \ldots, c_n, \phi_1, \ldots, \phi_n)$. For instance, for $n = 2$ we have $u^{(2)} = 12\partial_y^2 \log(\tau^{(2)})$ where

$$\begin{aligned}\tau^{(2)} =& 1 + \exp(\sqrt{c_1}(y + \phi_1)) + \exp(\sqrt{c_2}(y + \phi_2)) \\ &+ \left(\frac{\sqrt{c_1} - \sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}}\right)^2 \exp(\sqrt{c_1}(y + \phi_1) + \sqrt{c_2}(y + \phi_2)).\end{aligned} \tag{5}$$

The time-dependent 2-soliton solution of KdV then is

$$u_{\vec{c}}(x, t; \vec{\phi}) = u^{(2)}(x; c_1, c_2, \phi_1 - c_1 t, \phi_2 - c_2 t).$$

Below we shall often omit the phases $\vec{\phi}$ when they are not important, for instance in the evaluation of conserved quantities like E and H.

There is an important difference for the notion of stability of the families of n-solitons for $n = 1$ and $n \geq 2$. For $n = 1$ and given c, the time orbit of a 1-soliton, or equivalently the orbit of its spatial translates, traverses the full family $M_1(c)$, while for $n \geq 2$ and given $\vec{c}$, the time orbit and the spatial translates only traverse (different) one-dimensional submanifolds of $M_n(\vec{c})$. Consequently, for $n \geq 2$ there is a somewhat different notion of orbital stability of KdV n-solitons, namely that solutions stay close to n-soliton profiles with given $\vec{c}$ but varying $\vec{\phi}$. For (1) with $\varepsilon > 0$ and $n = 2$ (see Remark 1.6) we then have:

Theorem 1.5 *Let $\vec{c}_\star \in \mathbb{R}^2_+$. For all $C_0, \delta_2 > 0$ there exist $\delta_1, T_0, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. Let $\|u_0(\cdot) - u^{(2)}(\cdot, \vec{c}_\star, \vec{\phi})\|_{H^2} \le \delta_1$ for some $\vec{\phi} \in \mathbb{R}^2$, $\|u_0\|_{H^3} \le C_0$, and let u be the solution of (1) with $u(x, 0) = u_0(x)$. Then*

$$\sup_{t\in[0,T_0/\varepsilon]} \inf_{\vec{\phi}\in\mathbb{R}^2} \|u(\cdot, t) - u^{(2)}(\cdot, \vec{c}_\star, \vec{\phi})\|_{H^2} \le \delta_2. \tag{6}$$

The proof of Theorem 1.5 uses the same idea as sketched for Theorem 1.1 in fig. 3, namely the fact [MS93] that 2-solitons are minimizers of the next integral

$$H_2(u) = \int \left(\frac{1}{2}(\partial_x^2 u)^2 - \frac{5}{6} u(\partial_x u)^2 + \frac{5}{72} u^4 \right) \mathrm{d}x \tag{7}$$

of the KdV equation under the constraints $E(u), H(u) = \text{const}$.

Remark 1.6 The generalization of Theorem 1.5 to $n \ge 2$ is true for all n, with constants independent of ε, but these constants depend on n. For instance, T_0 typically decreases with increasing n. Therefore, for large n the result will be more of theoretical interest, while for smaller n the $O(1/\varepsilon)$ time scale n-soliton dynamics can be well traced also in numerical simulation of (1), i.e., T_0 can be chosen rather large in (2) and (6). Moreover, for large n the computations become lengthy. Therefore, here we restrict to $n = 2$; further explications for the general case are given in sec. 2.3. ⌋

Theorem 1.5 itself does not imply that the solitons really interact, cf. the discussion in [MS93] for the unperturbed KdV equation. However, a soliton interaction that happens on an $O(1)$ time scale in the unperturbed KdV equation also occurs in the KS–perturbed KdV equation due to the following approximation theorem. Numerical illustrations of local-in-time 2-soliton dynamics in (1) are given in figures 4 and 5.

Theorem 1.7 *Fix an integer $s \ge 2$. For all $C_1, T_0 > 0$ there exist $\varepsilon_0, C_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds. For all solutions $v \in C([0, T_0], H^{s+4})$ of the KdV–equation $\partial_t v = -\partial_x^3 v - \frac{1}{2}\partial_x(v^2)$ satisfying $\sup_{t\in[0,T_0]} \|v(t)\|_{H^{s+4}} \le C_1$ there is a solution $u \in C([0, T_0], H^s)$ of (1) with*

$$\sup_{t\in[0,T_0]} \|u(t) - v(t)\|_{H^s} \le C_2 \varepsilon .$$

Proof. A solution u of (1) is a sum of the KdV solution v and an error function εR, i.e., $u = v + \varepsilon R$. We find

$$\partial_t R = -\partial_x^3 R - \partial_x(vR) - \frac{\varepsilon}{2}\partial_x(R^2) - \varepsilon(\partial_x^2 + \partial_x^4)R - (\partial_x^2 + \partial_x^4)v,$$

