On the interaction of NLS-described wave packets with different carrier waves

Martina Chirilus-Bruckner, Christopher Chong, Guido Schneider

Institut für Analysis, Dynamik und Modellierung,
Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart

Hannes Uecker

Institut für Mathematik,
Universität Oldenburg, D-26111 Oldenburg

December 14, 2007

Abstract

We give a detailed analysis of the interaction of two NLS-described wave packets with different carrier waves for a nonlinear wave equation. By separating the internal dynamics of each wave packet from the dynamics caused by the interaction we prove that there is almost no interaction of such wave packets. We also prove the validity of a formula for the envelope shift caused by the interaction of well-prepared pulses and invalidate this formula by numerical experiments in case of non-well prepared pulses.

Contents

1 Introduction

We consider the nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u + u^3 \tag{1}$$

with $t \geq 0, x \in \mathbb{R}$, and $u = u(x,t) \in \mathbb{R}$. For this equation the ansatz

$$u(x,t) = \varepsilon A(X,T) e^{i(kx - \omega t)} + \text{c.c.} + O(\varepsilon^2), \quad X = \varepsilon(x - ct), \quad T = \varepsilon^2 t, \tag{2}$$

where $k, \omega \in \mathbb{R}$ satisfy the linear dispersion relation $\omega^2 = k^2 + 1$, where $c = d\omega/dk = k/\omega$ is the linear group velocity, and where $0 < \varepsilon \ll 1$ is a small perturbation parameter, leads to the Nonlinear Schrödinger (NLS) equation

$$2i\omega \partial_T A = (1 - c^2)\partial_X^2 A + 3|A|^2 A$$

describing slow modulations in time and space of the underlying carrier wave $e^{i(kx - \omega t)}$. This procedure is common in nonlinear optics and allows to reduce the dimension of the problem in numerical simulations by a factor up to 10^5, cf. [Ag01]. In modern fiber optics, however, not only a single carrier wave, but a number of different carrier waves is used, cf. [HK95].

In the particular case of two different carrier waves, i.e., $k_A \neq k_B$, the ansatz is given by

$$u(x,t) = \varepsilon A\big(\varepsilon(x - c_A t), \varepsilon^2 t\big) e^{i(k_A x - \omega_A t)} + \varepsilon B\big(\varepsilon(x - c_B t), \varepsilon^2 t\big) e^{i(k_B x - \omega_B t)} + \text{c.c.} + O(\varepsilon^2),$$

leading to a system of coupled NLS equations

$$\begin{aligned} 2i\omega_A \partial_T A &= (1 - c_A^2)\partial_{X_A}^2 A + 3A|A|^2 + 6A|B|^2, \\ 2i\omega_B \partial_T B &= (1 - c_B^2)\partial_{X_B}^2 B + 3B|B|^2 + 6B|A|^2. \end{aligned}$$

Since $X_A = \varepsilon(x - c_A t) = \varepsilon(x - c_B t) - \varepsilon(c_A - c_B)t = X_B - \frac{c_A - c_B}{\varepsilon} T$ and since the group velocities $c_A \neq c_B$ of the wave packets are different, this system has still the multiple scale character of the original problem. However, the interaction of localized wave packets will only happen on a very short time scale, such that asymptotically the interaction terms

$$6A(X_A, T)|B(X_B, T)|^2 = 6A(X_A, T)\ \overline{B}\ X_A - \frac{(c_B - c_A)}{\varepsilon} T, T\Big)^2$$

and

$$6B(X_B, T)|A(X_A, T)|^2 = 6B(X_B, T)\ \overline{A}\ X_B - \frac{(c_A - c_B)}{\varepsilon} T, T\Big)^2$$

are negligible. As a consequence, in lowest order we have a system of uncoupled NLS equations

$$\begin{aligned} 2i\omega_A \partial_T A &= (1 - c_A^2)\partial_{X_A}^2 A + 3A|A|^2, \\ 2i\omega_B \partial_T B &= (1 - c_B^2)\partial_{X_B}^2 B + 3B|B|^2, \end{aligned}$$

or, in other words, each band is described independently by a single NLS equation.

In applications the neglection of the coupling terms is a common procedure, cf [Ag01]. There exist a number of mathematical papers [PW96, BF06, CBS07] which validate this

procedure rigorously. Our research is dedicated to an improvement of existing estimates for wave interaction aiming towards applications in optical communication lines which use wavelength division multiplexing technologies, cf.[HK95].

In our previous work [CBSU07] we presented improved bounds for two waves modulated by NLS 1-solitons (in the following called well-prepared pulses, see Figure 1). Here, we further extend our results to waves whose envelopes are general localized profiles evolving according to the NLS equation (in the following called non-well prepared pulses, see Figure 1). We show for these general wave packets that the interaction leads to an $O(\varepsilon)$-phase shift of the carrier wave and to an $O(\varepsilon)$-shift of the envelope. Thus, we improve the bound for the possible envelope shift caused by the interaction of general localized NLS-described wave packets from $O(1)$, cf. [PW96], to $O(\varepsilon)$ and generalize the $O(\varepsilon)$-bound for the interaction of wave packets with NLS 1-solitons as envelope to general NLS-described wave packets. Moreover, we invalidate by numerical experiments a formula for the envelope shift for general wave packets, but prove analytically the validity of this formula for pulses in the form of NLS 1-solitons.

Notation. Many possibly different constants which can be chosen independently of $0 < \varepsilon \ll 1$ are denoted by C. The space $H^s(m)$ consists of s-times weakly differentiable functions for which $\|u\|_{H^s(m)} = \|u\,\langle x\rangle^m\|_{H^s} = \big(\sum_{j=0}^{s} \|\partial_x^j(u\,\langle x\rangle^m)\|^2 dx\big)^{1/2}$ with $\langle x\rangle = \sqrt{1+x^2}$ is finite, where we do not distinguish between scalar and vector-valued functions or real- and complex-valued functions. The space C_b^s consists of s-times continuously differentiable functions for which $\|u\|_{C_b^s} = \sum_{j=0}^{s} \sup_{x\in\mathbb{R}} |\partial_x^j u|$ is finite. We sometimes write, e.g., $\|u(x)\|_{C_b^s}$ for the C_b^s-norm of the function $x \mapsto u(x)$.