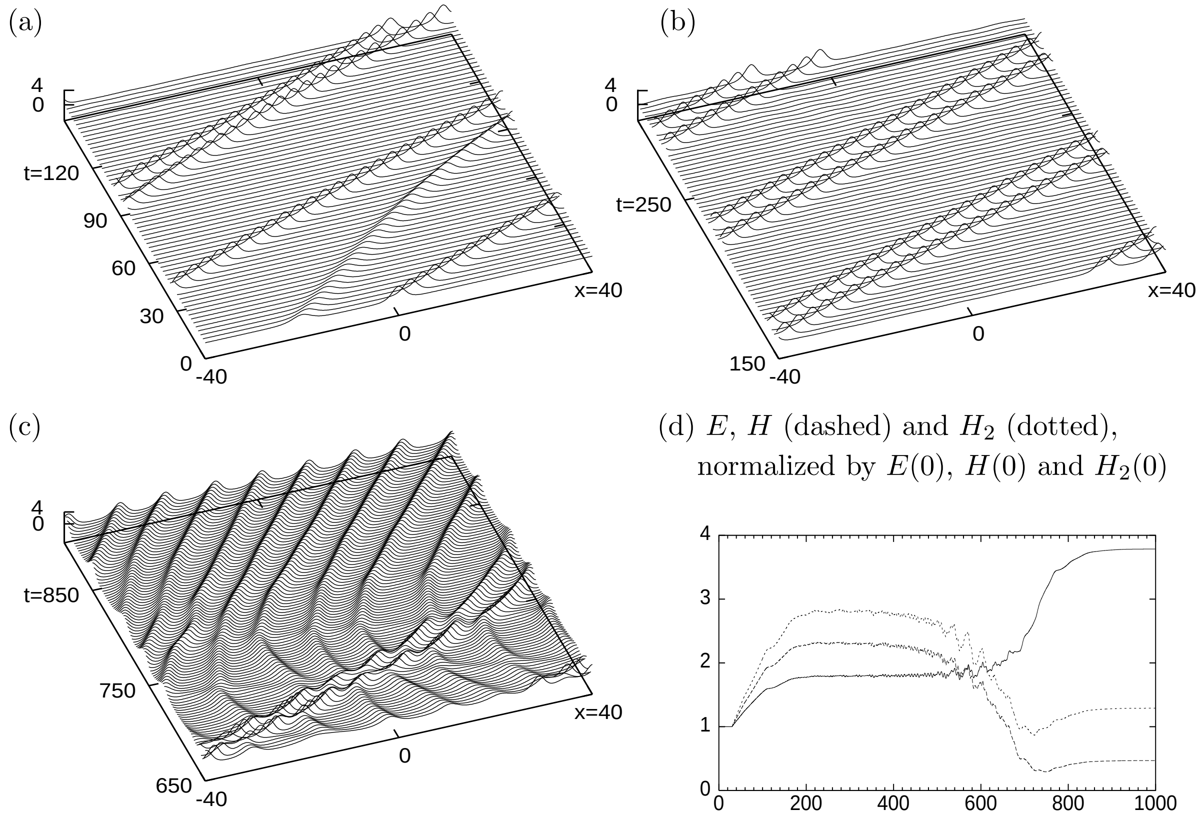

Figure 4: Illustration of local-in-time 2-soliton dynamics (and the convergence to the traveling pulse-train) in (1). The initial condition $u_0(x) = u_{c_1}(x+20) + u_{c_2}(x)$ with $c_1 = 0.6$ and $c_2 = 1.2$ is an approximation of a 2-soliton profile. First we set $\varepsilon = 0$ until $t = 30$ and then switch to $\varepsilon = 0.2$. Subsequently the slower pulse takes up mass and speeds up, i.e., $c_1(t)$ increases, while $c_2(t)$ roughly stays constant. At $t \approx 100$ the two pulses meet, but the interaction is *not* dominantly of KdV type. Instead, the slower pulse takes mass from the larger pulse and further speeds up. The two pulses then travel together for a long time (b), during which periodic waves grow on the unstable background. This again leads to a train of boosted copies of u^ε at large time (c). Panel (d) shows E, H and H_2 for this simulation, normalized by their initial values $E(0) = 21.35 \approx 12(c_1^{3/2} + c_2^{3/2})$, $H(0) = -13.37 \approx -\frac{36}{5}(c_1^{5/2} + c_2^{5/2})$ and $H_2(0) = 10.6 \approx \frac{36}{7}(c_1^{7/2} + c_2^{7/2})$. For $\varepsilon = 0$, these quantities are conserved well by the numerical scheme. The total mass is exactly conserved, also for $\varepsilon > 0$. Switching to $\varepsilon = 0.2$ at $t = 30$ we see a linear behavior of E, H, H_2 up to $t \approx 100$. At $t \approx 200$ a plateau is reached which corresponds to the two pulses traveling together in (b). For $t > 300$ the growing periodic waves can be seen in E, H and H_2, leading to the transition to the traveling pulse train for $t > 800$, where E, H and H_2 become constant again. However, in the present paper we are only concerned with the time interval $0 \le \tilde{t} \le t_0/\varepsilon$, $\tilde{t} = t - 30$, during which E, H and H_2 in (d) show linear growth. Running the simulation with different ε shows that this time interval indeed scales with $1/\varepsilon$. This figure and figures 2 and 5 have been produced using 512 spatial points and a split-step method: the KdV part $\partial_t u = -\partial_x^3 u - \frac{1}{2}\partial_x(u^2)$ has been integrated using finite difference and an explicit leap-frog scheme [ZK65], while for the dissipative part $\partial_t u = -\varepsilon(\partial_x^2 + \partial_x^4)u$ we used an implicit spectral method.