Acknowledgement. The authors would like to thank Kurt Busch and his working group at the University of Karlsruhe for raising this question. The work is partially supported by the Deutsche Forschungsgemeinschaft DFG and the Land Baden-Württemberg through the Graduiertenkolleg GRK 1294/1: Analysis, Simulation und Design nanotechnologischer Prozesse.

2 Approximate description of internal and interaction dynamics

In this and in the next section we derive approximation equations in order to describe the internal and interaction dynamics of the wave packets. In order to make the concept of internal and interaction dynamics more precise let S_t be the nonlinear evolution operator of the nonlinear wave equation (1). The evolution $S_t(u_A)$ of one single initial wave packet u_A is called internal dynamics. The solution to the sum of two single initial wave packets u_A and u_B evolves as $S_t(u_A + u_B)$. The interaction dynamics is then the difference $S_t(u_A + u_B) - S_t(u_A) - S_t(u_B)$. It is the purpose of this paper to give a precise description of this difference. We are especially interested in improved estimates for carrier and envelope shifts caused by the interaction.

Approximate description of internal dynamics. In the case of one single wave packet with a wavenumber k_A the dynamics can be described approximately by the ansatz (2). By adding higher order terms to the ansatz the formal error, or more precisely the later on introduced residual, can be made arbitrarily small. The NLS equation is then accompagnied by a system of linear PDEs and algebraic equations.

Approximate description of interaction dynamics. In the case of two-wave propagation the nonlinearity leads to an interaction between the wave packets which in turn result in a modification of the pure internal dynamics. We improve the ansatz from [CBSU07] and seek solutions of the form

$$u_\varepsilon = (\varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3)E + (\varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3)F + \text{c.c.} + \varepsilon^3 M_{\text{mixed}} \tag{3}$$

where the term M_{mixed} serves to cancel mixed and higher order harmonic terms in the formal error and where

$$\begin{aligned}
E &= \exp\big(i(k_A x - \omega_A t + \varepsilon \sigma_{A;1}(Z_B;T) + \varepsilon^2 \sigma_{A;2}(Z_B;T))\big);\\
F &= \exp\big(i(k_B x - \omega_B t + \varepsilon \sigma_{B;1}(Z_A;T) + \varepsilon^2 \sigma_{B;2}(Z_A;T))\big);\\
Z_A &= \varepsilon(x - c_A t + \varepsilon \sigma_A(X_B;T)); && (4)\\
Z_B &= \varepsilon(x - c_B t + \varepsilon \sigma_B(X_A;T)); && (5)\\
A_j &= A_j(Z_A;T); \quad B_j = B_j(Z_B;T); \quad X_A = \varepsilon(x - c_A t); \quad X_B = \varepsilon(x - c_B t): && (6)
\end{aligned}$$

The internal dynamics of the wave packets will be described by the variables $A_j, B_j, j = 1,2$, whereas the interaction dynamics is described by the phase shifts $\sigma_{A;j}, \sigma_{B;j}, j = 1,2$ and the envelope shifts σ_A, σ_B. The terms A_3, B_3 play a crucial role in this work, since it turns out that depending on the special choice for A_1, B_1 { well- or non-well prepared { they include contributions to the envelope shift and hence make the envelope shift formulas invalid.

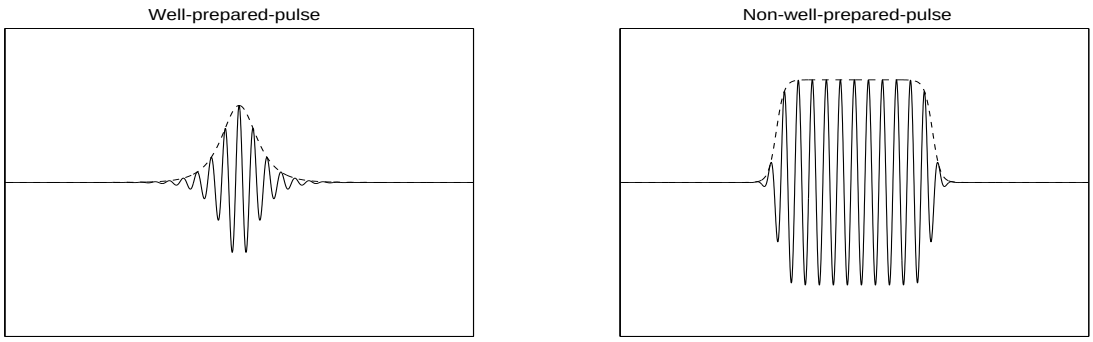

Figure 1: Left: A well-prepared pulse. The envelope (dashed line) is a NLS 1-soliton. Right: A non-well prepared pulse. The envelope (dashed line) can be anything \pulse like". Here we chose an almost rectangular envelope.

Remark 2.1 a) The ansatz (3) is more general than the one in [CBSU07] where we essentially chose A_1 and B_1 in the form of NLS 1-solitons. Here we allow A_1 and B_1 to be more general

solutions of the respective NLS equation, see Figure 1. This therefore requires the introduction of $\varepsilon^2 A_2 E$; $\varepsilon^2 B_2 F$ to describe the internal dynamics.

b) The phase shifts $\gamma_{A,1}$; $\gamma_{B,1}$ turn out to be real functions. In order to describe the interaction dynamics in more detail than in [CBSU07] we additionally introduce phase shift corrections $\gamma_{A,2}$; $\gamma_{B,2}$, which turn out to be imaginary, and the envelope shifts δ_A; δ_B. These last ones have already been introduced in earlier works like [Yan74] or [TPB04] where they are called pulse shifts. Our aim is to validate or invalidate the formulas for the envelope shifts not only formally, but by rigorous estimates.

c) Furthermore, we change the notation: The variables Y_A, Y_B from [CBSU07] are now called A_3, B_3, whereas A_3, B_3 from [CBSU07] are here contained in the term M_{mixed}.

d) Finally, in the following we replace the arguments X_B, X_A of δ_A; δ_B by Z_B, Z_A. More rigorously we may define Z_A, Z_B implicitly by

$$Z_A = \varepsilon(x - c_A t + \varepsilon \tilde{\delta}_A(Z_B)), \qquad Z_B = \varepsilon(x - c_B t + \varepsilon \tilde{\delta}_B(Z_A)). \tag{7}$$