which via partial integration implies

$$\frac{1}{2}\partial_t \int (\partial_x^s R)^2 \,\mathrm{d}x \;=\; -\int (\partial_x^s R)\partial_x^{s+1}(vR)\,\mathrm{d}x - \frac{\varepsilon}{2}\int (\partial_x^s R)\partial_x^{s+1}(R^2)\,\mathrm{d}x$$
$$+\int \varepsilon((\partial_x^{s+1}R)^2 - (\partial_x^{s+2}R)^2)\,\mathrm{d}x - \int (\partial_x^s R)\partial_x^s(\partial_x^2 + \partial_x^4)v\,\mathrm{d}x\;.$$

Next

$$\int (\partial_x^s R)\partial_x^{s+1}(vR)\,\mathrm{d}x = -\frac{1}{2}\int (\partial_x^s R)^2(\partial_x v)\,\mathrm{d}x + O(\|v\|_{H^{s+1}}\|R\|_{H^s}^2)\;,$$
$$\int (\partial_x^s R)\partial_x^{s+1}(R^2)\,\mathrm{d}x = -\int (\partial_x^s R)^2(\partial_x R)\,\mathrm{d}x + O(\|R\|_{H^s}^3)\;,$$
$$\int (\partial_x^{s+1}R)^2 - (\partial_x^{s+2}R)^2\,\mathrm{d}x \le \frac{1}{4}\,\|\partial_x^s R\|_{L^2}^2.$$

Thus the Cauchy–Schwarz inequality yields

$$\partial_t(\|R\|_{H^s}^2) \le C(\|R\|_{H^s}^2 + \varepsilon\|R\|_{H^s}^3 + C_1^2)$$

with a constant C independent of $0 < \varepsilon \ll 1$. For all $t \ge 0$, as long as $\varepsilon\|R(t)\|_{H^s}^3 \le 1$, Gronwall's inequality implies

$$\sup_{t\in[0,T_0]} \|R(t)\|_{H^s} \le C(1 + C_1^2)T_0\, e^{CT_0} =: \tilde{C}\;.$$

We are done by choosing $\varepsilon > 0$ so small that $\varepsilon\tilde{C}^3 \le 1$. □

Remark 1.8 The phenomena explained in this paper occurs at a time of order $O(1/\varepsilon)$ which is beyond the $O(1)$ time interval of validity of (1) for the inclined-film problem. Except for special limits, (1) only serves as a phenomenological model for going beyond the pure KdV dynamics valid on the $O(1)$-time interval. ⌋

2 The proofs

2.1 A priori estimates

Let $C_u = \hat{C}C_0$ with $\hat{C} > 0$ chosen below. First we prove that there is a $T_0 > 0$ independent of $0 < \varepsilon \ll 1$ such that

$$\sup_{t\in[0,T_0/\varepsilon]} \|u(t)\|_{H^2} \le C_u. \tag{8}$$

In order to do so we prove upper bounds for the time derivatives of the first three integrals of the unperturbed KdV equation, using the convention that $H_0(u){=}E(u)$. The estimates are obtained in such a way that for the j-th integral we only use estimates