Then δ_A and $\tilde{\delta}_A$ resp. δ_B and $\tilde{\delta}_B$ differ by $O(\varepsilon)$ terms which we may discard for our purposes. Therefore, from now on we write Z_B, Z_A for the arguments of δ_A; δ_B, respectively. c

Remark 2.2 At this point the notion of an envelope shift is somewhat ambiguous since by Taylor-expansion w.r.t. δ_A and δ_B we have with $X_A = \varepsilon(x - c_A t), X_B = \varepsilon(x - c_B t)$,

$$\begin{aligned}\varepsilon\,(x,t) = {} & \varepsilon A_1(X_A,T) + \varepsilon^2 A_2(X_A,T) + \varepsilon^3 (A_3(X_A,T) + \delta_A \partial_1 A_1(X_A,T))\, E \\ & + \varepsilon B_1(X_B,T) + \varepsilon^2 B_2(X_B,T) + \varepsilon^3 (B_3(X_B,T) + \delta_B \partial_1 B_1(X_B,T))\, F + O(\varepsilon^4).\end{aligned}$$

The terms $\varepsilon^2 A_2$ and $\varepsilon^2 B_2$ do not contribute to envelope shifts caused by interaction since they are determined by internal dynamics of the individual pulses (see (12)). The term A_3 is of the same order as the envelope shift term $\delta_A \partial_1 A_1$, i.e., it accounts for both internal and interaction dynamics, but it is neither clear to which amount A_3 describes the interaction, nor in which way { as phase or envelope correction. In other words, it has to be checked, if the derived formulas really quantify the entire envelope shift in the particular order. The validity of the envelope shift formula is investigated numerically in Sec. 5 and explained analytically in Sec. 6. This expansion obviously gives an $O(\varepsilon)$-bound for the envelope shift if we can prove an $O(\varepsilon^3)$-bound in L^1 for the terms indicated with $O(\varepsilon^4)$ and an $O(1)$-bound for A_3 and B_3. Then the vertical bound $O(\varepsilon^3)$ only allows a 'horizontal error' of $O(\varepsilon^1)$. The required bounds will be proven in Proposition 4.3 and Lemma 4.7. c

Remark 2.3 Since $\gamma_{A,2}$; $\gamma_{B,2}$ are supposed to describe interaction dynamics we may assume that $\gamma_{A,2} = \gamma_{B,2} = 0$ initially. Moreover, due to the fact that $\gamma_{A,2}$ and $\gamma_{B,2}$ turn out to be spatially localized, also after interaction E and F contain only phase shifts for $|x| \to \infty$, i.e. $\gamma_{A,2}$ and $\gamma_{B,2}$ play no role for the envelope shift. In detail, in Lemma 4.6 we prove the $O(1)$-boundedness of $\gamma_{A,1}$ and $\gamma_{B,1}$ in L^1 and that $\gamma_{A,2}$ and $\gamma_{B,2}$ are $O(1)$-bounded in $H^s(m)$. Thus, for instance, $|\gamma_{A,2}(\delta_B, T)| \le C/(1 + \varepsilon|x - c_B t|)^m$ due to Sobolev's embedding theorem for $s > 1/2$. For the same reason we have $|A_j(\delta_A, T)| \le C/(1 + \varepsilon|x - c_A t|)^m$ and so, for large t, i.e. for $t > 1/\varepsilon$,

$$|A_j(\delta_A, T)|\,\gamma_{A,2}(\delta_B, T) = O((\varepsilon t)^{-m}).$$

Moreover, for well prepared pulses $|A_j(\cdot_A;T)| \ll |_{A;2}(\cdot_B;T)|$ is exponentially small. According to the last remark, $|A_j(\cdot_A;T)| \ll |_{A;2}(\cdot_B;T)$ has to be $o(\varepsilon^\infty)$, except during interaction. Thus we require $(\Delta t)^m = O(\varepsilon^{1+m})$ with > 0 arbitrary small but fixed. This yields $t \ll \varepsilon^{-(1+1/m+)}$ ε^{-2} for $m \geq 2$. In summary, for $C_1\varepsilon^{-(1+1/m+)} \leq t \leq C_2\varepsilon^{-2}$ the corrections $_{A;2}$ and $_{B;2}$ play no role for the envelope shifts. In case of well prepared pulses this can be sharpened to $C_1\ln(\varepsilon)\varepsilon^{-1} \leq t \leq C_2\varepsilon^{-2}$. c

3 Derivation of approximation equations

The so-called residual

$$\mathrm{Res}(\psi) = \partial_t^2\psi(\varepsilon^\cdot) + \partial_x^2\psi(\varepsilon^\cdot) - \psi(\varepsilon^\cdot) + \psi(\varepsilon^\cdot)^3 \tag{8}$$

describes how much an ansatz fails to satisfy the nonlinear wave equation (1). Plugging in the ansatz (3) into the residual

$$\mathrm{Res}(\psi) = \sum_{l;m;n} \varepsilon^l \mathrm{Res}_{l;m;n} E^m F^n \tag{9}$$

leads to a number of conditions in order to make the residual as small as possible, in particular to Nonlinear Schrödinger equations for A_1 and B_1.

Remark 3.1 The term $M_{\mathrm{mixed}} = M_{\mathrm{mixed}}(A_1;A_2;A_3;B_1;B_2;B_3;E;F)$ accounts for terms involving higher order or mixed harmonics, i.e. for the frequencies which are generated by the nonlinearity according to the formula

$$\begin{aligned}(\varepsilon A_1E + \varepsilon^2A_2E + \varepsilon^3A_3E + \varepsilon B_1F + \varepsilon^2B_2F + \varepsilon^3B_3F + \mathrm{c.c.})^3 \\ = \sum_{k_1+\cdots+k_{12}=3;k_j\geq 0} \frac{3!}{k_1!\cdots k_{12}!}(\varepsilon A_1E)^{k_1}\cdots(\varepsilon^3\overline{B_3F})^{k_{12}};\end{aligned}$$

however without the nonlinear terms generated at E or F. At ε^3E^2F for example the term $A_1^2B_1$ appears. To cancel this term we extend the ansatz by $_{21}\varepsilon^3A_1^2B_1E^2F$ and get an algebraic equation for $_{21}$ of the form

$$1 + (2i\omega_A + i\omega_B)^2 + (2ik_A + ik_B)^2 \ll_{21} = 3:$$

The procedure is essentially the same for each such term leading to equations of the form

$$1 + (l\omega_A + j\omega_B)^2 + (lk_A + jk_B)^2 \ll_{lj} = \ll_{lj}:$$

Now M_{mixed} contains all these extensions. Thus, we can concentrate on the remaining terms of the residual. c

Remark 3.2 Since $A_j; j = 1;2;3;$ depend on the same variables and belong to the same harmonic, the subsequent hierarchy of conditions (10){(15) appears shifted in order, i.e. the residual actually contains much more terms, for example $i(-k_A^2 + \omega_A^2 - 1)A_j(Z_A;T)E; j = 1;2;3$, which we only listed for $j = 1$. Hence choosing the dispersion relation as solvability condition cancels all terms. The exact same mechanism holds for the entire system of equations (10){(15), so we tacitly left all these terms out to simplify the exposition. c

Using the notation $\mathrm{Res}_{l;m;n}$ from (8) for the coefficients of $\varepsilon^l E^m F^n$ we find the subsequent hierarchy of equations.