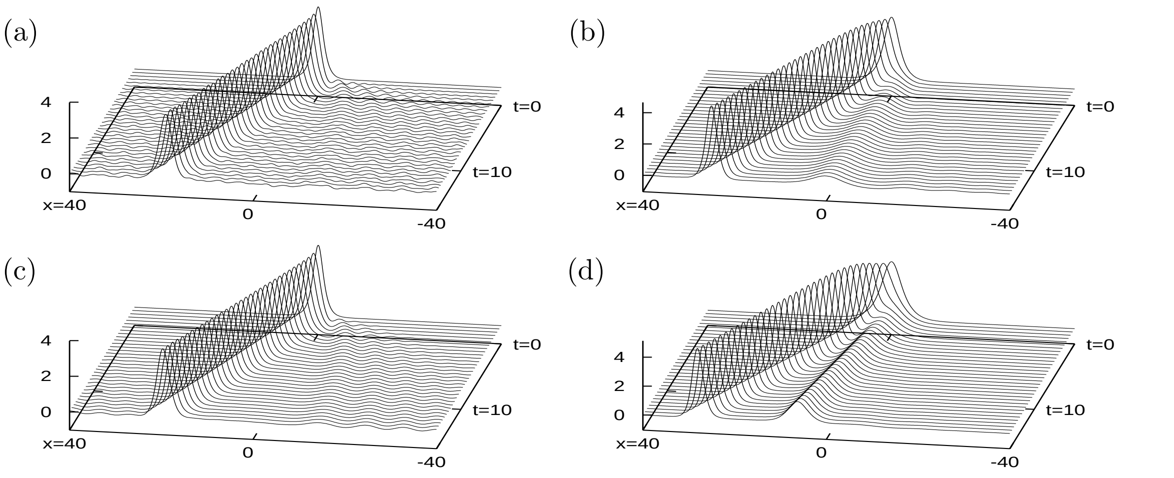

Figure 5: On the $O(1)$ time-scale, the KdV dynamics explain different possible behaviors of, for instance, similar pulses with different masses $M(u) = \int u(x)\,\mathrm{d}x$. Here $\varepsilon = 0$ in (a,b) and $\varepsilon = 0.2$ in (c,d), and initial conditions are $u_0(x) = 4\mathrm{sech}(x/a)$, with $a = 1$ ($M = 4\pi$) in (a,c) and $a = 1.5$ ($M = 6\pi$) in (b,d). In (a) this leads to a KdV 1-soliton and a dispersive tail (which re-enters the domain at $x = 40$ near $t = 4$ due to the periodic boundary conditions), while the higher mass in (b) gives a KdV 2-soliton (and a small dispersive tail). Consequently, this also yields two qualitatively different evolutions for $\varepsilon > 0$, i.e., two different ways for the pulse to "drain excess mass" [CDK98].

for derivatives $\partial_x^k u$ with $0 \le k \le j$. In sec. 2.2 the estimate (8) is then used to additionally prove a bound on $|\frac{\mathrm{d}}{\mathrm{d}t}E(u)|$ which yield the estimate (4) for the proof of Theorem 1.1. Similarly, to prove Theorem 1.5 we first show upper bounds on $\frac{\mathrm{d}}{\mathrm{d}t}H_3(u)$ (the 4^{th} integral), to obtain $\sup_{t\in[0,T_0/\varepsilon]} \|u(t)\|_{H^3} \le C_u$ for some $C_u = \hat{C}C_0$.

We start with

$$E(u) = H_0(u) = \int \frac{1}{2}u^2\,\mathrm{d}x.$$

Implicitly exploiting that $\frac{\mathrm{d}}{\mathrm{d}t}E(u) = 0$ for $\varepsilon = 0$, by Parseval's identity we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}u^2\,\mathrm{d}x &= \int u\partial_t u\,\mathrm{d}x = \int u\left(-\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 u + \partial_x^4 u)\right)\mathrm{d}x \\
&= \varepsilon\int \left((\partial_x u)^2 - (\partial_x^2 u)^2\right)\mathrm{d}x = 2\pi\varepsilon\int (k^2 - k^4)|\hat{u}|^2\,\mathrm{d}k \\
&\le 2\pi\varepsilon\int \frac{1}{4}|\hat{u}(k)|^2\,\mathrm{d}k = \frac{\varepsilon}{4}\int u^2\,\mathrm{d}x.
\end{aligned}$$

For

$$H(u) = H_1(u) = \int \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{6}u^3\right)\mathrm{d}x.$$

we find, using $\frac{\mathrm{d}}{\mathrm{d}t}H(u) = 0$ for $\varepsilon = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{6}u^3\right)\,\mathrm{d}x = \int \left((\partial_x u)(\partial_x\partial_t u) - \frac{1}{2}u^2\partial_t u\right)\,\mathrm{d}x$$
$$= \int \left((\partial_x u)\partial_x\left(-\varepsilon(\partial_x^2{+}\partial_x^4)u\right) - \frac{1}{2}u^2\left(-\varepsilon(\partial_x^2{+}\partial_x^4)u\right)\right)\,\mathrm{d}x = \varepsilon(s_0 + s_1 + s_2)$$

with

$$s_0 = \int (\partial_x^2 u)^2 - (\partial_x^3 u)^2\,\mathrm{d}x, \quad s_1 = \int \frac{1}{2}u^2(\partial_x^2 u)\,\mathrm{d}x, \quad s_2 = \int \frac{1}{2}u^2(\partial_x^4 u)\,\mathrm{d}x.$$

Presuming $\|u(t)\|_{H^1} \le C_u$ for the t under consideration shows $|s_1| = |-\int u(\partial_x u)^2\,\mathrm{d}x| \le C_u^3$. Moreover, using $|ab| \le \frac{1}{2}(\eta a^2 + \eta^{-1}b^2)$, $\eta > 0$, we obtain

$$|s_2| = \left|-\int u(\partial_x u)(\partial_x^3 u)\,\mathrm{d}x\right| \le C_u\left|\int \left(\eta^{-1}(\partial_x u)^2 + \eta(\partial_x^3 u)^2\right)\,\mathrm{d}x\right| \le C_\delta + \delta\|\partial_x^3 u\|_{L^2}^2$$

with a constant $C_\delta \to \infty$ for $\delta \to 0$. Choosing $\delta = 1/2$ and estimating $k^4 - k^6/2 \le C$ with a constant C independent of k as in the estimate for $\frac{\mathrm{d}}{\mathrm{d}t}E$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{6}u^3\right)\,\mathrm{d}x &\le \varepsilon\int \left((\partial_x^2 u)^2 - \frac{1}{2}(\partial_x^3 u)^2\right)\,\mathrm{d}x + \varepsilon(C_{1/2} + C_u^3)\\ &= 2\pi\varepsilon\int (k^4 - \frac{1}{2}k^6)|\hat{u}(k)|^2\,\mathrm{d}k + \varepsilon(C_{1/2} + C_u^3) \le \varepsilon(C\|u\|_{L^2}^2 + C_{1/2} + C_u^3)\\ &\le C\varepsilon \end{aligned} \tag{9}$$

for a $C > 0$.

Next we consider

$$H_2(u) = \int \left(\frac{1}{2}(\partial_x^2 u)^2 - \frac{5}{6}u(\partial_x u)^2 + \frac{5}{72}u^4\right)\,\mathrm{d}x$$

and t such that $\|u(t)\|_{H^2} \le C_u$. We have, using $\frac{\mathrm{d}}{\mathrm{d}t}H_2 = 0$ for $\varepsilon = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) = \int \partial_t u\left(\partial_x^4 u - \frac{5}{6}(\partial_x u)^2 + \frac{5}{6}\partial_x^2(u^2) + \frac{5}{18}u^3\right)\,\mathrm{d}x = \varepsilon(s_0 + s_1 + s_2 + s_3)$$

with $s_0 = \int (\partial_x^3 u)^2 - (\partial_x^4 u)^2\,\mathrm{d}x$ and

$$\begin{aligned}|s_1| &= \left|\int (\partial_x^2 u)\left(\frac{5}{18}u^3 + \frac{5}{18}\partial_x^2(u^3)\right)\,\mathrm{d}x\right| \le 5C_u^4,\\ |s_2| &= \frac{5}{6}\left|\int (\partial_x^2 u + \partial_x^4 u)\partial_x^2(u^2)\,\mathrm{d}x\right|\\ &\le 2C_u^3 + \frac{5}{12}\int \left(\eta(\partial_x^4 u)^2 + \eta^{-1}(\partial_x^2(u^2))^2\right)\,\mathrm{d}x \le \alpha\|\partial_x^4 u\|_{L^2}^2 + C_\alpha,\\ |s_3| &= \frac{5}{6}\left|\int \left((\partial_x u)^2\partial_x^2 u - (\partial_x(\partial_x u)^2)(\partial_x^3 u)\right)\,\mathrm{d}x\right|\\ &\le C_u^3 + \frac{5}{12}\int \left(\eta(\partial_x^3 u)^2 + \eta^{-1}(\partial_x(\partial_x u)^2)^2\right)\,\mathrm{d}x \le \beta\|\partial_x^3 u\|_{L^2}^2 + C_\beta.\end{aligned}$$

Thus, choosing $\alpha = \beta = 1/2$ and estimating $\frac{3}{2}k^6 - \frac{1}{2}k^8 \le C$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_2(t) \le \varepsilon \int \left((\partial_x^3 u)^2 (1 + \frac{1}{2}) - (\partial_x^4 u)^2 (1 - \frac{1}{2}) \right) \mathrm{d}x + \varepsilon C C_u^4 \le C\varepsilon. \qquad (10)$$

Therefore, provided that $\|u(t)\|_{H^2} \le C_u$ we found a constant $C_5 = C_5(C_u)$ such that for all $T_0 > 0$, all $\varepsilon \in (0,1)$, all $t \in [0, T_0/\varepsilon]$ and $j = 0,1,2$ we have $\frac{\mathrm{d}}{\mathrm{d}t} H_j(u) \le C\varepsilon$ i.e., $H_j(u(t)) \le H_j(u(0)) + C_5 \varepsilon t$. To close the argument we define

$$F_1(t) = 2\big[H_0(u(t)) + H_1(u(t))\big] + \frac{2}{9} H_0^2(u(t)),$$
$$F_2(t) = 2\big[H_0(u(t)) + H_1(u(t)) + H_2(u(t))\big] + \frac{5}{3} F_1(t)^{3/2}.$$

Then $\|u(t)\|_{H^j}^2 \le F_j(t)$, $j = 1,2$, and, as long as $\|u(t)\|_{H^2} \le C_u$, $\frac{\mathrm{d}}{\mathrm{d}t} F_j \le C C_5 \varepsilon t$. In particular

$$\|u(t)\|_{H^2}^2 \le F_2(t) \le F_2(0) + C_6 \varepsilon t \le 2F_2(0)$$

for all $t \in [0, T_0/\varepsilon]$, for sufficiently small $T_0 > 0$. Since also $F_2(0) \le C C_0$ for all u_0 with $\|u_0\|_{H^2} \le C_0$ this implies (8), i.e., $\sup_{t\in[0,T_0/\varepsilon]} \|u(t)\|_{H^2} \le C_u = \hat{C} C_0$ for some $\hat{C} > 0$.