At εE we find

$$\mathop{\mathrm{Res}}_{1;1;0} = (-k_A^2 + \omega_A^2 - 1)A_1(Z_A;T) \stackrel{!}{=} 0$$

which yields the linear dispersion relation

$$\omega_A^2 = k_A^2 + 1:$$

At $\varepsilon^2 E$ we find

$$\mathop{\mathrm{Res}}_{2;1;0} = 2i(-k_A c_A + \omega_A)\partial_1 A_1(Z_A;T) \stackrel{!}{=} 0$$

which yields the linear group velocity

$$c_A = k_A/\omega_A:$$

At $\varepsilon^3 E$ we find

$$\mathop{\mathrm{Res}}_{3;1;0} = s_{31} + s_{32}$$

with

$$\begin{aligned} s_{31} &= 2i\omega_A\partial_2 A_1(Z_A;T) + (1 - c_A^2)\partial_1^3 A_1(Z_A;T) + 3|A_1(Z_A;T)|^2 A_1(Z_A;T); \\ s_{32} &= -2(\omega_A c_B - k_A)(Z_A;T)\partial_1 \Phi_{A;1}(Z_B;T) + 6|B_1(Z_B;T)|^2 - A_1(Z_A;T): \end{aligned}$$

Then $s_{31} \stackrel{!}{=} 0$ yields the NLS equation

$$2i\omega_A\partial_2 A_1(Z_A;T) = (1 - c_A^2)\partial_1^3 A_1(Z_A;T) + 3|A_1(Z_A;T)|^2 A_1(Z_A;T); \qquad (10)$$

and $s_{32} \stackrel{!}{=} 0$ yields the phase shift formula

$$\Phi_{A;1}(Z_B;T) = \frac{3}{k_A - \omega_A c_B} \int^{Z_B} |B_1(\zeta;T)|^2 \, d\zeta; \qquad (11)$$

so $\Phi_{A;1}$ is a real quantity and therefore a pure phase correction.

At $\varepsilon^4 E$ we find

$$\mathop{\mathrm{Res}}_{4;1;0} = 2\omega_A i\partial_2 A_2(Z_A;T) + (1 - c_A^2)\partial_1^3 A_2(Z_A;T) + s_{41} + s_{42} + s_{43} + s_{44} + s_{45} + s_{46}$$

where

$$\begin{aligned} s_{41} &= -6A_2(Z_A;T)\overline{A}_1(Z_A;T) + 3\overline{A}_2(Z_A;T)A_1(Z_A;T) - A_1(Z_A;T); \\ s_{42} &= 2c_A\partial_1\partial_2 A_1(Z_A;T); \\ s_{43} &= 6 - B_2(Z_B;T)\overline{B}_1(Z_B;T) + \overline{B}_2(Z_B;T)B_1(Z_B;T) - A_1(Z_A;T); \\ s_{44} &= -2\omega_A A_1(Z_A;T)\partial_2 \Phi_{A;1}(Z_B;T); \\ s_{45} &= 2i\partial_1 A_1(Z_A;T)((k_A - c_B\omega_A)\partial_1 \Phi_A(Z_B;T) + (1 - c_A c_B)\partial_1 \Phi_{A;1}(Z_B;T)); \\ s_{46} &= i(1 - c_B^2)\partial_1^3 \Phi_{A;1}(Z_B;T) + 2\partial_1 \Phi_{A;2}(Z_B;T)(\omega_A c_B - k_A) - A_1(Z_A;T): \end{aligned}$$

The terms in s_{43}, s_{44} are interaction terms in the sense that they are products of functions such that both Z_A and Z_B appear as arguments. Thus, since we consider localized solutions, they are $O(1)$ only on an $O(\varepsilon)$ time-scale and will, therefore, be moved into the equations for A_3 at $\varepsilon^5 E$.

We are now left with a linear inhomogeneous evolution equation for A_2,

$$2i\omega_A \partial_2 A_2(Z_A, T) = (1 - c_A^2)\partial_1^3 A_2(Z_A, T) + s_{41} + s_{42}. \tag{12}$$

Here, no coupling with terms involving B-variables occurs such that A_2 describes internal dynamics of a single pulse.

The terms in s_{45} together with (11) give the envelope shift formula

$$\gamma_A(Z_B, T) = \frac{3(1 - c_A c_B)}{(c_B - \omega_A k_A)^2} \int^{Z_B} |B_1(\zeta, T)|^2 \, d\zeta. \tag{13}$$

The terms in s_{46} yield the second order correction to the phase shift in the form

$$\gamma_{A,2}(Z_B, T) = \frac{i(1 - c_B^2)}{2(k_A - \omega_A c_B)} \partial_2 \gamma_{A,1}(Z_B, T) = \frac{3i(1 - c_B^2)}{2(k_A - \omega_A c_B)^2} |B_1(Z_B, T)|^2; \tag{14}$$

so $\gamma_{A,2}$ is purely imaginary and therefore an amplitude correction, which however is algebraically small w.r.t. ε except during collision of wave packets.