For the proof of Theorem 1.5 (the 2–soliton case) we also need to bound $\|u(t)\|_{H^3}$. Therefore we let $\|u_0\|_{H^3} \le C_0$, $C_u = \hat{C} C_0$ for some $\hat{C} > 0$ chosen below, and additionally estimate $\frac{\mathrm{d}}{\mathrm{d}t} H_3(u)$ with

$$H_3(u) = \int \left(\frac{1}{2} (\partial_x^3 u)^2 - \frac{7}{6} u (\partial_x^2 u)^2 + \frac{35}{36} u^2 (\partial_x u)^2 - \frac{7}{216} u^5 \right) \mathrm{d}x.$$

Exactly as above, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} H_3(t) \le C\varepsilon,$$

with $C = O(C_u^5)$, as long as $\|u(t)\|_{H^3} \le C_u$. Defining, for instance,

$$F_3(t) = 2\big[H_0(t) + H_1(t) + H_2(t) + H_3(t)\big] + \frac{5}{3} F_1(t)^{1/2} F_2(t) + \frac{7}{216} F_1(t)^{5/2},$$

we obtain, with some $\hat{C} > 0$,

$$\|u(t)\|_{H^3}^2 \le F_3(t) \le F_3(0) + C_6 \varepsilon t \le 2F_1(0) \le \hat{C} C_0. \qquad (11)$$

for all $t \in [0, T_0/\varepsilon]$, for sufficiently small $T_0 > 0$, as long as $\sup_{0\le\tau\le t} \|u(\tau)\|_{H^3} \le C_u$. This again yields a $T_0 > 0$ independent of $0 < \varepsilon \ll 1$ with

$$\sup_{t\in[0,T_0/\varepsilon]} \|u(t)\|_{H^3} \le C_u. \qquad (12)$$

The same estimates are possible for all integrals H_j of the unperturbed KdV equation with $j \in \mathbb{N}$ since H_j is quadratic in the highest derivative $\partial_x^j u$. However, as already indicated, for the j^{th} integrals the relevant constant $C(C_u) = O(C_u^{j+2})$ grows faster for larger C_u. Therefore, and also to keep notations and computations to a reasonable level, we restrict to the case $n = 2$ in Theorem 1.5, cf. Remark 1.6.

2.2 Near a 1-soliton

Like above, but now using (8), we first have the upper *and lower* a priori bound

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}u^2\,\mathrm{d}x\right| = \left|\int u\partial_t u\,\mathrm{d}x\right| = \left|\int u\left(-\partial_x^3 u - \frac{1}{2}\partial_x(u^2) - \varepsilon(\partial_x^2 u + \partial_x^4 u)\right)\,\mathrm{d}x\right|$$
$$= \varepsilon\left|\int \left((\partial_x u)^2 - (\partial_x^2 u)^2\right)\,\mathrm{d}x\right| \leq \varepsilon C_u^2. \tag{13}$$

Combining (13) with the upper bound (9) for $\frac{\mathrm{d}}{\mathrm{d}t}H$ we have a constant $C_5 = C_5(C_u)$ such that for all $T_0 > 0$, all $\varepsilon \in (0,1)$, and all $t \in [0, T_0/\varepsilon]$ we have $\left|\frac{\mathrm{d}}{\mathrm{d}t}E(u)\right| \leq C_5\varepsilon$ and $\frac{\mathrm{d}}{\mathrm{d}t}H(u) \leq C_5\varepsilon$, i.e.,

$$|E(u(t)) - E(u(0))| \leq C_5\varepsilon t \quad \text{and} \quad H(u(t)) - H(u(0)) \leq C_5\varepsilon t. \tag{14}$$

Next we use a bootstrap-type argument to estimate ②, ③ and ④ in fig. 3, first in L^2 and then in H^1. Since $E(u_c) = 12c^{3/2}$, to each $E = E(u(t)) > 0$ there corresponds exactly one $c = c(t)$ with $E(u(t)) = E(u_{c(t)})$. In the following we assume (without loss of generality) that $\inf_\phi \|u_0 - \tau_\phi u_{c_\star}\|_{H^1} = \|u_0 - u_{c_\star}\|_{H^1}$, i.e., that at $t = 0$ the infimum is attained at $\phi = 0$. Then

$$|c(0)^{3/2} - {c_\star}^{3/2}| = \frac{1}{12}|E(u_0) - E(u_{c_\star})| = \frac{1}{24}\left|\int \left(u_0^2 - u_{c_\star}^2\right)\,\mathrm{d}x\right|$$
$$= \frac{1}{24}\left|\int (u_0 + u_{c_\star})(u_0 - u_{c_\star})\,\mathrm{d}x\right| \leq C(u_{c_\star})\|u_0 - u_{c_\star}\|_{L^2} \leq C\delta_1. \tag{15}$$