At $\varepsilon^5 E$ we find

$$\operatorname*{Res}_{5,1,0} = 2\omega_A i \partial_2 A_3(Z_A, T) + (1 - c_A^2)\partial_1^3 A_3(Z_A, T) + s_{51} + s_{52} + s_{53} + s_{54} + s_{55} + s_{56}$$

where

$$\begin{aligned}
s_{51} &= -6A_3(Z_A, T)\overline{A}_1(Z_A, T) + 3\overline{A}_3(Z_A, T)A_1(Z_A, T) - A_1(Z_A, T);\\
s_{52} &= -\partial_2^3 A_1(Z_A, T) + 2c_A \partial_1 \partial_2 A_2(Z_A, T);\\
s_{53} &= -6A_1(Z_A, T)|A_2(Z_A, T)|^2;\\
s_{54} &= -6B_3(Z_B, T)\overline{B}_1(Z_B, T) + \overline{B}_3(Z_B, T)B_1(Z_B, T) - A_1(Z_A, T);\\
s_{55} &= (1 - c_A^2)\partial_1 A_1(Z_A, T)\partial_1^3 \gamma_A(Z_B, T) + 2(1 - c_A c_B)\partial_1^3 A_1(Z_A, T)\partial_1 \gamma_A(Z_B, T)\\
&\quad + 2i\left[(1 - c_B^2)A_1(Z_A, T)\partial_1^3 \gamma_{A,2}(Z_B, T) + 2(1 - c_A c_B)\partial_1 A_1(Z_A, T)\partial_1 \gamma_{A,1}(Z_B, T)\right];\\
s_{56} &= -2\omega_A\left(A_1(Z_A, T)\partial_2 \gamma_{A,2}(Z_B, T) + A_2(Z_A, T)\partial_2 \gamma_{A,1}(Z_B, T)\right)\\
&\quad + i\left[2c_B \partial_2 A_1(Z_A, T)\partial_1 \gamma_{A,1}(Z_B, T) + 2c_A \partial_1 A_1(Z_A, T)\partial_2 \gamma_{A,1}(Z_B, T)\right.\\
&\quad \left. + 2c_B A_1(Z_A, T)\partial_1 \partial_2 \gamma_{A,1}(Z_B, T)\right].
\end{aligned}$$

The terms s_{51}, s_{52} and s_{53} describe internal dynamics, whereas s_{54}, s_{55} and s_{56} are interaction terms in the same sense as s_{43} and s_{44}. We choose A_3 to satisfy the linear PDE

$$\begin{aligned}
2i\omega_A \partial_2 A_3(Z_A, T) &= (1 - c_A^2)\partial_1^3 A_3(Z_A, T) + M_0[A_3, B_3]\\
&\quad + I(A_1, A_2, B_1, B_2, \gamma_A, \gamma_A) + \varepsilon^{-1}(s_{43} + s_{44})
\end{aligned} \tag{15}$$

where $M_0[A_3, B_3] = s_{51} + s_{54}$ is linear in its arguments and $I(A_1, A_2, B_1, B_2, \gamma_A, \gamma_A) = s_{52} + s_{53} + s_{55} + s_{56}$ contains inhomogeneous terms which are $O(1)$ bounded on the $O(1/\varepsilon^2)$-time scale if $A_1, \ldots, \gamma_B$ are $O(1)$ bounded (up to second derivatives for A_1, B_1).

Finally we choose $B_1, B_2, B_3, \gamma_{B,1}, \gamma_{B,2}$, and γ_B to satisfy the counterparts to (10)–(15).

4 Validity of the approximation

As a consequence of the perturbation analysis of the last section, the first non-vanishing terms in the residual are formally of order $O(\varepsilon^6)$. Below we will prove

Lemma 4.1 Let $s_A \geq s+4$; $k_A \geq k_B$; $k_A, k_B > 0$, and let $A|_{T=0}, B|_{T=0} \in H^{s_A}(\mathbb{R}) \cap H^{s_A+4}(0)$. Then for all $T_0 > 0$ there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|\mathrm{Res}(\cdot)\|_{H^s} \leq C\varepsilon^{11/2}.$$

The difference between the exponents of the formal error $O(\varepsilon^6)$ and $O(\varepsilon^{11/2})$ in the lemma follows from the scaling properties of the L^2-norm. The weighted spaces $H^s(m)$ are used to describe analytically the condition that the wave packets are spatially localized. This is needed to estimate the interaction terms like for instance s_{43} and s_{44}.

As a direct consequence of Lemma 4.1 and of the fact that our original system (1) does not contain quadratic terms, with a simple application of Gronwall's inequality [KSM92] it follows that the original system really behaves as predicted by the approximation.

Theorem 4.2 (similar to [CBSU07, Theorem 3.6]) Let $s_A \geq s+4$; $k_A \geq k_B$; $k_A, k_B > 0$, and let $A|_{T=0}, B|_{T=0} \in H^{s_A}(\mathbb{R}) \cap H^{s_A+4}(0)$. Then for all $T_0 > 0$ there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|u(x,t) - \varepsilon\psi(x,t)\|_{H^s} \leq C\varepsilon^{7/2}.$$

From Theorem 4.2 we obtain by Sobolev's embedding theorem

Proposition 4.3 Under the assumptions of Theorem 4.2 we have

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|u(x,t) - \varepsilon\psi(x,t)\|_{C_b^{s-1}} \leq \varepsilon^{7/2}. \tag{16}$$

As explained in Remark 2.2 this last estimate together with the subsequent Lemma 4.7 allows us to bound the magnitude of the envelope shift by $C\varepsilon$.

Hence it remains to give the Proof of Lemma 4.1. The assertion obviously follows if we prove that the approximation equations (10)–(15) possess order O(1)-bounded solutions on the $O(1/\varepsilon^2)$-time scale. We have to solve three different kinds of equations. The first set of equations, (10) and (12), describes internal dynamics. Since these two equations are independent of the small parameter $0 < \varepsilon \ll 1$ we have

Lemma 4.4 For all $s \geq 2, m \geq 0$, and initial condition $A_1|_{T=0} \in H^s(m) \cap H^{s+2m}(0)$ there exists a time $T_0 > 0$ such that (10) has a unique solution

$$A_1 \in C^1([0, T_0], H^s(m) \cap H^{s+2m}(0)).$$

Proof: We apply the variation of constant formula and use the fact that $i\partial_x^2$ is the generator of a strongly continuous semigroup in $H^s(m) \cap H^{s+2m}(0)$, cf. [CKS95]. ∎

Note that T_0 is independent of the weight. This can be proven like in [SW00, Lemma 6.4] such that the existence time is determined only by the local existence and uniqueness in H^s-spaces.