Therefore $|c(0) - c_\star| \leq C\delta_1$, thus $\|u_{c(0)} - u_{c_\star}\|_{H^1} \leq C\delta_1$, $\inf_\phi \|u_0 - \tau_\phi u_{c(0)}\|_{H^1} \leq C\delta_1$, and finally, using (3), $|H(u_0) - H(u_{c(0)})| \leq C\delta_1^2$. Similarly, using

$$|E(u_{c(t)}) - E(u_{c(0)})| = |E(u(t)) - E(u(0))| \leq C_5\varepsilon t$$

we have $|c(t) - c(0)| \leq C\varepsilon t$, thus

$$\|u_{c(t)} - u_{c(0)}\|_{H^1} \leq C\varepsilon t \quad \text{and} \quad |H(u_{c(t)}) - H(u_{c(0)})| \leq (C\varepsilon t)^2,$$

due to (3). Therefore, using (3) again, (14) and the inequalities above, we may estimate ⑤ in fig. 3 as

$$C_3 \inf_{\phi\in\mathbb{R}} \|u(t) - \tau_\phi u_{c(t)}\|_{H^1}^2 \leq H(u(t)) - H(u_{c(t)})$$
$$\leq (H(u(t)) - H(u(0))) + (H(u(0)) - H(u_{c(0)})) + (H(u_{c(0)}) - H(u_{c(t)}))$$
$$\leq C_5\varepsilon t + C\delta_1^2 + C(\varepsilon t)^2. \tag{16}$$

We introduce the deviation v from the orbit $\{\tau_\phi u_{c_\star} : \phi \in \mathbb{R}\}$ by $u = \tau_\phi u_{c_\star} + v$ with $\|v(t)\|_{H^1} = \inf_{\phi\in\mathbb{R}} \|u(t) - \tau_\phi u_{c_\star}\|_{H^1}$. Then, by (16),

$$\|v(t)\|_{H^1} = \inf_{\phi_1,\phi_2\in\mathbb{R}} \|u(t) - \tau_{\phi_2}u_{c(t)} + \tau_{\phi_2}u_{c(t)} - \tau_{\phi_1}u_{c_\star}\|_{H^1}$$
$$\leq \inf_\phi \|u(t) - \tau_\phi u_{c(t)}\|_{H^1} + \inf_\phi \|u_{c(t)} - \tau_\phi u_{c_\star}\|_{H^1}$$
$$\leq C_3^{-1/2}(C_5\varepsilon t + C\delta_1^2 + C(\varepsilon t)^2)^{1/2} + C\delta_1 + C\varepsilon t$$

where the terms $C\delta_1 + C\varepsilon t$ correspond to ② + ④ in fig. 3. Therefore $\|v(t)\|_{H^1} \leq \delta_2$ by choosing $\varepsilon_0 > 0$ sufficiently small, and δ_1, T_0 sufficiently small but independent of $\varepsilon \in (0, \varepsilon_0)$. This completes the proof of Theorem 1.1. □

2.3 Near a 2-soliton

To generalize Theorem 1.1 to n-soliton dynamics we want to use the fact that the n-soliton profiles $u^{(n)}(\cdot; \vec{c})$ minimize the n^{th}integral $H_n(u)$ under the n constraints $H_j(u) = H_j(u^{(n)}(\cdot, \vec{c}))$, j=0,…,n−1. Indeed, from the results of [MS93],

$$C_3^{(n)} \inf_{\vec{\phi}\in\mathbb{R}^n} \|u - u^{(n)}(\cdot; \vec{c}, \vec{\phi})\|^2_{H^n} \leq H_n(u) - H_n(u^{(n)}(\cdot; \vec{c})) \tag{17}$$

under these constraints. Therefore we need to generalize two steps: first, under the assumption that $\|u(t)\|_{H^{n+1}} \leq C_u$ we need a priori estimates for the first $n+1$ integrals of the KdV, i.e., up to H_n, in the sense of (14). Second, given $\vec{a} = (E(u), \dots, H_{n-1}(u)) \in \mathbb{R}^n$ we need to calculate $\vec{c}$ from

$$f(\vec{c}) := (E(u_{\vec{c}}), H(u_{\vec{c}}), \dots, H_{n-1}(u_{\vec{c}})) = \vec{a} \tag{18}$$

and, moreover, control $|\vec{c} - \vec{b}|_{\mathbb{R}^n}$ in terms of $|f(\vec{c}) - f(\vec{b})|_{\mathbb{R}^n}$ and control

$$\inf_{\vec{\phi}\in\mathbb{R}^n} \|u^{(n)}(\cdot; \vec{c}, \vec{\phi}) - u^{(n)}(\cdot; \vec{b}, \vec{\phi}_0)\|_{H^n}$$

in terms of $|\vec{c} - \vec{b}|_{\mathbb{R}^n}$, where $\vec{\phi}_0$ is arbitrary and is only introduced for notational consistence. In principle these two steps are possible for all n: step 1 since the n^{th}integral is quadratic in the highest derivative $\partial_x^n u$, cf. sec. 2.1, and step 2 since for all n we have the explicit formula

$$H_j(u^{(n)}) = \frac{36(-1)^j}{2j+3} \sum_{i=1}^{n} c_i^{(2j+3)/2}, \tag{19}$$

obtained from taking the limits $|\phi_m - \phi_i| \to \infty$, $i, m = 1, \dots, n$. However, as already said (sec. 2.1 and Remark 1.6) the relevant constants (in both steps) become large for large n which is why we restrict to $n = 2$.

Thus we write $u_{\vec{c}}(\cdot; \vec{\phi}) = u^{(2)}(\cdot; \vec{c}, \vec{\phi})$, and continue omitting $\vec{\phi}$ where it is not important, i.e., in the evaluation of E and H. As in sec. 2.2 but now using (12) we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t} H_1(u)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t} \int \left(\frac{1}{2}(\partial_x u)^2 - \frac{1}{6}u^3\right) \mathrm{d}x\right| \leq \varepsilon \left|\int \left((\partial_x^2 u)^2 - \frac{1}{2}(\partial_x^3 u)^2\right) \mathrm{d}x + s_1 + s_2\right| \\ &\leq \varepsilon(C_u^2 + CC_u^3).\end{aligned}$$

Hence, in addition to (14) we have, with a new C_5,

$$|H_1(u(t)) - H_1(u_0))| \leq C_5 \varepsilon t. \tag{20}$$

Using (19), the equation (18) for $\vec{c}(t)$ is

$$E(u_{\vec{c}(t)}) = 12(c_1^{3/2} + c_2^{3/2}) = E(u(t)), \quad H(u_{\vec{c}(t)}) = -\frac{36}{5}(c_1^{5/2} + c_2^{5/2}) = H(u(t)).$$

This yields a unique solution $\vec{c}(t)$ as long as $H(u(t)) < 0$, which is guaranteed by (20) for $t \le t_0 = T_0/\varepsilon$ for sufficiently small T_0. Next, from

$$\begin{aligned} 12\left[c_1(t)^{3/2} + c_2(t)^{3/2} - (c_1(0)^{3/2} + c_2(0)^{3/2})\right] &= E(u(t)) - E(u(0)) = O(\varepsilon t), \\ -\frac{36}{5}\left[c_1(t)^{5/2} + c_2(t)^{5/2} - (c_1(0)^{5/2} + c_2(0)^{5/2})\right] &= H(u(t)) - H(u(0)) = O(\varepsilon t), \end{aligned}$$

we obtain $|\vec{c}(t) - \vec{c}(0)|_{\mathbb{R}^2} \le C\varepsilon t$, hence $\inf_{\vec{\phi}} \|u_{\vec{c}(t)}(\cdot;\vec{\phi}) - u_{\vec{c}(0)}(\cdot;\vec{\phi}_0)\| \le C\varepsilon t$. Similarly, $|\vec{c}_\star - \vec{c}(0)|_{\mathbb{R}^2} \le C\delta_1$ by estimates as in (15), hence $\inf_{\vec{\phi}} \|u_{\vec{c}(0)}(\cdot;\vec{\phi}) - u_{\vec{c}_\star}(\cdot;\vec{\phi}_0)\| \le C\delta_1$, and consequently, using (10) and (17),

$$\begin{aligned} C_3^{(2)} \inf_{\vec{\phi}} \|u(t) - u_{\vec{c}(t)}(\cdot;\vec{\phi})\|_{H^2}^2 &\le H_2(u(t)) - H_2(u_{\vec{c}(t)}) \\ &\le (H_2(u(t)) - H_2(u(0))) + (H_2(u(0)) - H_2(u_{\vec{c}(0)})) + (H_2(u_{\vec{c}(0)}) - H_2(u_{\vec{c}(t)})) \\ &\le C_5\varepsilon t + C\delta_1^2 + C(\varepsilon t)^2. \end{aligned} \tag{21}$$

as in (16).

The remainder of the proof of Theorem 1.5 now works as the proof of Theorem 1.1. Let $u = u_{\vec{c}_\star} + v$ with $\|v(t)\|_{H^2} = \inf_{\vec{\phi}\in\mathbb{R}^2} \|u(\cdot,t) - u_{\vec{c}_\star}(\cdot;\vec{\phi})\|_{H^2}$. Then

$$\begin{aligned} \|v(t)\|_{H^2} &= \inf_{\vec{\phi},\vec{\psi}\in\mathbb{R}^2} \|u(t) - u_{\vec{c}(t)}(\cdot;\vec{\phi}) + u_{\vec{c}(t)}(\cdot;\vec{\phi}) - u_{\vec{c}_\star}(\cdot,\vec{\psi})\|_{H^2} \\ &\le \inf_{\vec{\phi}} \|u(t) - u_{\vec{c}(t)}(\cdot;\vec{\phi})\|_{H^2} + \inf_{\vec{\psi}} \|u_{c(t)}(\cdot;\vec{\psi}) - u_{\vec{c}_\star}(\cdot;\vec{\phi}_0)\|_{H^1} \\ &\le \frac{1}{\sqrt{C_3^{(2)}}}(C_5\varepsilon t + C\delta_1^2 + C(\varepsilon t)^2)^{1/2} + C\delta_1 + C\varepsilon t. \end{aligned}$$

Therefore, choosing again $\varepsilon_0 > 0$ sufficiently small, and δ_1, T_0 sufficiently small but independent of $\varepsilon \in (0, \varepsilon_0)$, the proof of Theorem 1.5 is complete. □

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