Since (12) is a linearized NLS equation for A_2 with $O(1)$-bounded inhomogeneous terms $s_{41} + s_{42}$ with exactly the same arguments we find

Lemma 4.5 Let $A_1 \in C([0,T_0]; H^s(m) \cap H^{s+2m}(0))$ with $s \geq 2$ be a solution of (10). Then for all initial conditions $A_2|_{T=0} \in H^s(m) \cap H^{s+2m}(0)$ there exists a unique solution of (12) with

$$A_2 \in C\left([0,T_0]; H^s(m) \cap H^{s+2m}(0)\right).$$

The second group of equations, namely (11), (13), and (14), describes the essential interaction dynamics. By pure integration we find

Lemma 4.6 Let $A_1, B_1 \in C([0,T_0]; H^s(m) \cap H^{s+2m}(0))$ be a solution of (10). Then

$$\partial_{\xi_{A,1}}, \partial_{\xi_{B,1}}, \partial_{\xi_B}\partial_{\xi_A}, \partial_{\xi_A}\partial_{\xi_B}, \partial_{\xi_{A,2}}, \partial_{\xi_{B,2}} \in C([0,T_0]; H^s(m) \cap H^{s+2m}(0)),$$

and $\xi_{A,1}, \xi_{B,1}, \xi_A, \xi_B \in C([0,T_0]; C_b^{s+2m})$.

In terms of local existence and uniqueness and $O(1)$-boundedness of solutions the only nontrivial equation is (15) which is a linearized NLS equation for A_3 with $O(1)$-bounded inhomogeneous terms and terms $\varepsilon^{-1}(s_{43} + s_{44})$. Since the last terms are only $O(\varepsilon^{-1})$ on an $O(\varepsilon)$-scale w.r.t. T we find

Lemma 4.7 For all $s \geq 2$ there exists a $C > 0$ such that for all $\varepsilon \in (0,1]$ the following holds. System (15) with zero initial data has a unique solution $A_3, B_3 \in C([0,T_0]; H^s(m) \cap H^{s+2m}(0))$. It satisfies

$$\sup_{0 \leq T \leq T_0} \|(A_3, B_3)(T)\|_{H^s(m) \cap H^{s+2m}(0)} \leq C.$$

Proof: [CBSU07, Lemma 4.2] ∎

5 Numerical simulations

Before we discuss the validity of the envelope shift formula (13) in case of well-prepared pulses we provide some numerical experiments to illustrate the above analysis. As a result of our experiments we invalidate the formula for the envelope shift in case of non-well prepared pulses.

The numerical scheme used is accurate enough so that the true differences between the analytical approximate solutions and the actual (numerical in this case) solution can be detected. The scheme also conserves energy which is necessary to have precise estimates, according to the

lower bound $|u - u_{approx}| \geq |u| - |u_{approx}|$. Due to the multiscale character of the problem such numerical computations are CPU and memory intensive and require extended periods of time to run. Therefore, the analytical approximation solution is clearly preferable to the numerical one, which is computed here only to draw comparisons. The initial value of the numerical solution is defined on a large grid of equally spaced points x_m, $m \in 1, \ldots, N$, and is defined as

$$u_{num}(x_m, 0) = \varepsilon(x_m, 0), \tag{17}$$

where $\varepsilon(x, t)$ is defined in (3). The numerical solution is generated at equally spaced values of time t_n, $n \in N$, by integrating (1). In the examples below k_A, k_B, x_A, and x_B are chosen such that pulse A will travel through pulse B.

Theorem 4.2 is confirmed numerically by computing the difference

$$r(t_n) = \sup_m |u_{num}(x_m, t_n) - \varepsilon(x_m, t_n)|$$

as a function of time. Moreover, in order to numerically compute the phase shifts and the envelope shifts, the two-pulse solution is compared with the sum of two corresponding single pulse solutions. The phase shift was computed by finding the average difference between adjacent roots of the shifted (two-pulse solutions) and non-shifted solution (two single pulse solutions). For $k_B = 0$ the envelope shift can be estimated by looking at the position of the maximal amplitude. This is due to the fact that the carrier wave with $k_B = 0$ will be identical to its modulating envelope, which makes it easier to detect the actual envelope shift. In the case that $k_B \neq 0$ the envelope was fit with an appropriate function including a parameter for the envelope shift.

The confirmation and quantification of analytical results is as follows

> It was shown in [CBSU07] that if ε is the sum of two well-prepared pulses with the corrections $\varepsilon_{A,1}, \varepsilon_{B,1}$ for the phase shifts taken into account, then
>
> $$\sup_{t \in [0, t_0/\varepsilon^2]} r(t) = O(\varepsilon^3);$$
>
> see also the discussion following Lemma 6.2 for the relation of the ansatz in [CBSU07] to our ansatz ε in (3). We first numerically confirm this result and compare it to the standard ansatz where no phase shift corrections are taken into account. In the left panel of Figure 2 a plot of $r(t)$ is shown. Before interaction the difference between the standard and the improved solution is negligible. For times after the interaction the approximate solution with the $\varepsilon_{A,B}$ corrections is $O(\varepsilon^3)$ accurate and only $O(\varepsilon^2)$ without it. This procedure was carried out for various values of ε in order to deduce the asymptotic behavior as $\varepsilon \to 0$. The results are plotted in the right panel of Figure 2, with the improved approximation clearly superior. Figure 3 shows a comparison of numerically computed shifts with those predicted by the formulas given in (11) and (13).
>
> We turn our attention to the case where the envelope is not described by a 1-soliton, but rather an arbitrary solution to the NLS equation. These solutions are called non-well prepared pulses, see Figure 1 for an example. Using the ansatz (3) again achieve $O(\varepsilon^3)$-order accuracy without the assumption that the envelope is well-prepared, see the

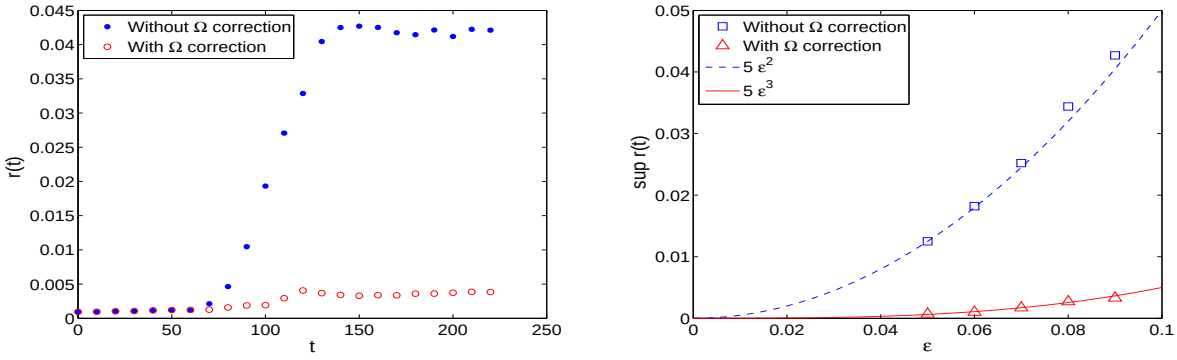

Figure 2: Left: Plot of error function, $r(t)$, for $\varepsilon = 0.9$. Right: Plot of $\sup_{x \in \mathbb{R}} r(t)$ for various ε.

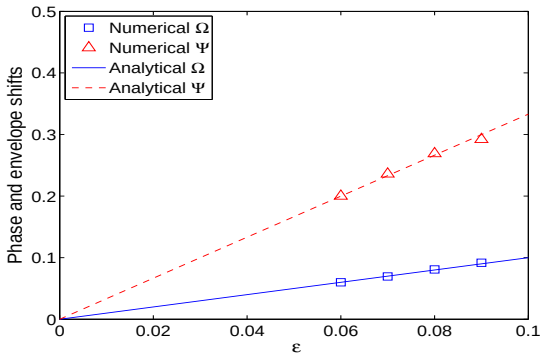

Figure 3: Plot of numerically computed phase and envelope shifts (markers) and the analytical values (lines); both shifts are $O(\varepsilon)$ and the analytical shifts are close to the computed ones.

left panel of Figure 4. The phase and envelope shifts are computed in the same way as described above. Both exhibit an $O(\varepsilon)$-trend but the formula for the envelope shift is invalid, see the right panel of Figure 4.

Summary. We find an $O(\varepsilon)$-shift of the phase and of the envelope. The formula for the phase shift is valid in case of well and non-well prepared pulses. In contrast, the formula for the envelope shift is only valid in case of well prepared pulses, but makes wrong predictions in case of non-well prepared pulses.

6 The validity of the envelope shift formula

As already alluded to in Remark 2.2, it is not clear if the correction term A_3 contributes to the description of the envelope shift or not. Therefore, in order to distinguish between the parts of A_3, which account for internal and interaction dynamics respectively, we introduce the following definition.

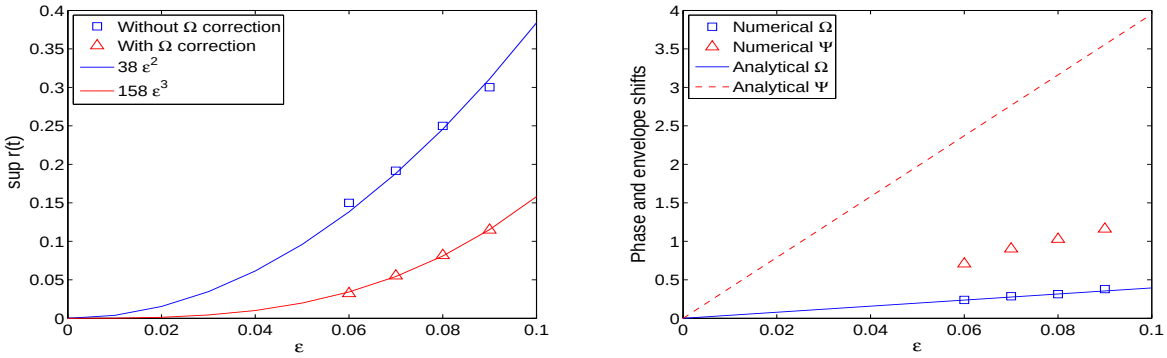

Figure 4: Left: Plot of error for non-well prepared pulses. When taking into account the phase shift again an $O(\varepsilon^3)$ trend is shown. Right: Plot of numerically computed phase and envelope shifts (markers) and analytical values (lines) for non-well prepared pulses; both shifts are $O(\varepsilon)$, but only the predicted phase shift value is valid.

Definition 6.1 Let $A_3^{(c)}, B_3^{(c)}$ be a solution to the coupled system

$$2i\omega_A \partial_2 A_3(Z_A, T) = (c_A^2 \partial_1^2 - 1)\partial_1^3 A_3(Z_A, T) - M_0[A_3, B_3] - I(A_1, A_2, B_1, B_2, \gamma_A, \gamma_A) - \varepsilon^{-1} L_A,$$
$$2i\omega_B \partial_2 B_3(Z_B, T) = (c_B^2 \partial_1^2 - 1)\partial_1^3 B_3(Z_B, T) - M_0[B_3, A_3] - I(B_1, B_2, A_1, A_2, \gamma_B, \gamma_B) - \varepsilon^{-1} L_B,$$

where $L_A = s_{43} + s_{44}$ and L_B respectively. Let $A_3^{(u)}, B_3^{(u)}$ be a solution to the uncoupled system

$$2i\omega_A \partial_2 A_3(X_A, T) = (c_A^2 \partial_1^2 - 1)\partial_1^3 A_3(X_A, T) - M_0[A_3, 0] - I(A_1, A_2, 0, 0, 0, 0),$$
$$2i\omega_B \partial_2 B_3(X_B, T) = (c_B^2 \partial_1^2 - 1)\partial_1^3 B_3(X_B, T) - M_0[B_3, 0] - I(B_1, B_2, 0, 0, 0, 0).$$

We call the envelope shift formula (13) valid, if

$$\| A_3^{(u)} - A_3^{(c)} \|_{C_b^s} \leq C\varepsilon^\gamma, \tag{18}$$

for an $\gamma > 0$, and respectively for $B_3^{(u)}, B_3^{(c)}$.

So the envelope shift formula (13) is only valid, if the correction terms A_3, B_3 only describe internal dynamics (at least in leading order).

In [CBSU07] we constructed well-prepared pulses as follows.

Lemma 6.2 Let $s \geq 2$, $k_0 > 0$ and $\omega_0 < 0$. For sufficiently small $\varepsilon > 0$ there exists a two-dimensional family of approximate modulating pulse solutions to (1) of the form

$$u(x,t) = \varepsilon v_{k_0}(x - c_g t + x_0; k_0 x - \omega t + \phi), \tag{19}$$

parameterized by the envelope shift $x_0 \in \mathbb{R}$ and phase shift $\phi \in [0, 2\pi)$, where v_k is 2π-periodic in its second argument, $\omega = \omega_0 + \omega_0\varepsilon^2 + O(\varepsilon^4) = k_0 c_p$ with phase velocity $c_p = c_p^0 + \omega_1\varepsilon^2 + O(\varepsilon^4)$, where $c_p^0 = \omega_0/k_0$ is the linear phase velocity, $\omega_1 = \omega_0/k_0$, and with group velocity $c_g = k_0/\omega = 1/c_p$. Moreover,

$$\varepsilon v_{k_0}(\xi, y) = \varepsilon A_{\text{pulse}}(\varepsilon\xi) e^{iy} + \text{c.c.} + O(\varepsilon^3 e^{-r|\xi|}) \tag{20}$$

with $\tilde{A}_{pulse}$ given by the homoclinic solution

$$\tilde{A}_{pulse}(X) = \left(\frac{2C_1}{C_2}\right)^{1=2} \operatorname{sech}(C_1^{1=2}X) \tag{21}$$

of

$$\partial_X^2 \tilde{A} = C_1\tilde{A} \quad C_2\tilde{A}^3; \quad C_1 = \quad 2\ _0!\ _0=(1 \quad c_0^2); \quad C_2 = 3=(1 \quad c_0^2); \tag{22}$$

where $c_0 = k_0=(1 + k_0^2)^{1=2}$ is the linear group velocity and

$$j\tilde{A}_{pulse}(X)j \quad Ce^{\ rjXj}; \quad r = \sqrt{\frac{2\ _0!\ _0}{1 \quad c_0^2}}:$$

Finally, the residual fulfills

$$kRes(\ 'v_{k_0})k_{H^s} \quad C"^{11=2}: \tag{23}$$

Following this construction we find $!^2 = k^2 + 1 + \sim(")"^2$ with $\sim(0) < 0$. By this choice the NLS-equation changes into

$$2i!_A\partial_2 A_1(Z_A;T) = (1 \quad c_A^2)\partial_1^3 A_1(Z_A;T) + \sim_1(0)A_1 + 3jA_1(Z_A;T)j^2A_1(Z_A;T);$$

and similar in the equations for A_2 and A_3. The well prepared pulses are constructed via the stationary solutions of the last equations, i.e. in case of well prepared pulses we find $s_{42} = 0$. As a consequence, we can choose $A_2 = B_2 = 0$ such that $s_{43} = 0$. Since $\partial_2\ _{A;1} = 0$ for such pulses, we also have $s_{44} = 0$. Now the coupled and uncoupled version of the evolution equations for A_3 only differ through the coupling terms, which are $O(")$ on an $O(1)$ time scale w.r.t. T. Hence, the envelope shift formula is valid in this special case.

Remark 6.3 For general wave packets we have $s_{42} \neq 0$, such that we need the correction given by A_2 and hence neither s_{43} nor s_{44} vanish. Since $"^{\ 1}\int_0^T ks_{43} + s_{44}k\ d\ = O(1)$ already for $T = O(")$ we have that

$$kA_3^{(u)} \quad A_3^{(c)}k_{C_b^s} = O(1);$$

also already for $T = O(")$ and the envelope shift formula can be expected to be invalid for the general situation of non-well prepared pulses as our numerical experiments confirm.

Conclusions. In leading order the two-wave propagation is given by a linear superposition of the individual waves as long as they are well separated. Then nonlinear behavior appears during collision which causes a phase shift that translates into an envelope shift. Hence, the solitary wave interaction is elastic in leading order { a scenario reminiscent of integrable equations. Since

$$"g("(x + "a)) \quad "g("x) = "g'("x)"^2a + O\ "("^2a)^2\ = O("^3)$$

the estimate (16) immediately shows that an envelope shift larger than $O(")$ is not possible. This estimate is valid both for well- and non-well prepared pulses, however, quantitatively, the derived envelope shift formula (14) is only valid for well-prepared pulses, whereas for general wave packets additional contributions must be taken into account.

References

[Ag01] G.P. Agrawal. Nonlinear fiber optics. Academic Press, 3. ed., 2001.

[BF06] A. Babin, A. Figotin. Linear superposition in nonlinear wave dynamics. Rev. Math. Phys., 18:971–1053, 2006.

[CBSU07] M. Chirilus-Bruckner, G. Schneider, and H. Uecker. On the interaction of NLS-described modulating pulses with different carrier waves. Math. Meth. Appl. Sci., 30:1965–1978, 2007.

[CKS95] W. Craig, T. Kappeler, W. Strauss. Microlocal dispersive smoothing for the Schrödinger equation. Comm. Pure Appl. Math. 48:769–860, 1995.

[Den93] J. Denzler. Nonpersistence of breather families for the perturbed sine Gordon equation. Comm. Math. Phys., 158(2):397–430, 1993.

[HK95] A. Hasegawa, Y. Kodama. Solitons in optical communications. Clarendon Press, 1995.

[KSM92] P. Kirrmann, A. Mielke, G. Schneider. The validity of modulation equations for extended systems with cubic nonlinearities. Proc. Roy. Soc. Edin. A, 122, 85–91, 1992.

[OY74] M. Oikawa, N. Yajima. A Perturbation Approach to Nonlinear Systems. 2: Interaction of Nonlinear Modulated Waves. J. Phys. Soc. Japan, 37:486–496, 1974.

[PW96] R.D. Pierce and C.E. Wayne. On the validity of mean field amplitude equations for counterpropagating wavetrains. Nonlinearity , 8:433–457, 1996.

[SW00] G. Schneider and C.E. Wayne. The long-wave limit for the water wave problem. I. The case of zero surface tension. Comm. Pure Appl. Math. 53:1475–1535, 2000.

[SS99] C. Sulem, P.-L. Sulem. The nonlinear Schrödinger equation. Appl. Math. Sci., Springer-Verlag, Vol.139 ,1999.

[TPB04] L. Tkeshelashvili, S. Pereira, and K. Busch. General theory of nonresonant wave interaction: Giant soliton shift in photonic band gap materials. Europhys. Lett., 86(2):205–211, 2004.